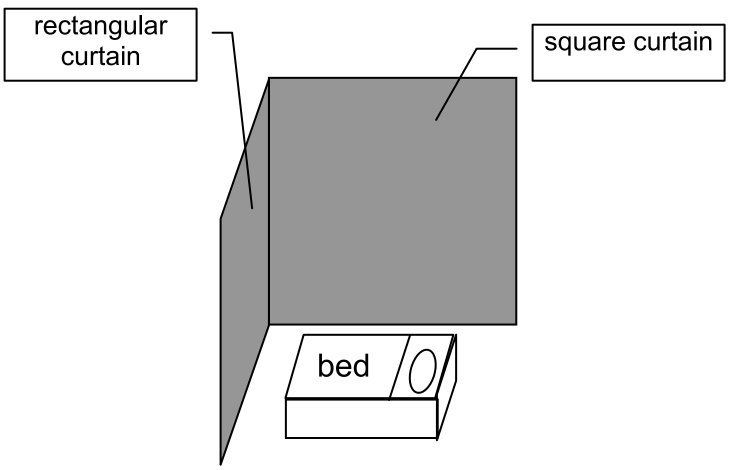
Using the Derivative to Solve an Optimization Problem

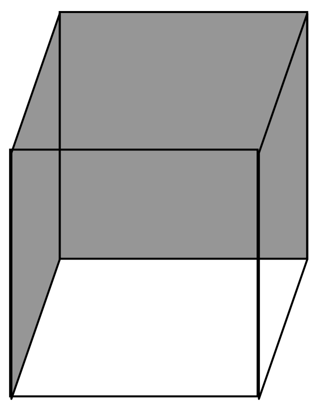
Sami is a maintenance worker at a hospital in Baghdad. He has been asked to curtain off the area around a patient bed for privacy. The bed is in a corner of one ward, so the curtain needs to shield only two of its sides: there will be a square drape shielding one long side, and a smaller rectangular drape shielding the foot of the bed. Both lengths of curtain will drape from curtain rods suspended at equal heights from the ceiling of the hospital ward.

At al-Mansour Hospital in Baghdad, a malnourished boy suffering from whooping cough is comforted by his mother before receiving an injection.

Photo: Dan DeLong, *Seattle Post-Intelligencer*, January 20, 2001. Used by permission.



The drapes, together with the walls, will help to enclose what can be thought of as a rectangular "box of privacy" around the bed.

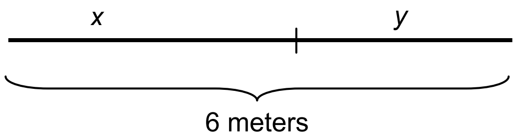


But this is where Sami faces a problem. Because of wartime devastation and supply shortages in Baghdad, the only material that he has available for the curtain rods is a 6-meter wooden pole, which he can cut into the two pieces that he needs. Sami is concerned to know how large a rectangular volume he will be able to enclose with these two pieces of curtain rod.

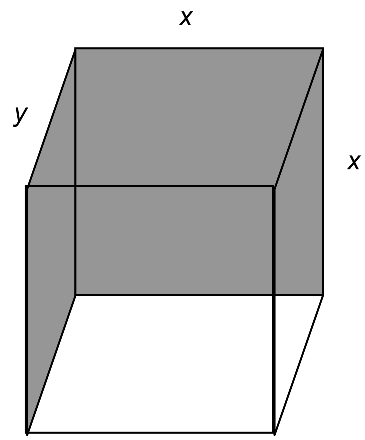
Such problems, known as **solid problems**, have been studied and solved since ancient Greek times. In the medieval world, Arabic-speaking scholars in Baghdad developed new tools for solving such problems, as we will explore further below.

Schwartz, Randy K., "Combining Strands of Many Colors: Episodes from Medieval Islam for the Mathematics Classroom,"
Loci: Convergence (August 2010), DOI: 10.4169/loci003546

Let's signify the lengths of Sami's two pieces of curtain rod by the letters x and y:



The rectangular box will have corresponding dimensions, as shown at right.



Exercise 1. First, let's do some numerical experiments.

(a) Write down an equation for y in terms of x, using the constraint that the rod is only 6 meters long. This is called the **constraint equation**.

(b) In the table below, a variety of sample values of x are listed. Find the corresponding square area, x^2. Then use the constraint equation to fill in the corresponding values of y.

x (meters)	x^2 (square meters)	y (meters)	V (cubic meters)
0			
1			
2			
3			
4			
5			
6			

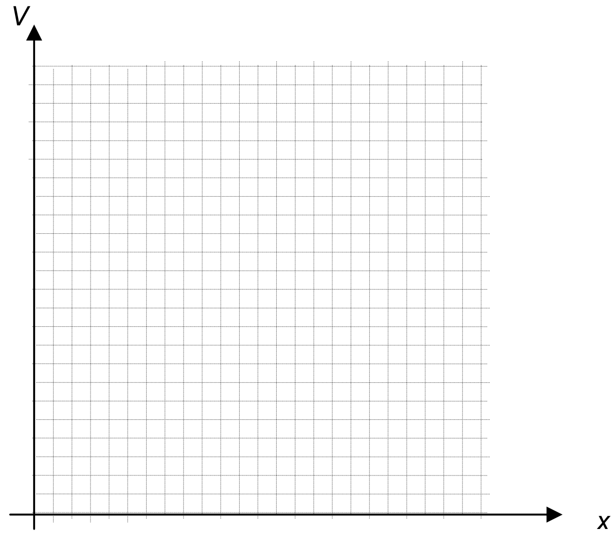
(c) Write a formula for the volume of the rectangular box, V, in terms of x and y. Since Sami's objective is to determine which volumes are possible, this volume is called the **objective function.**

(d) Use the objective function to fill in the corresponding values of V in the table above.

Schwartz, Randy K., "Combining Strands of Many Colors: Episodes from Medieval Islam for the Mathematics Classroom," *Loci: Convergence* (August 2010), DOI: 10.4169/loci003546

Exercise 2. Now, let's do some graphing to see how the volume V varies as x varies.

(a) Using the graph paper below, plot one point for each of the 7 rows in your chart. Choose the scales carefully. You won't be able to use the same scale on both axes.



(b) Connect your 7 plotted points with a smooth curve.

(c) Recall that the wooden pole is only 6 meters long, so that the points on the graph outside of the interval $0 \le x \le 6$ are extraneous to the curtain design problem. Keeping this in mind, use your graph to make these predictions:

The number of curtain designs that enclose a volume of 26 cubic meters is ________.

The number of curtain designs that enclose a volume of 40 cubic meters is ________.

(d) To view the graph on your calculator, first combine the objective and constraint equations to write the function V in terms of x alone. Leave your answer as a cubic function in descending degree form.

(e) Type the function from part (d) into your calculator. Check the TABLE feature: the numbers there should match your table from Exercise 1.

Exercise 3. Use the other graphing features of your calculator to answer these questions, accurate to 2 decimal digits:

$V = 26$ cubic meters when $x =$ ______ and $y =$ ______ meters

OR when $x =$ ______ and $y =$ ______ meters.

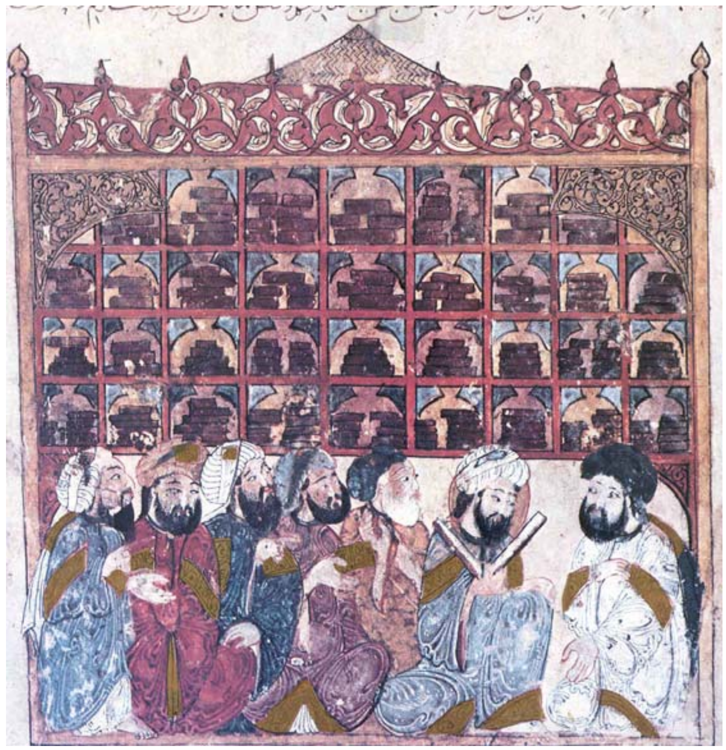
$V = 28$ cubic meters when $x =$ ______ and $y =$ ______ meters

OR when $x =$ ______ and $y =$ ______ meters.

Sami wants to know the *maximum* volume that he can enclose using his curtain rods. The maximum volume is said to be **optimal**, and the problem is called an **optimization** problem.

Based on your numbers in Exercise 1 and your graph in Exercise 2, you might have guessed that x = 2 meters gives a curtain size of optimal volume. However, such guesses aren't completely reliable. Usually, the optimal value of a variable isn't a nice whole number like 2, but a hard-to-guess fraction like 19/7, or even an irrational number like $\sqrt{7}$. The *graphical* and *numerical* approaches can *suggest estimates* of the optimal values. But in general, finding the exact values, and being certain that they give us the very best solution, requires that we use an *analytic* approach based on algebra and calculus.

Important breakthroughs in developing this analytic approach were made in the Middle Ages by scholars associated with libraries, referred to in Arabic at the time as "houses of wisdom." These were essentially government-supported research centers, the most famous one located in Baghdad in what is now Iraq. Scholars speaking many different languages arrived there from throughout the Middle East, but they conversed with each other in Arabic (that's why many of our mathematical terms, such as "algebra," "algorithm" and "zero," came to us from Arabic).



In al-Tūsī's day, Iraq was one of the world's leading centers for scientific inquiry. This miniature painting from the year 1236 depicts scholars at a house of wisdom in Basra, in southern Iraq. It was painted by Yahyā ibn Mahmūd al-Wāsitī, of Baghdad, copying from the original by al-Harīrī (1054-1122) in the manuscript *Al-Maqāmāt*.

Bibliothèque nationale Française, Paris, Mss or., Arabe 5847.

A discussion of what we would today call the **derivative** of a function has been found in an algebra treatise written in Baghdad in the year 1209 by **Sharaf al-Dīn al-Tūsī** (1135-1213). Al-Tūsī was born in Persia (now Iran), and had already spent his life teaching in Damascus, Aleppo, Mosul and other cities before arriving in Baghdad. Al-Tūsī did not call his newly discovered function a "derivative." In fact, he gave it no special name, and there is no evidence that he explored the concept thoroughly. It was just a tool he fashioned for solving a cubic polynomial problem much like Sami's curtain problem. He did not record in his treatise his path of discovery, but modern scholars have reconstructed a plausible path, as outlined in Exercise 4 below.

Like all other medieval mathematicians, al-Tūsī did not have algebraic symbols; instead, equations were written out *rhetorically*, that is, in words. But here, we use modern notation to abbreviate his work.

Exercise 4. To find the maximum volume, we need to analyze $V(x)$ in order to see whether the volume rises or falls when we change x by a small amount h.

(a) Copy the cubic polynomial $V(x)$ that you found in Exercise 2(d) above.

(b) Substitute $x + h$ for x in the volume function, and expand the resulting terms.

$$V(x + h) =$$

(c) Collect like terms according to powers of h:

$$V(x + h) = (\underline{\qquad\qquad\quad}) + (\underline{\qquad\qquad\quad})h + (\underline{\qquad\quad})h^2 + (\underline{\quad})h^3$$

(d) Compare with the original volume:

$$V(x + h) - V(x) = (\underline{\qquad\qquad\qquad})h + (\underline{\qquad\quad\;})h^2 + (\underline{\quad})h^3$$

(e) Based on our work in Exercises 1 and 2, we suspected that values of x near 4 meters are important. Substituting this into your answer to part (d),

$$\text{If } x = 4, \qquad \text{then } V(x + h) - V(x) = \underline{\quad}h + \underline{\quad}h^2 + \underline{\quad}h^3$$

(f) Factor your answer to part (e):

$$\text{If } x = 4, \qquad \text{then } V(x + h) - V(x) = (\underline{\qquad\quad})h^2$$

(g) By determining the signs of the two factors in part (f), complete the following (circle the correct inequality symbol in each case):

$$\text{If } x = 4 \text{ and } h \approx 0, \quad \text{then} \quad V(x + h) - V(x) \quad < \quad > \quad 0$$

$$\text{so} \qquad V(x) \quad < \quad > \quad V(x + h)$$

Since $x = 4$ makes $V(x)$ bigger than $V(x + h)$ for all small values of h, then $V(x)$ is the highest value of V "in the neighborhood." We call such a value a **relative maximum**.

(h) Let's remember that $x = 4$ is a relative maximum only because, in part (e), it made the coefficient of h that we'd found in part (d) vanish, that is, equal zero:

$$(\underline{\qquad\qquad\qquad}) = 0$$

Hey, cool! Compare this with the objective function $V(x)$ from part (a) and notice that it's exactly what we call *the derivative*.

Now solve that equation:

$$x = \underline{\qquad\qquad}$$

(i) What would you advise Sami to do in order to curtain off the largest volume of privacy?

Schwartz, Randy K., "Combining Strands of Many Colors: Episodes from Medieval Islam for the Mathematics Classroom," *Loci: Convergence* (August 2010), DOI: 10.4169/loci003546