The Rule of Double False Position

On a certain toll road, the traffic averages 36,000 vehicles per day when the toll is set at $1 per car. Experience has shown that increasing the toll will result in 300 fewer vehicles for each penny of increase. Highway officials who are planning future budgets assume that this trend will continue. They have a couple of questions:

(a) Traffic would dwindle to 0 vehicles if how high a toll were set?

(b) How many vehicles would be expected if there were no toll at all?

Our answers to questions (a) and (b) will supply the missing quantities (the intercepts) in this table:

average daily traffic (number of vehicles)	toll (dollars)
36,000	1.00
35,700	1.01
0	
	0

Of course, we could use algebra to find these intercepts. First, we would select a pair of letters like x and y to represent the two variables. Then, we would figure out an equation for the linear function relating the two variables. Finally, we would substitute zero for either variable, and solve for the corresponding intercept.

But there is an easier way to figure this out. It was used by Arab and Chinese mathematicians over 1,000 years ago, long before anyone began to use symbols like x and y to represent variables. In English, this technique is known as the "Rule of Double False Position", or the "Double False Method."

To use Double False Position, first we arrange the numbers in a grid:

36,000　　1.00

35,700　　1.01

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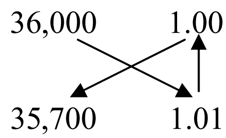
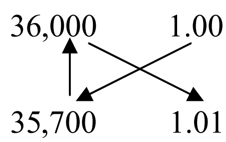
To find the intercept for the left column, we will follow this path through the grid:



(Notice that an arrow is “missing” in the left-hand column: that reminds us that we’re calculating the missing intercept for that column.) Along the diagonal arrows we’ll multiply, and along the vertical arrow we’ll subtract. The calculation is arranged like this:

$$\frac{36000 \times 1.01 - 1.00 \times 35{,}700}{1.01 - 1.00} = \frac{660}{.01} = 66{,}000.$$

That’s the intercept for the left column. To find the intercept for the *right* column, we use a path where the *right* column is missing an arrow:



$$\frac{1.00 \times 35{,}700 - 36{,}000 \times 1.01}{35{,}700 - 36{,}000} = \frac{-660}{-300} = 2.20$$

We can now fill in the missing intercepts:

average daily traffic (number of vehicles)	toll (dollars)
36,000	1.00
35,700	1.01
0	2.20
66,000	0

So, traffic would dwindle to 0 vehicles if we set the toll to $2.20 or higher. And we expect that traffic would jump to 66,000 vehicles per day if there were no toll at all.

Practice Problems

Exercise 1. An artist is planning to sell signed prints of her latest works. If 50 prints are offered for sale, she can charge \$400 each. However, if she makes more than 50 prints, she must lower the price of all the prints by \$5 for each print in excess of the 50.

Arrange the information in this table (don't forget to write the column headings on the first row):

(a) Her sales would dwindle to 0 prints if she set how high a price?

Arrange the numbers in a grid with arrows:

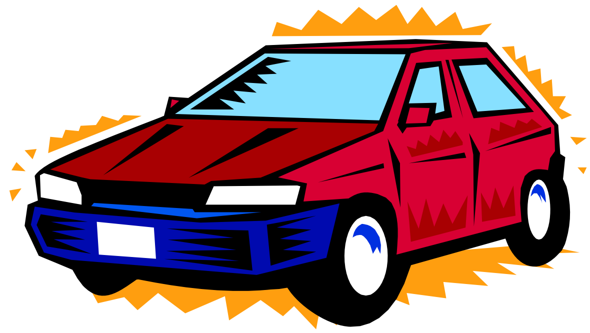
Calculate the answer:

(b) If she were to give her prints away for free, how many prints would she distribute?

Arrange the numbers in a grid with arrows:

Calculate the answer:

Exercise 2. To stimulate sales during a recession, General Motors Corp. decided to temporarily lower its financing rate for new vehicle purchases. When the rate was lowered to 8¾ %, sales jumped by 12% compared to those that were being recorded under the standard financing terms. When the rate was lowered all the way to 3½ %, sales rose 19% higher than those under standard financing.



Arrange the information in this table:

(a) If the trend continues in a linear fashion, what sales increase can be expected under zero-percent financing?

Arrange the numbers in a grid with arrows:

Calculate the answer:

(b) Approximately what standard interest rate is offered by GM to its customers?

Arrange the numbers in a grid with arrows:

Calculate the answer:

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Exercise 3. A management consultant charges a base fee for each consultation, plus an hourly rate. Her records for two different consultations reveal the following data:

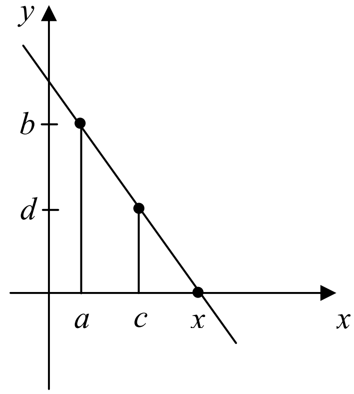
time	fee
3 hrs 20 mins	$281.50
2 hrs 45 mins	$249.30

(a) Use the Rule of Double False Position to determine her base fee, in dollars (round to the nearest cent).

(b) Use the rise-over-run procedure to determine her hourly rate, in dollars per hour (round to the nearest cent).

Why Does the Double False Method Work?

The Double False Method gives us the correct intercepts if we assume that the trend we have observed will continue *in a linear or straight-line fashion*.



Since the curve is straight, the triangles in this figure are proportional, so [fill in the missing denominator]:

$$\frac{b}{x-a} = \frac{d}{[\qquad]}.$$

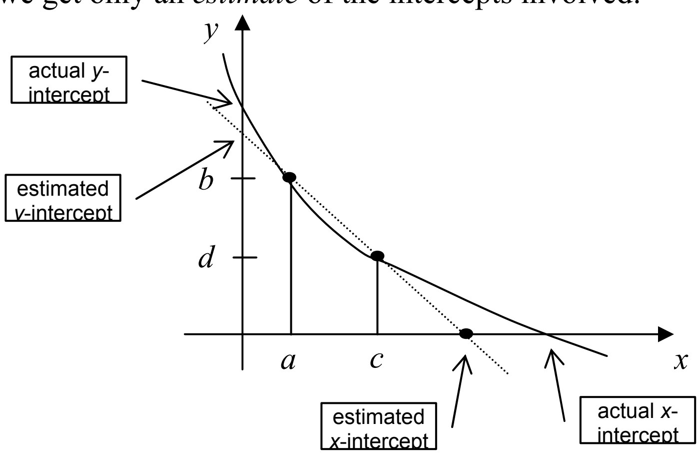
Exercise 4.

(a) Solve the above equation for x.

(b) Use the Double False Method to find the x-intercept for the data below. Does the resulting formula match what you found in part (a)?

x-coordinate	y-coordinate
a	b
c	d

Notice that if we use the Rule of Double False Position in a situation where the trend is not a perfectly straight line, then we get only an *estimate* of the intercepts involved:



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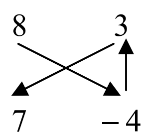
"Excess and Deficiency" in China

The earliest known use of the Rule of Double False Position was by the Chinese. They usually began with two guesses of the desired intercept, one guess too big and the other guess too small. That's why they called their method *ying bu-tsu*, literally "too much and not enough," often translated as Excess and Deficiency.

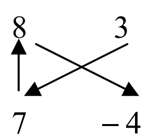
There was a whole chapter on Excess and Deficiency in the classic mathematical text ***Chiu Chang Suan Shu*** ("Nine Chapters on the Mathematical Art"), which was written by one or more unknown Chinese authors no later than 100 CE. Here is the first problem from that chapter:

> A group of people buy hens together. Each person contributes 8 *wen*, and 3 *wen* are left over; 7 are contributed, and 4 is the deficit. How many people, and what is the cost of the hens?

Notice that we write the deficit of 4 as a negative number in the grid:



$$\frac{8\times -4-3\times 7}{-4-3}=\frac{-53}{-7}=7\frac{4}{7}.$$



$$\frac{3\times 7-8\times -4}{7-8}=\frac{53}{-1}=-53$$

donation per person (*wen*)	left over money (*wen*)
8	3
7	– 4
0	– 53
$7\,{}^{4}/_{7}$	0

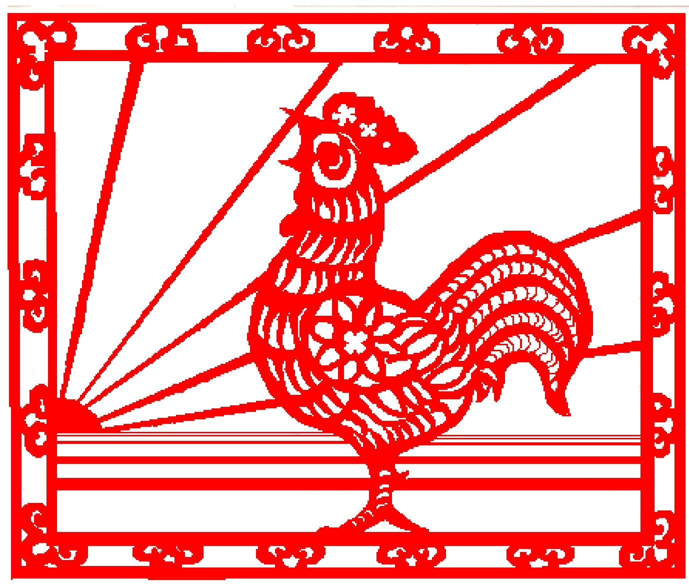
So, if no one donates anything, they would need 53 *wen*; this is the cost of the hens. And if everyone donates $7\,{}^{4}/_{7}$ *wen*, it will exactly pay for the hens. The number of people must be $53 \div 7\,{}^{4}/_{7} = 7$.

Paper was one of ancient China's major gifts to the world, invented during the Han dynasty in the early 2nd Century CE or earlier. Here, papermakers post freshly-made sheets to dry.

Image: http://en.wikipedia.org/wiki/History_of_paper

Here is the next problem from the same chapter of *Chiu Chang Suan Shu*. See if you can solve it.

Exercise 5. A group of people buy hens together. Each person contributes 9 *wen*, and 11 *wen* are left over; 6 are contributed, and 16 is the deficit. How many people, and what is the cost of the hens?



A Chinese-style papercut.
(Artist: Fanghong, Wikimedia Commons)

donation per person (*wen*)	left over money (*wen*)

Total cost of the hens: ________________

Number of people: _________________

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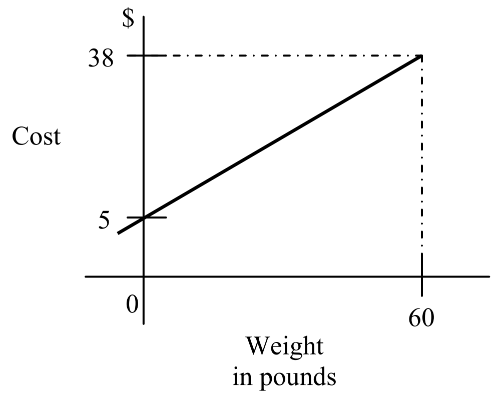
The following two exercises are modeled after questions used on the CPA and GMAT professional exams.

Exercise 6. The figure to the right gives the cost of shipping a package from coast to coast. How heavy a package can be shipped for $24.25?

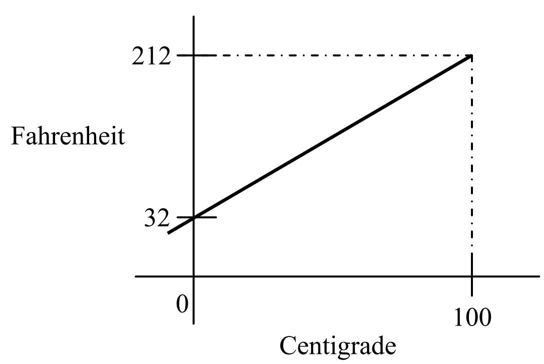


Compared to our budget of $24.25, the $38 charge for a 60-pound package represents an "excess" of $13.75, while the $5 base charge represents a "deficit" of $19.25. We store these results in a table:

weight (lbs.)	budget overrun ($)
60	13.75
0	-19.25

We must determine the weight that corresponds to a budget overrun of 0. Use the Double False Method to solve the problem:

Exercise 7. The figure to the right gives temperature conversions between Centigrade and Fahrenheit. Use the above procedure to convert 95º F. to Centigrade.



The "Rule of Elchatayn" in the Arab World and Europe

Al-Khowārizmī

Drawing from Mohammad Tūfīq Haydār, *Tārīkh al-'ulūm 'inda-l-'arabī* (Beirut: Dār al-Tanshi'at al-Lubnānīyyat, 1990)

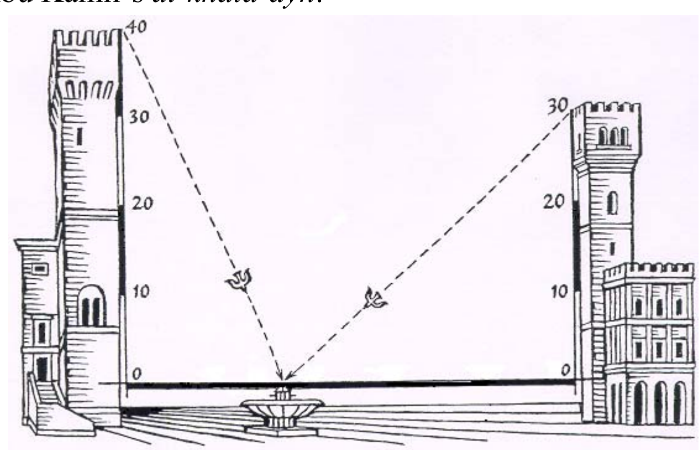
During the Middle Ages, it was the Arab world that was pre-eminent in science and mathematics. In fact, our word "algebra" came from the title of an important Arabic text, *Hisāb al-jabr wa-l-muqābala*, written by the mathematician **al-Khowārizmī** in Baghdad around 825 CE.

Al-Khowārizmī solved several of the problems discussed in his book by using the Rule of Double False Position, possibly borrowed from his reading of translations of earlier mathematical works from China, India or Egypt. In Arabic, the method is called *al-khatā'ayn* meaning "two falses" or "double falsehood," again because it is based on taking two guesses that are likely to be incorrect. Al-Khowārizmī's disciple **Abū Kāmil**, known as "the Egyptian Calculator," wrote an entire book on the use of *al-khatā'ayn*.

Many of the Arab mathematical advances, including the arithmetic of 10-digit Arabic numerals and especially the algebra of Abū Kāmil, were introduced to Europe in the 1200s by **Leonardo Fibonacci** of Pisa, Italy. By that time, Italy was emerging as the first capitalist region in the world, and new mathematical techniques were needed in order to keep track of things like barter, foreign trade, pricing, profit margins, interest, currency exchange, weights and measures, partnerships, inheritance and other financial matters. The Italians invented some of their own techniques (like double-entry bookkeeping), and they borrowed many more from the Arabs. Fibonacci, for instance, learned how to use Arabic numerals when he was a boy, working in the counting house where his father was a customs official on the coast of what is now Algeria. He picked up other Arab techniques in his later travels to Egypt, Syria and other places.

In Fibonacci's home town of Pisa there were many, many pigeons and many, many towers (in fact, the famous Leaning Tower of Pisa was designed by one of his contemporaries). See whether you can solve this story problem taken from Fibonacci's most important book, written in 1202. It is from the section on what he called "the rule of elchatayn," his direct translation of Abū Kāmil's *al-khatā'ayn*.

Exercise 8. Two birds start flying from the tops of two towers 50 feet apart, one 40 feet high, the other 30, starting at the same time and flying at the same rate, and reaching the center of a fountain between the two towers at the same moment. How far is the fountain from each tower?



Drawing by Enrico Arno from Joseph and Frances Gies, *Leonard of Pisa and the New Mathematics of the Middle Ages* (New York: Thomas Y. Crowell Company, 1969)

Thinking about the problem, would you agree that the distances flown by the two birds must be equal?_____ We'll use this fact to see how far off our two guesses are.

(a) Write two wild guesses in the first column of the table on the next page.

distance from fountain to base of tall tower (feet)	distance from fountain to base of short tower (feet)	squared distance from fountain to top of tall tower (square feet)	squared distance from fountain to top of short tower (square feet)	discrepancy between squared distances (column 4 minus column 3)

(b) Use the fact that the towers are 50 feet apart to fill in the second column of the table.

(c) Use the Pythagorean Theorem to fill in columns 3 and 4.

(d) Subtract to find the discrepancy in column 5. (A lucky guess in column 1 would lead to a 0 in column 5.)

(e) Our goal is to get a 0 in column 5. Apply the Rule of Double False Position to turn your two guesses in column 1 into the correct answer 0 in column 5.

Distance from fountain to base of tall tower: __________feet

(f) Write that distance in the table below, and check that you get 0 in the last column.

distance from fountain to base of tall tower (feet)	distance from fountain to base of short tower (feet)	squared distance from fountain to top of tall tower (square feet)	squared distance from fountain to top of short tower (square feet)	discrepancy between squared distances (column 4 minus column 3)

Exercise 9. In the last Exercise we used squared quantities, which usually suggests parabolas or other curves, not straight lines. Recall that the Double False Method is not guaranteed to give an exact answer unless the trend involved is linear. Yet our answer was exact. Why?

(a) To see why, let x represent the distance from the fountain to the base of the tall tower. Fill out the rest of the table below, *simplifying your answers fully.*

distance from fountain to base of tall tower (feet)	distance from fountain to base of short tower (feet)	squared distance from fountain to top of tall tower (square feet)	squared distance from fountain to top of short tower (square feet)	discrepancy between squared distances (column 4 minus column 3)
x				

(b) Were there any squares remaining in your answer in column 5? _______________

(c) Is your answer in column 5 a linear function of x? ______ What is its x-intercept? _______ feet.

Exercise 10. This exercise explores the relation between the "Christian" year used in the West, and the "Islamic" year used among Muslims. For example, the Christian year 1492 roughly corresponded to the Muslim year 897, while the Christian year 1990 roughly corresponded to the Muslim year 1410.

The years given by these two calendars differ from one another for two main reasons. First, their counting began with different events: the birth of Christ in one case, and the Hijra, or emigration of Muhammad from Mecca to Medina in the other case, over 600 years later. Second, the Christian calendar is based on a solar year of about 365 days, while the Islamic calendar is based on a lunar year of about 354 days. This means that the relation between Christian and Muslim dates is not fixed but shifts gradually over time.

The use of solar and lunar calendars flourished side by side in the Middle East. Nomads tended to rely on a lunar calendar, because they keyed the movements of their desert caravans to the phases of the moon. By contrast, "sedentary" peoples (those living in villages and towns based on settled agriculture) tended to rely on a solar calendar, because they timed their farming practices according to the seasons of the sun.

Because calendars are so rooted in culture, the use of solar and lunar calendars persists today even in regions where few people are farmers or nomads. Those who conduct business both in predominantly Christian and predominantly Muslim countries must be able to translate dates back and forth.

(a) Arrange in this table the data given in the first paragraph above:

Muslim year, x	Christian year, y

(b) The relation between Muslim and Christian years is very close to being linear, $y = mx + b$. Use the data in your table and the definition of slope to estimate m with a high degree of precision.

(c) Use *al-khata'ayn* (the double false method) to estimate b with a high degree of precision.

(d) Use your model $y = mx + b$ to complete this table; round your answers to the nearest year.

Muslim year, x	Christian year, y
1	
1000	
	2020

Based on the top row, you estimate that the Hijra took place in the Christian year _____________.

(e) Use your model $y = mx + b$ to estimate when the Muslim and Christian years will be the same (round your answer to the nearest year).