The Rule of Double False Position

Much of our work in this course has focused on the use of matrices in solving linear equations. The first people known to have used them in this way were Chinese scholars of the Han dynasty, about 2000 years ago. Their name for matrix was *fangcheng*, which can be translated as "solution by tabulation" or "rectangular array." Centuries later, Europeans "rediscovered" many of the ancient Chinese techniques, and augmented them with discoveries of their own.

As a review and warm-up, see if you can solve the following three problems that are taken from a classic mathematical text called *Chiu Chang Suan Shu* ("Nine Chapters on the Mathematical Art"), which was written by one or more unknown Han scholars in China sometime before the year 100 CE.

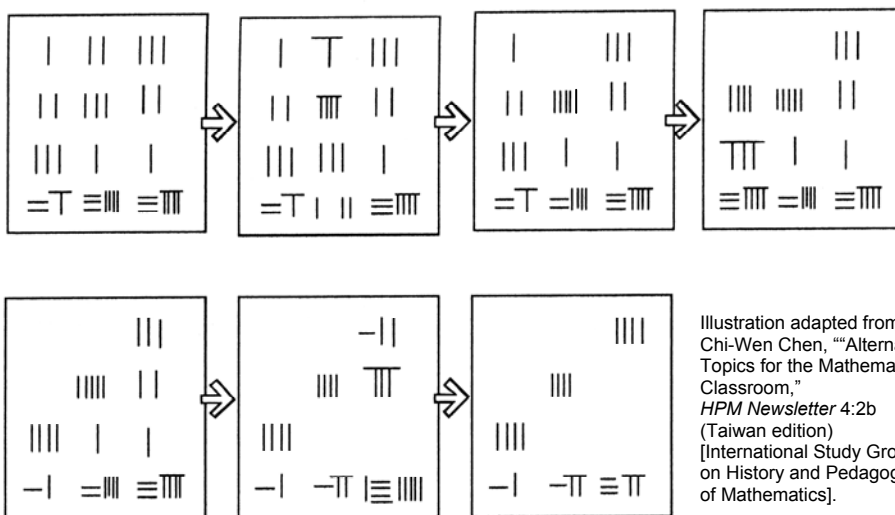Emperor Wu led Han China during 141 – 87 BCE, its period of greatest expansion.

Image: http://en.wikipedia.org/wiki/Emperor_Wu_of_Han

Exercise 1. This is the first problem from the chapter on rectangular arrays in the "Nine Chapters" text. Solve it using the method that Europeans named "Gauss-Jordan Elimination."

The total yield of 3 sheaves of superior grain, 2 sheaves of medium grain and 1 sheaf of inferior grain is 39 *dou* of rice. The total yield of 2 sheaves of superior grain, 3 sheaves of medium grain and 1 sheaf of inferior grain is 34 *dou*. The total yield of 1 sheaf of superior grain, 2 sheaves of medium grain and 3 sheaves of inferior grain is 26 *dou*. What is the yield of one sheaf of each grade of grain?

This is how the author(s) of the "Nine Chapters" solved Exercise 1. As you can see, they arranged the equations on columns instead of rows. Then they performed a series of operations, much as you did, in order to reduce the matrix to echelon form. Numbers were represented by clusters of carved wooden sticks (counting rods) placed horizontally or vertically. The author(s) avoided using negative numbers in this chapter of their text, while zeroes were represented with blank spaces. Thus, the last matrix can be read as follows: "4 sheaves of superior grain yields 37 *dou*, 4 sheaves of medium grain yields 17 *dou*, and 4 sheaves of inferior grain yields 11 *dou*."

Illustration adapted from Chi-Wen Chen, ""Alternative Topics for the Mathematics Classroom," *HPM Newsletter* 4:2b (Taiwan edition) [International Study Group on History and Pedagogy of Mathematics].

Does this answer agree with your own answer to Exercise 1? _____

The Chinese also used some of what we call determinants. Solve the next two problems by using the method that Europeans named "Cramer's Rule."

Exercise 2. Five large containers and one small container have a total capacity of 3 *hu*. One large container and 5 small containers have a capacity of 2 *hu*. Find the capacities of one large container and one small container.

Exercise 3. The yield of 2 sheaves of superior grain, 3 sheaves of medium grain and 4 sheaves of inferior grain is each less than 1 *dou*. But if one sheaf of medium grain is added to the superior grain or if one sheaf of inferior grain is added to the medium, or if one sheaf of superior grain is added to the inferior, then in each case the yield is exactly one *dou*. What is the yield of one sheaf of each grade of grain?

Our main goal in this activity is for you to learn a method that you can use to quickly find the intercepts of a line if you, know any two points on the line. In English, this technique is known as the "Rule of Double False Position," or the "Double False Method." It was used both by Chinese and by Arab mathematicians over 1,000 years ago. In the next exercise, you'll derive the method from Cramer's Rule.

Exercise 4. Consider the general linear equation $px + qy = r$.

(a) Express the intercepts in terms of p, q and r.

x-intercept = _____

y-intercept = _____

(b) Suppose that we know points (a, b) and (c, d) lie on the line. Write the pair of equations that results when these points are plugged into the linear equation.

(c) Use the result of part (b) and Cramer's Rule of Determinants to find p and q in terms of r, a, b, c, and d. Show your work.

$p =$ _____

$q =$ _____

(d) Combine the results of parts (a) and (c) to express each intercept purely in terms of a, b, c, and d.

x-intercept = _____

y-intercept = _____

The pair of formulae that you came up with in Exercise 4(d) is one way to express the Rule of Double False Position. But Chinese and Arab mathematicians discovered this method in their own way, without formulae. They didn't use Cramer's Rule of Determinants like you just did, because Cramer's Rule came much later (in the 1700s in Europe). They didn't even have symbols like x or y to represent variables. But that's not a problem for us, because their method works even if the equation of the line isn't known, *as long as any two points on the line are known.* Once it was discovered, this method could be used very easily. Here's an example that shows how.

Sample Problem

On a certain toll road, the traffic averages 36,000 vehicles per day when the toll is set at $1 per car. Experience has shown that increasing the toll will result in 300 fewer vehicles for each penny of increase. Highway officials who are planning future budgets assume that this trend will continue. They have a couple of questions:

(a) Traffic would dwindle to 0 vehicles if how high a toll were set?

(b) How many vehicles would be expected if there were no toll at all?

Our answers to questions (a) and (b) will supply the missing quantities (the intercepts) in this table:
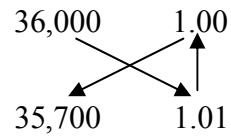
average daily traffic (number of vehicles)	toll (dollars)
36,000	1.00
35,700	1.01
0	
	0

Of course, we could use algebra to find these intercepts. First, we would select a pair of letters like x and y to represent the two variables. Then, we would figure out an equation for the linear function relating the two variables. Finally, we would substitute zero for either variable, and solve for the corresponding intercept.

But the Double False Method is easier. First, we arrange the numbers in a grid:

36,000   1.00

35,700   1.01

To find the intercept for the left column, we will follow this path through the grid:
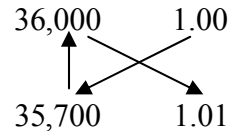
$$36{,}000 \qquad 1.00$$
$$35{,}700 \qquad 1.01$$

(Notice that an arrow is "missing" in the left-hand column: that reminds us that we're calculating the missing intercept for that column.) Along the diagonal arrows we'll multiply, and along the vertical arrow we'll subtract. The calculation is arranged like this:

$$\frac{36000 \times 1.01 - 1.00 \times 35{,}700}{1.01 - 1.00} = \frac{660}{.01} = 66{,}000 \,.$$

That's the intercept for the left column. Do you see how it matches up with the formula you got in Exercise 4(d)? The grid with the arrows is just a way to memorize the procedure without memorizing the formula. This is an example of what some have called "algebra without variables."

To find the intercept for the *right* column, we use a path where the *right* column is missing an arrow:

$$36{,}000 \qquad 1.00$$
$$35{,}700 \qquad 1.01$$

$$\frac{1.00 \times 35{,}700 - 36{,}000 \times 1.01}{35{,}700 - 36{,}000} = \frac{-660}{-300} = 2.20$$

We can now fill in the missing intercepts:

average daily traffic (number of vehicles)	toll (dollars)
36,000	1.00
35,700	1.01
0	2.20
66,000	0

So, traffic would dwindle to 0 vehicles if we set the toll to $2.20 or higher. And we expect that traffic would jump to 66,000 vehicles per day if there were no toll at all.

Practice Problem

Exercise 5. A traffic engineer is studying how the road-construction delays along one stretch of Highway 230 are affecting the amount of daily traffic along the highway. When the delay was 22 minutes, the average daily traffic was 25,300 vehicles. When the delay increased to 30 minutes, the average daily traffic fell to 17,700 vehicles. Assume that the trend continues in a straight line.

(a) Arrange the information in this table (don't forget to write the column headings on the first row):

(b) Average daily traffic would dwindle to 0 vehicles if the construction delay was how long?

Arrange the numbers in a grid with arrows:

Calculate the answer (round to nearest one-tenth of a minute):

(c) If there were no construction delay at all, how much average daily traffic would you expect?

Arrange the numbers in a grid with arrows:

Calculate the answer (round to nearest whole vehicle):

Method of the Eliminant

In addition to Cramer's Rule, the method of the *eliminant* (which we studied in section 11.1) is another determinant-based technique that can be used to derive the Rule of Double False Position.

Exercise 6. Let's rewrite the general linear equation $px + qy = r$ in the form $px + qy - r = 0$.

(a) Suppose that we know points (a, b) and (c, d) lie on the line. Write the pair of equations that result when these points are plugged into the linear equation:

$$px + qy - r = 0$$

$(a, b) \longrightarrow$ _____

$(c, d) \longrightarrow$ _____

(b) Notice that the result is a 3 x 3 homogeneous linear system in the variables p, q, r. Write the 3 x 3 coefficient matrix below.

(c) We want a nontrivial solution of this system. Recall that a square homogeneous linear system has a nontrivial solution if and only if its square coefficient matrix is noninvertible (singular). What is the determinant of a singular matrix? Write the 3 x 3 matrix and that value of its determinant below.

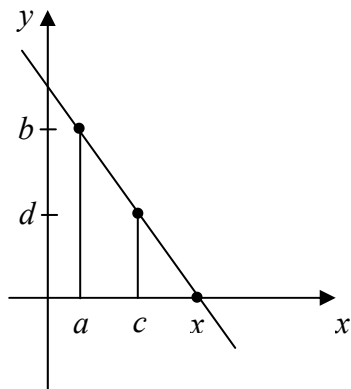$$\begin{vmatrix} & & \\ & & \\ & & \end{vmatrix} = \underline{\quad}$$

(d) Evaluate the left-hand side of the above equation by cofactor expansion along the top row. Write the resulting equation below. You have succeeded in "eliminating" the unknown coefficients p, q, r of the linear equation $px + qy - r = 0$ by expressing them in terms of the coordinates a, b, c, d of the given points on the line.

(e) Find the x-intercept of the above equation. Does your answer agree with the earlier Rule of Double False Position?

(f) Find the y-intercept of the above equation. Does your answer agree with the earlier Rule of Double False Position?

The Graphical Perspective

The Rule of Double False Position gives us the correct intercepts if we assume that the trend we've observed continues *in a linear or straight-line fashion*.

Since the curve is straight, the triangles in this figure are proportional, so

$$\frac{b}{x-a} = \frac{d}{x-c}.$$

Now solving this equation for the *x*-intercept yields the same formula that we got from Cramer's Rule,

$$x = \frac{ad-bc}{d-b}.$$

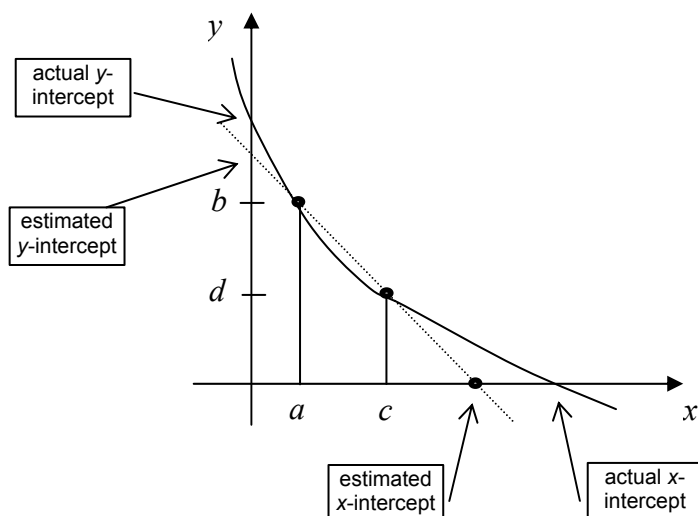And the proportionality between a second pair of similar triangles likewise yields the formula for the *y*-intercept,

$$y = \frac{ad-bc}{a-c}.$$

Exercise 7. What happens when we use the Rule of Double False Position with a line that is horizontal? (You might want to look first at a numerical and/or graphical example to figure this out.)

Exercise 8. What happens when we use the Rule of Double False Position with a line that is vertical? (You might want to look first at a numerical and/or graphical example to figure this out.)

Exercise 9. What happens when we use the Rule of Double False Position with a line that passes through the origin? (You might want to look first at a numerical and/or graphical example to figure this out.)

Notice that if we use the Rule of Double False Position in a situation where the trend is not a perfectly straight line, then we get only an *estimate* of the intercepts involved:

"Excess and Deficiency" in China

The earliest known use of the Rule of Double False Position was by the Chinese. They usually began with two guesses of the desired intercept, one guess too big and the other guess too small. That's why they called their method *ying bu-tsu*, literally "too much and not enough," often translated as Excess and Deficiency.

There was a whole chapter on Excess and Deficiency in the *Chiu Chang Suan Shu* ("Nine Chapters on the Mathematical Art"). Here is the ninth problem from that chapter. See if you can solve it.

Exercise 10. A tub of full capacity 10 *dou* contains a certain quantity of husked rice. Unhusked rice is added to fill up the tub. When the rice is all husked, it is found that the tub contains 7 *dou* of husked rice altogether. (Assume each *dou* of unhusked rice produces 3/5 *dou* of husked rice.) Find the original amount of husked rice in the tub.

(a) Write two wild guesses in the first column of this table.

Husked rice (*dou*)	Unhusked rice (*dou*)	Total amount of rice when all is husked (*dou*)	Actual amount of rice when all is husked (*dou*)	discrepancy between volumes (column 4 minus column 3)
			7	
			7	

(b) Use the fact that the tub's full capacity is 10 *dou* to fill in the second column of the table.

(c) Use the fact that each *dou* of unhusked rice produces 3/5 *dou* of husked rice to fill in column 3.

(d) Subtract to find the discrepancy in column 5. (A lucky guess in column 1 would lead to a 0 in column 5.)

(e) Our goal is to get a 0 in column 5. Apply the Rule of Double False Position to turn your two guesses in column 1 into the correct answer 0 in column 5.

Original amount of husked rice: _____*dou*

(f) Write that amount in column 1 of the table below, and check that you get 0 in the last column.

Husked rice (*dou*)	Unhusked rice (*dou*)	Total amount of rice when all is husked (*dou*)	Actual amount of rice when all is husked (*dou*)	discrepancy between volumes (column 4 minus column 3)
			7	

The "Rule of Elchatayn" in the Arab World and Europe

During the Middle Ages, it was the Arab world that was pre-eminent in science and mathematics. In fact, our word "algebra" came from the title of an important Arabic text, *Hisāb al-jabr wa-l-muqābala*, written by the mathematician **al-Khowārizmī** in Baghdad around 825 CE.

Al-Khowārizmī solved several of the problems discussed in his book by using the Rule of Double False Position, possibly borrowed from his reading of translations of earlier mathematical works from China, India or Egypt. In Arabic, the method is called *al-khatā'ayn* meaning "two falses" or "double falsehood," again because it is based on taking two guesses that are likely to be incorrect. Al-Khowārizmī's disciple **Abū Kāmil**, known as "the Egyptian Calculator," wrote an entire book on the use of *al-khatā'ayn*.
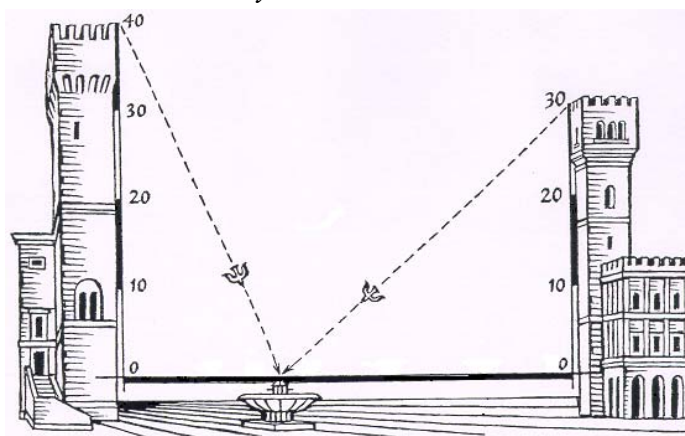

Al-Khowārizmī

Drawing from Mohammad Tūfīq Haydār,
Tārīkh al-'ulūm 'inda-l-'arabī
(Beirut: Dār al-Tanshi'at al-Lubnānīyyat, 1990)

Many of the Arab mathematical advances, including the arithmetic of 10-digit Arabic numerals and especially the algebra of Abū Kāmil, were introduced to Europe in the 1200s by **Leonardo Fibonacci** of Pisa, Italy. By that time, Italy was emerging as the first capitalist region in the world, and new mathematical techniques were needed in order to keep track of things like barter, foreign trade, pricing, profit margins, interest, currency exchange, weights and measures, partnerships, inheritance and other financial matters. The Italians invented some of their own techniques (like double-entry bookkeeping), and they borrowed many more from the Arabs. Fibonacci, for instance, learned how to use Arabic numerals when he was a boy, working in the counting house where his father was a customs official on the coast of what is now Algeria. He picked up other Arab techniques in his later travels to Egypt, Syria and other places.

In Fibonacci's home town of Pisa there were many, many pigeons and many, many towers (in fact, the famous Leaning Tower of Pisa was designed by one of his contemporaries). See whether you can solve this story problem taken from Fibonacci's most important book, written in 1202. It is from the section on what he called "the rule of elchatayn," his direct translation of Abū Kāmil's *al-khatā'ayn*.

Exercise 11. Two birds start flying from the tops of two towers 50 feet apart, one 40 feet high, the other 30, starting at the same time and flying at the same rate, and reaching the center of a fountain between the two towers at the same moment. How far is the fountain from each tower?

Drawing by Enrico Arno from Joseph and Frances Gies, *Leonard of Pisa and the New Mathematics of the Middle Ages* (New York: Thomas Y. Crowell Company, 1969)

Thinking about the problem, would you agree that the distances flown by the two birds must be equal? _____ We'll use this fact to see how far off our two guesses are.

(a) Write two wild guesses in the first column of the table on the next page.

distance from fountain to base of tall tower (feet)	distance from fountain to base of short tower (feet)	squared distance from fountain to top of tall tower (square feet)	squared distance from fountain to top of short tower (square feet)	discrepancy between squared distances (column 4 minus column 3)

(b) Use the fact that the towers are 50 feet apart to fill in the second column of the table.

(c) Use the Pythagorean Theorem to fill in columns 3 and 4.

(d) Subtract to find the discrepancy in column 5. (A lucky guess in column 1 would lead to a 0 in column 5.)

(e) Our goal is to get a 0 in column 5. Apply the Rule of Double False Position to turn your two guesses in column 1 into the correct answer 0 in column 5.

Distance from fountain to base of tall tower: _____feet

(f) Write that distance in the table below, and check that you get 0 in the last column.

distance from fountain to base of tall tower (feet)	distance from fountain to base of short tower (feet)	squared distance from fountain to top of tall tower (square feet)	squared distance from fountain to top of short tower (square feet)	discrepancy between squared distances (column 4 minus column 3)

Exercise 12. In the last Exercise we used squared quantities, which usually suggests parabolas or other curves, not straight lines. Recall that the Double False Method is not guaranteed to give an exact answer unless the trend involved is linear. Yet our answer was exact. Why?
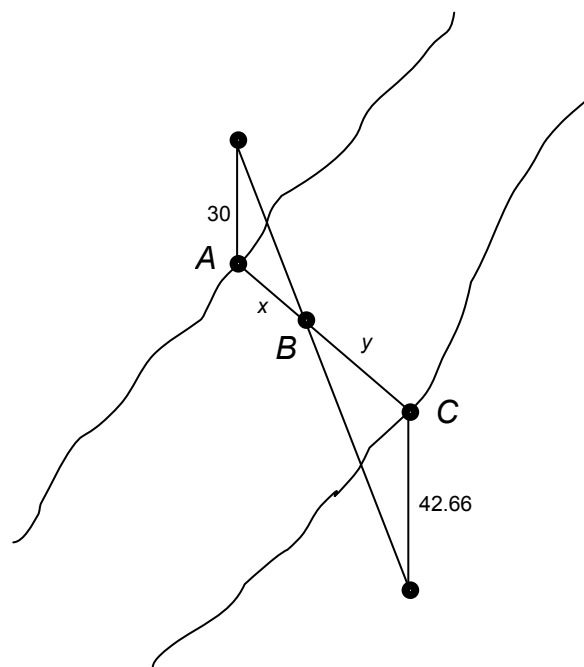
(a) To see why, let x represent the distance from the fountain to the base of the tall tower. Fill out the rest of the table below, *simplifying your answers fully.*

distance from fountain to base of tall tower (feet)	distance from fountain to base of short tower (feet)	squared distance from fountain to top of tall tower (square feet)	squared distance from fountain to top of short tower (square feet)	discrepancy between squared distances (column 4 minus column 3)
x				

(b) Were there any squares remaining in your answer in column 5? _____

(c) Is your answer in column 5 a linear function of x? _____ What is its x-intercept? _____ feet.

Exercise 13. A buoy B was positioned in a water channel many years ago. Now, it's desired to know how far it sits from the sides of the channel. When rangefinders are placed at opposite points A and C, it's found that the channel width there is 97.61 meters. When one of the rangefinders is moved 30 meters due north of A, it's found that the other rangefinder must be moved 42.66 meters due south of C to remain aligned with the buoy.

(a) Use the similarity of the two triangles to complete this proportion:

$$\frac{x}{30} = \frac{[\quad\quad]}{[\quad\quad]}$$

(b) Write the result in standard form:

$$\underline{\quad\quad}x + \underline{\quad\quad}y = 0$$

(c) Our goal is to make the linear function on the left-hand side equal zero. Write this function as the heading of the third column of the table below.

distance x to near side (meters)	distance y to far side (meters)	

d) Write two wild guesses in the first column of the table.

(e) Use the fact that the channel width was 97.61 meters to fill in the second column of the table.

(f) Complete the third column of the table.

(g) Our goal is to get a 0 in column 3. Apply the Rule of Double False Position to turn your two guesses in column 1 into the correct answer 0 in column 3.

Distance from buoy to near side: _____meters