Finite Mathematics

Binomial Coefficients and Subsets

In the worksheet "Combinations and Their Sums," we saw how ibn Mun'im, living in Morocco around 1200, used **combinations** $C(n,r)$ to count the number of ways to select colored threads or other things from a pool of items, assuming that repeats aren't allowed and the order of selection doesn't matter. Let's do some practice and warm-up.

Exercise 1.

a) In how many different ways can 3 employees be selected from an office of 7 employees?

$C(\_\_\_\_,\_\_\_\_) = \_\_\_\_\_\_$

b) Use the "row-sum pattern" to simplify:

$C(3,3)+C(4,3)+C(5,3)+C(6,3) = C(\_\_\_\_,\_\_\_\_)$

$= \_\_\_\_\_\_$

c) Use the "column-sum pattern" to simplify:

$C(9,5)+C(9,6) = C(\_\_\_\_,\_\_\_\_)$

$= \_\_\_\_\_\_$

Ibn Mun'im wasn't the first to explore combinations like these, but apparently they'd never been studied by mathematicians in ancient Greece and Mesopotamia, who excelled in many other topics. In the late 900s, an Indian poet named Halayudha presented the arithmetical triangle as a tool in categorizing the different kinds of poetic meter, the sequences of stressed or unstressed syllables in one line of poetry (see Exercise 9 below). A little later, around 1010, the Arab mathematician **Abū Bakr al-Karajī** in Baghdad used the arithmetical triangle to solve an algebra problem, that of figuring out the coefficients of binomial powers like $(x+y)^4$ (see Exercises 2-4 below).

During the Middle Ages, it was the Arab world that was pre-eminent in science and mathematics. In fact, our word "algebra" came from the title of an important Arabic text, *Hisāb al-jabr wa-l-muqābala*, written by the mathematician **al-Khowārizmī** in Baghdad around 825. Scholars such as al-Khowārizmī and al-Karajī were associated with libraries, referred to in Arabic at the time as "houses of wisdom." These were essentially government-supported research centers, the one in Baghdad being the most famous. Scholars speaking many different languages arrived there from throughout the Middle East, but they conversed with each other in Arabic (that's why many of our mathematical terms, such as "algebra," "algorithm" and "zero," came to us from Arabic).

Al-Khowārizmī

Drawing from Mohammad Tūfīq Haydār, *Tārīkh al-'ulūm 'inda-l-'arabī* (Beirut: Dār al-Tanshi'at al-Lubnānīyyat, 1990)

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Exercise 2. Let's start with one of the simplest binomial powers:

$$(x+y)^2 = (x+y)(x+y) \quad \text{(use distributivity)}$$

$$= \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_$$

F O I L

use commutativity to combine the like terms

$$= \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_$$

$$= \_\_\_ x^2 + \_\_\_ xy + \_\_\_ y^2$$ (fill these blanks with the numerical coefficients)

Since $(x+y)$ is a binomial (two-term) expression, we call $(x+y)^4$ a **binomial power**.
We say that we have **expanded** the binomial power, and the numerical coefficients (1, 2, and 1) are called **binomial coefficients**.

Exercise 3. Determine the binomial coefficients of $(x+y)^3$ in the same way:

$$(x+y)^3 = (x+y)^2(x+y)$$

$$= (x^2 + 2xy + y^2)(x+y)$$

$$= \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_ + \_\_\_\_\_\_ + \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_$$

$$= \_\_\_ x^3 + \_\_\_ x^2 y + \_\_\_ xy^2 + \_\_\_ y^3$$

To expand a binomial power like $(x+y)^{10}$, we could keep going like this, but it would take a lot of time and effort to work our way up to 10! To simplify things, we look for patterns. Let's summarize what we know from the first four powers of $(x+y)$:

$$(x+y)^0 = 1$$
$$(x+y)^1 = 1x + 1y$$
$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

Look at the triangular arrangement of coefficients above. Compare this to ibn Mun'im's arithmetical triangle, shown at right. Do you see the similarity? It seems as if each set of binomial coefficients is exactly the same as one *column* of the arithmetical triangle, the triangle whose entries were obtained by calculating combinations $C(n,r)$.

For a tassel of 10 colors										1	1
For a tassel of 9 colors									1	9	10
For a tassel of 8 colors								1	8	36	45
For a tassel of 7 colors							1	7	28	84	120
For a tassel of 6 colors						1	6	21	56	126	210
For a tassel of 5 colors					1	5	15	35	70	126	252
For a tassel of 4 colors				1	4	10	20	35	56	84	210
For a tassel of 3 colors			1	3	6	10	15	21	28	36	120
For a tassel of 2 colors		1	2	3	4	5	6	7	8	9	45
For a tassel of 1 color	1	1	1	1	1	1	1	1	1	1	10
	using color #1	using color #2 (& earlier)	using color #3 (& earlier)	using color #4 (& earlier)	using color #5 (& earlier)	using color #6 (& earlier)	using color #7 (& earlier)	using color #8 (& earlier)	using color #9 (& earlier)	using color #10 (& earlier)	Total no. of tassels

This suggests that we can predict binomial coefficients simply by using our combination formula $C(n,r)$. But which n and r do we plug in? Remember that along each column of ibn Mun‘im's triangle, the number of colors available, n, was a constant. So, we'll write each coefficient to have the same first number n, such as 3:

$$\underline{\text{ex.}}\quad (x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$= C(3,0)x^3 + C(3,1)x^2y + C(3,2)xy^2 + C(3,3)y^3$$

The second number, r, which varies from term to term, tells you exactly how many of the 3 variables in the term are y. For example, next to xy^2 we write $C(3,2)$, since 2 of its 3 variables are y. It makes sense that this is the correct way to predict the coefficient, because the xy^2 term combines the like terms *xyy*, *yxy* and *yyx*. Since the number of such terms is the number of ways to fill any 2 of the 3 spots with y (and the rest of them with x), the correct coefficient must be $C(3,2)$ which is 3.

From now on, we'll write $C(n,r)$ in the more compact **binomial coefficient notation**, $\binom{n}{r}$. Although calculating the binomial coefficient $\binom{n}{r}$ involves dividing, you don't write the fraction bar there, because that would make it look like $\frac{n}{r}$, whereas the formula for $\binom{n}{r}$ involves a lot more than just dividing n by r. As a review,

$$\binom{n}{r} \quad = \quad nCr \quad = \quad \frac{n!}{r!(n-r)!}$$

$\binom{n}{r}$	nCr	$\frac{n!}{r!(n-r)!}$
The fastest way to write it by hand	The way to calculate it on a TI calculator	The way to calculate it "from scratch"

Look how much more compact the expansion becomes in this new notation:

$$(x+y)^3 = C(3,0)x^3 + C(3,1)x^2y + C(3,2)xy^2 + C(3,3)y^3 \quad \text{(old notation)}$$

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3 \quad \text{(new notation)}$$

If you stare at this for a while, you notice patterns that can be used to shortcut the harder problems such as $(x+y)^4$:

a) Each term in the expansion has the same total number of variables, 3. This is called the "total degree" of the term.
b) Each binomial coefficient uses the same number on top, 3.
c) The number on the bottom, r, increases systematically from 0 to 3.
d) The bottom number, r, tells you exactly how many of the 3 variables in the term are y.
e) The final coefficients are symmetrical: they read the same forwards as backwards, e.g., 1, 3, 3, 1.

These patterns can be summarized in the form of a formula, often called **the Binomial Theorem**:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

Exercise 4. Use these patterns to complete the following expansions (fill in all blanks):

a) $(x+y)^4 = \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{\_\_}x^2y^2 + \binom{4}{\_\_}x^1y^3 + \binom{\_\_}{\_\_}x^0y^4$

$= x^4 + \_\_\,x^3y + \_\_\,x^2y^2 + \_\_\,xy^3 + y^4$ (Simplify your answer.)

b) $(x+y)^5 = \binom{\_\_}{\_\_}\_\_\_\_ + \binom{\_\_}{\_\_}\_\_\_\_ + \binom{\_\_}{\_\_}\_\_\_\_ + \binom{\_\_}{\_\_}\_\_\_\_ + \binom{\_\_}{\_\_}\_\_\_\_ + \binom{\_\_}{\_\_}\_\_\_\_$

= ______+ ______+ ______+ ______+ ______+ ______ (Simplify your answer.)

c) $(x+y)^7$ = _________+ _________+ _________$+\cdots$ (Find the first 3 terms only.)

= _________+ _________+ _________$+\cdots$ (Simplify your answer where possible.)

d) $(r+s)^9 = \cdots +$ _________+ _________+ _________+ _________ (Find the last 4 terms only.)

$= \cdots +$ _________+ _________+ _________+ _________ (Simplify where possible.)

Check your work in parts (a) – (d) by referring to the columns of ibn Mun‘im’s arithmetical triangle on page 2.

e) $(2+.01)^5$ = _________+ _________+ _________+ _________$+\cdots$ (Find first 4 terms only.)

= ________+ ________+ ________+ ________$+\cdots$ (Simplify where possible.)

= ____________ (Add it up.)

f) $(1+1)^5 = \binom{\_\_}{\_\_}\_\_\_\_ + \binom{\_\_}{\_\_}\_\_\_\_ + \binom{\_\_}{\_\_}\_\_\_\_ + \binom{\_\_}{\_\_}\_\_\_\_ + \binom{\_\_}{\_\_}\_\_\_\_ + \binom{\_\_}{\_\_}\_\_\_\_$

$2^5 = \binom{\_\_}{\_\_} + \binom{\_\_}{\_\_} + \binom{\_\_}{\_\_} + \binom{\_\_}{\_\_} + \binom{\_\_}{\_\_} + \binom{\_\_}{\_\_}$ (Simplify the powers only.)

32 = ______+ ______+ ______+ ______+ ______+ ______ (Now check your answers.)

g) CHALLENGE PROBLEM
In the expansion of $(5r^3 - 2s^2)^6$, the fully simplified coefficient of r^9s^6 is __________.
(Show all of your work.)

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Exercise 4e suggests that binomial expansions might be useful in numerically approximating fifth powers and fifth roots (and other powers and roots). This is exactly one of the ways they were used in medieval China.

Exercise 5. Shown at the right is the arithmetical triangle given in the algebra textbook *Precious Mirror of the Four Elements* (*Siyuan Yuchian*), written during the Yuan dynasty in 1303 by Chu Shi-Chieh. His text focused on solving systems of many equations and variables; the "four elements" of Chu's title referred to man, matter, heaven and earth, which symbolized four variables.

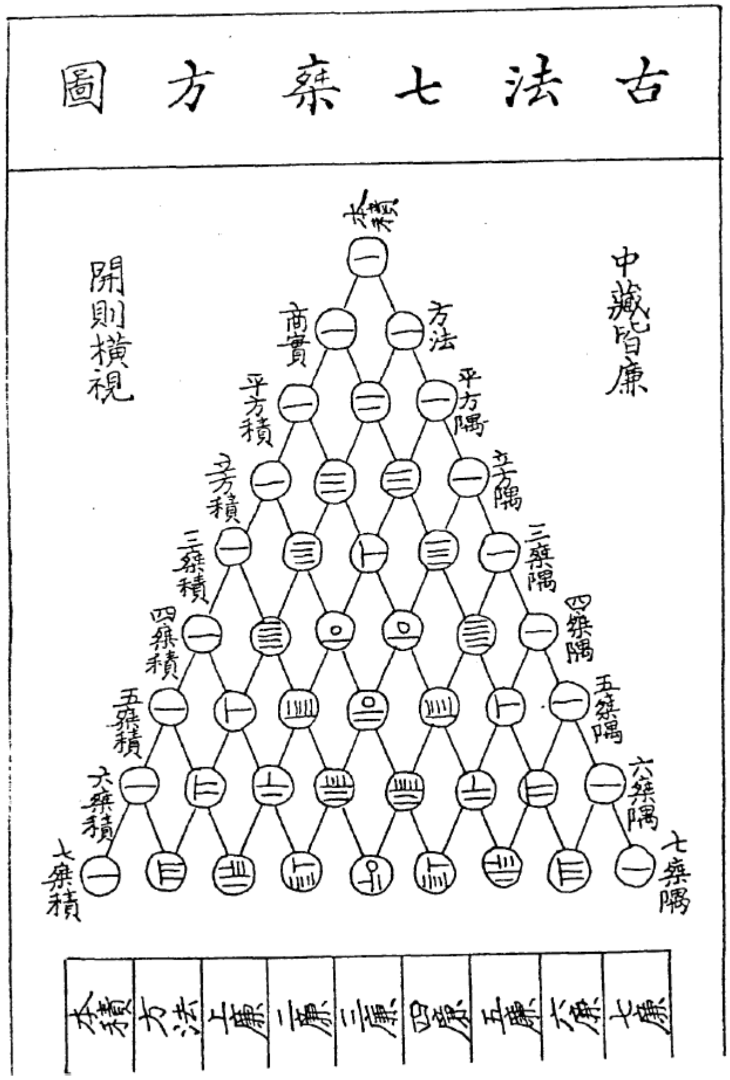


Photo: Joseph Needham, *Science and Civilization in China* (Cambridge University Press, 1959)

a) Comparing this triangle to those we've already worked with, translate the following numerals from Indo-Arabic into Chinese:

3 + 3 = 6   ◯ + ◯ = ◯

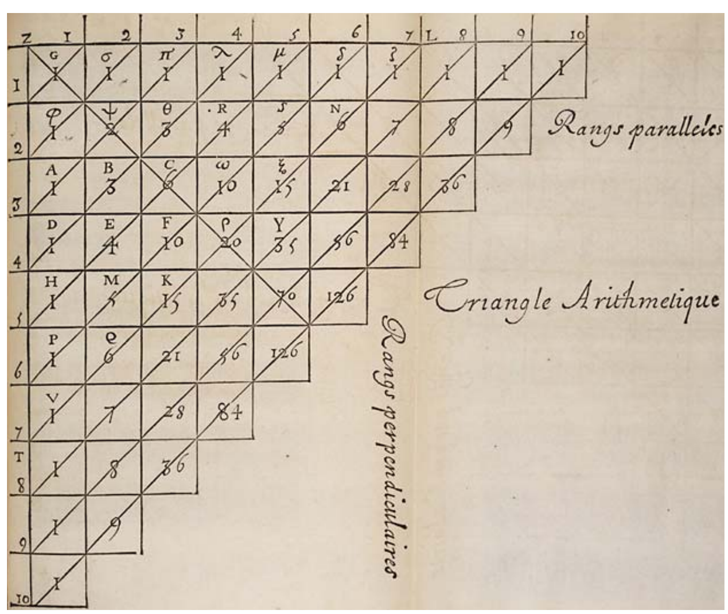
(fill in each circle)

b) Remembering that binomial coefficients are always symmetrical, see if you can discover a slight mistake that Chu (or his scribe) made in copying out his triangle. Record your discovery as the following violation of symmetry:

◯ ≠ ◯

(fill in each circle)

Triangles like these were also put to use later in Europe. Blaise Pascal (in France in 1654) used the triangle to solve a probability problem arising from a gambling situation, and Isaac Newton (in England in 1664) used it in conjunction with his development of calculus. In Europe and the Americas, the triangle is still generally known as "Pascal's Triangle."



Pascal's "Arithmetical Triangle" from his *Traité du Triangle Arithmétique* (1665)

Photo: Cambridge University Library

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Exercise 4f suggests that in any binomial expansion, the sum of the binomial coefficients is a power of 2. Let's confirm this.

Exercise 6. Shown below is an arithmetical triangle in the Chinese style.

		row no.	**row total**
$\binom{0}{0}$	1	0	1
$\binom{1}{0}$ $\binom{1}{1}$	1 1	1	2
$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$	1 2 1	2	___
$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$	1 3 3 1	3	___
$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$	1 4 6 4 1	4	___
$\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$	1 5 10 10 5 1	5	___

a) Fill in the missing row totals.
b) Is each row total a power of 2? _________
c) Based on your observation, it appears that if n stands for the row no., then the row total is

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=\_\_\_\_\_\_ \quad \text{(fill in the missing formula)}$$

Exercise 7. A telephone company offers 5 customer options in addition to its basic service. A customer can choose any (or all or none) of the options. In how many different ways can this be done? Let's solve this by two different methods.

a) (Addition Rule) Let's tally the subsets according to the size of the package purchased:

The number of ways to select 5 of the 5 options is $\binom{5}{5}$ = __1__.

The number of ways to select 4 of the 5 options is $\binom{5}{\_}$ = ______.

The number of ways to select 3 of the 5 options is $\binom{5}{\_}$ = ______.

The number of ways to select 2 of the 5 options is $\binom{5}{\_}$ = ______.

The number of ways to select 1 of the 5 options is $\binom{5}{\_}$ = ______.

The number of ways to select 0 of the 5 options is $\binom{5}{\_}$ = ______.

The number of ways to select some subset of the 5 options is ______ (total).

b) (Multiplication Rule) Let's tally the subsets according to the customer's decision process:

Step 1: decide whether to buy option 1 or not; $n_1 = \underline{\quad 2 \quad}$ choices.
Step 2: decide whether to buy option 2 or not; $n_2 = \underline{\qquad}$ choices.
Step 3: decide whether to buy option 3 or not; $n_3 = \underline{\qquad}$ choices.
Step 4: decide whether to buy option 4 or not; $n_4 = \underline{\qquad}$ choices.
Step 5: decide whether to buy option 5 or not; $n_5 = \underline{\qquad}$ choices.

Total number of ways to make decision = ______________________.

Comparing your answers to the two parts of Exercise 7, the fact that they match confirms why

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5 ,$$

or more generally,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n .$$

Whenever you add up all the binomial coefficients having the same upper number (number of options available), their total is 2 raised to that power. Interpreting this another way, if a set has n elements, the number of subsets of it (including the whole set and the empty set) is 2^n. This rule gives us a powerful shortcut for solving lots of counting problems.

ex. Suppose that in Exercise 7, the customer isn't allowed to choose more than 4 of the options. In how many different ways can this be done?

$$\left.\begin{array}{l} n = 5 \text{ options available} \\ r = 0, 1, 2, 3 \text{ or } 4 \text{ options chosen} \end{array}\right\} \begin{array}{l} \text{without repeats} \\ \text{without order} \end{array}$$

Long method:

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} = 1 + 5 + 10 + 10 + 5 = 31, \underline{\text{ans}}$$

Shortcut:

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} = 2^5 - \binom{5}{5} = 32 - 1 = 31, \underline{\text{ans}}$$

Use this shortcut to solve the rest of the exercises.

Exercise 8. About 150 CE, the Indian doctor Sushruta used combinations to compute the number of flavors composed of one or more of the 6 basic tastes (bitter, sour, salty, astringent, sweet, hot). Predict the answer that he found.

$$\left.\begin{array}{l} n = 6 \text{ tastes available} \\ r = 1, 2, 3, 4, 5 \text{ or } 6 \text{ tastes selected} \end{array}\right\} \begin{array}{l} \text{without repeats} \\ \text{without order} \end{array}$$

$$\text{number of combinations} = 2^6 - \binom{6}{\underline{\quad}}$$

$$= 64 - \underline{\qquad}$$

$$= \underline{\qquad}$$

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Exercise 9. The great astronomer and mathematician of south India, Bhāskara (1114-1185), illustrated the use of binomial coefficients by posing various problems from art and architecture in his classic work *Līlāvatī* (translated by H. T. Colebrooke, Allahabad, India: Kitab Mahal, 1967).

a) In classical Indian poetry and music, each line is a combination of long (L) and/or short (S) syllables. To systematically test out the sound quality of different meters, it was useful to know the number of variations possible. Bhāskara explained that in a 6-syllable line, the number of combinations having 4 long syllables (such as LLSSLL) is $\binom{6}{4} = 15$. Based on such methods, he asked for the number of variations possible (p. 62).

$$\left.\begin{array}{l} n = 6 \text{ syllables available in a line} \\ r = 0, 1, 2, 3, 4, 5 \text{ or } 6 \text{ long syllables} \end{array}\right\}$$

When we choose *which* of the 6 syllables will be long, the order of choice doesn't matter and repeats aren't allowed.

number of combinations = ______________________________

b) At one point, Bhāskara described "a pleasant, spacious and elegant edifice, with eight doors, constructed by a skilful architect, as a palace for the lord of the land" (p. 64). He then asked for the total number of combinations of palace doors that could be opened, assuming that at least one is opened.

$$\left.\begin{array}{l} n = \_\_\_\_\_\_\_\_ \text{ doors available} \\ r = \_\_\_\_\_\_\_\_ \text{ doors opened} \end{array}\right\}$$

number of combinations = ______________________

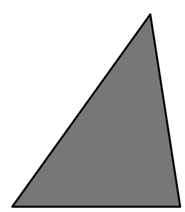
Taj Mahal photo: Richard IJzermans, http://www.flickr.com/photos/ironmanixs

Exercise 10. In 1291, the mathematician Abū-l-'Abbās Ahmad ibn al-Bannā' of Marrakech, Morocco, computed the number of various types of geometry problems. For example, he said that every triangle has 5 basic quantities (height, area, and the lengths of the 3 sides). In any given triangle problem, between 1 and 4 of these quantities are known, and the problem is to try to determine the rest. So, the number of possible types of triangle problem is:

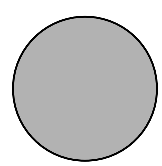
$$\left.\begin{array}{l} n = 5 \text{ total quantities} \\ r = 1, 2, 3 \text{ or } 4 \text{ known quantities} \end{array}\right\} \begin{array}{l} \text{without repeats} \\ \text{without order} \end{array}$$

$$\text{number of problems} = 2^5 - \binom{5}{0} - \binom{5}{5}$$

$$= 32 - 1 - 1 = 30, \underline{\text{ans}}$$



a) Ibn al-Bannā' said that every circle has 3 basic quantities (diameter, perimeter and area). How many different types of circle problem are possible?



b) Ibn al-Bannā' said that every rectangle has 4 basic quantities (length, width, diagonal and area). How many different types of rectangle problem are possible?

Exercise 11. A new kind of "combination lock" has ten pushbuttons (labeled 0 through 9). Each button can be toggled between its "up" and "down" position. The lock opens only if a certain combination of the buttons (the subset programmed by the owner) is pressed down, in any order.

a) Suppose the subset programmed by the owner must include at least one button in the down position. In how many different ways can such a lock be programmed?

b) Compare this to a conventional "combination lock." Suppose that such a lock can be opened only by the correct *sequence* (the order is important) of three digits, each digit between 0 and 49. In how many different ways can such a lock be programmed?

Exercise 12. In the *dim sum* restaurant Heavenly Garden, a trolley comes around with a selection of 7 dishes. In how many different ways can a customer choose as many as 5 of the dishes?

Exercise 13. Pratima is conducting a study examining the major difficulties faced by minority-owned businesses in Oakland County. As part of her study, she has identified 9 such businesses that declared bankruptcy in the last 18 months. If she wants to select a sample of between 3 and 8 of these as focuses of case studies, how many different samples are possible?

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