A Brief History of Quaternions and the Theory of Holomorphic Functions of Quaternionic Variables

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Abstract

In this paper I will give a brief history of the discovery (Hamilton, 1843) of quaternions. I will address the issue of why a Theory of Triplets (the original goal of Hamilton) could not be developed. Finally, I will discuss briefly the history of various attempts to define holomorphic functions on quaternionic variables.

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1 Introduction

The discovery of the quaternions is one of the most well documented discoveries in mathematics. In general, it is very rare that the date and location of a major mathematical discovery are known. In the case of quaternions, however, we know that they were discovered by the Irish mathematician, William Rowan Hamilton on October 16^{th}, 1843 (we will see later how we come to be so precise).

The early 19^{th} century was a very exciting time for Complex Analysis. Though complex numbers had been discussed in works published in the 1500s, the study of complex numbers was often dismissed as useless. It was not until centuries later that the practicality of complex numbers was really understood. In the late 1700s, Euler made significant contributions to complex analysis, but most of the fundamental results which now form the core of complex analysis were discovered between 1814 and 1851 by Cauchy, Riemann, among others [6]. Hamilton himself, before discovering quaternions, was involved with complex numbers. In 1833, he completed the Theory of Couplets, which was at the time regarded as a new algebraic representation of the Complex Numbers. He represented two real numbers a and b as a couple (a, b), and then defined the additive operation to be $(a, b) + (c, d) = (a + c, b + d)$ and the multiplicative operation to be $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$ thus defining the algebraic definition of complex numbers. From a modern perspective, this is nothing more than a well understood way to represent complex numbers, but in 1833, the understanding of the algebraic structure of what we now call the complex field was not fully grasped.

As his next natural step, Hamilton wanted to extend the complex numbers to a new algebraic structure with each element consisting of one real part and two distinct imaginary parts. This would be known as the Theory of Triplets. There are natural mathematical reasons why one would want to attempt such a construction, but Hamilton was guided, as well, by a desire to use these triplets to represent rotations in three-dimensional space, just like complex numbers could be used to represent rotations in the two-dimensional plane.

Hamilton worked unsuccessfully at creating this algebra for over 10 years, and finally had a breakthrough on October 16^{th}, 1843 while on a walk with his wife, Lady Hamilton. They had been walking along the Royal Canal in Dublin when it occurred to Hamilton that his new algebra would require three rather than two imaginary parts. In order to do this, he could create a new algebraic structure consisting of one real part and three imaginary parts i, j, and k. For this new structure to work, Hamilton realized that these new imaginary elements would have to satisfy the following conditions

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1}$$

Hamilton carved these results on the nearby Broome Bridge. Unfortunately the carvings no longer remain today [1]. However, his discovery was so significant that every year on October 16th, the Mathematics Department of the National University of Ireland, Maynooth, holds a Hamilton Walk to Broome Bridge commemorating his discovery. This sequence of events is documented in a famous letter that Hamilton wrote to his son, which I attach in the appendix.

In this paper, I will first describe the skew field of quaternions, and I will then attempt to explain why Hamilton had to abandon the Theory of Triplets. I will conclude with a section that traces one of the most important developments in the study of quaternions since Hamilton, namely the attempt to replicate, for quaternionic functions, the theory of holomorphic functions that has so much importance in the study of the complex plane.

2 The algebra of quaternions

In this section, I set the stage for the rest of the paper and I provide the basic algebraic definitions of quaternions [3]. To better understand and appreciate the discovery of quaternions, it is important to understand them as an algebraic structure. The quaternions, often denoted by $\mathbb{H}$, in honor of their discoverer, constitute a non-commutative field, also known as a skew field, that extends the field $\mathbb{C}$ of complex numbers.

In modern day mathematics, we use abstract algebra to describe algebraic structures as groups, rings, and fields. A group, for example, is a non-empty set together with an associative binary operation and which contains an identity, and an inverse for every element. Galois first used the term group around 1830, so this was a new and exciting branch of mathematics in Hamilton's time. In abstract algebra, a field is a set $\mathbb{F}$, together with two associative binary operations, typically referred to as addition and multiplication. In order to have a field, we require that $\mathbb{F}$ is an Abelian group under addition (i.e. addition is commutative and its identity is denoted by 0), that $\mathbb{F}\backslash\{0\}$ is an Abelian group under multiplication, and that multiplication must distribute over addition. In short, this guarantees the ability to perform addition, subtraction, multiplication, and division on all elements in the field (except division by zero). If $\mathbb{F}\backslash\{0\}$ is not an Abelian group under multiplication, then we call $\mathbb{F}$ a skew field. In this case, we are still able to perform addition, subtraction, multiplication, and division, but the multiplication is not commutative. While the terminology for fields and groups was not fully established until after Hamilton's discovery of quaternions, this abstract approach was already being used at this time.

To begin with, the field of complex numbers is defined by

$$\mathbb{C} = \{a + ib | a, b \in \mathbb{R}, i^2 = -1\}.$$

This means that every complex number can be written in the form $a + ib$ where a and b are real numbers and i is an imaginary unit, i.e. $i^2 = -1$. We now want to construct a new field, $\mathbb{H}$, such that $\mathbb{C}$ is a subset of $\mathbb{H}$, and such that the new operations in $\mathbb{H}$ are compatible with the old operations in $\mathbb{C}$. This means we are looking for a field extension of $\mathbb{C}$ according to the following definition.

Definition 2.1. *Given two fields, $\mathbb{E}$ and $\mathbb{F}$, we say that $\mathbb{E}$ is a field extension over $\mathbb{F}$ if $\mathbb{F}$ is a subset of $\mathbb{E}$ and the operations of $\mathbb{F}$ are those of $\mathbb{E}$ restricted to $\mathbb{F}$ [3].*

Remark 2.2. *For example, $\mathbb{C}$ is a field extension over $\mathbb{R}$. Indeed, $\mathbb{R}$ is a subset of $\mathbb{C}$ because every element of $\mathbb{R}$ can be written in the form $a + ib$ where $b = 0$. Finally, the additive and multiplicative operators of $\mathbb{R}$ are consistent with those of $\mathbb{C}$ when $b = 0$. In general, we say that $\mathbb{E}$ has degree n over $\mathbb{F}$ if $\mathbb{E}$ has dimension n as a vector space over $\mathbb{F}$. Because $\{1, i\}$ forms a 2 dimensional vector space of $\mathbb{C}$ over $\mathbb{R}$, the field of complex numbers is a degree 2 field extension over the field of real numbers.*

In the specific case of quaternions, $\mathbb{H}$ is constructed by adding two new elements j and k such that $i^2 = j^2 = k^2 = ijk = -1$. The field of quaternions can then be written as

$$\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k | q_t \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}.$$

We can now begin to talk about the additive and multiplicative operations that can be defined on elements of $\mathbb{H}$ and turn it into a field [5]. Let $p = p_o + p_1 i + p_2 j + p_3 k$ and $q = q_o + q_1 i + q_2 j + q_3 k$ be two elements in $\mathbb{H}$. We say that $p = q$ if and only if $p_t = q_t$ for all $t \in \{0, 1, 2, 3\}$. Addition is defined in the natural sense, that is $p + q = p_o + q_o + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)k$.

It takes a little bit more work to define the multiplication of p and q. Using the distributivity of multiplication over addition, the product

$$pq = (p_o + p_1 i + p_2 j + p_3 k)(q_o + q_1 i + q_2 j + q_3 k)$$

becomes

$$pq = p_o q_o + p_o q_1 i + p_o q_2 j + p_o q_3 k + p_1 q_0 i + p_1 q_1 i^2 + p_1 q_2 ij +$$
$$+ p_1 q_3 ik + p_2 q_0 j + p_2 q_1 ji + p_2 q_2 j^2 + p_2 q_3 jk + p_3 q_0 k + p_3 q_1 ki + p_3 q_2 kj + p_3 q_3 k^2$$

Notice that we must be careful simplifying this step because multiplication is commutative for real numbers but not for imaginary elements. Using the basic properties of quaternions and identities (1) we can rewrite the multiplication again.

$$pq = p_0 q_0 + p_0 q_1 i + p_0 q_2 j + p_0 q_3 k + p_1 q_0 i - p_1 q_1 + p_1 q_2 k - p_1 q_3 j +$$
$$+ p_2 q_0 j - p_2 q_1 k - p_2 q_2 + p_2 q_3 i + p_3 q_0 k + p_3 q_1 j - p_3 q_2 i - p_3 q_3$$

We regroup the terms according to the imaginary units, and we obtain

$$pq = p_0 q_0 - (p_1 q_1 + p_2 q_2 + p_3 q_3) + (p_o q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2) i +$$
$$+ (p_0 q_2 - p_1 q_3 + p_2 q_0 + p_3 q_1) j + (p_0 q_3 + p_1 q_2 - p_2 q_1 + p_3 q_0) k$$

We may also represent this product as the product of two matrices where

$$pq = r_0 + ir_1 + jr_2 + kr_3 \text{ such that } \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Notice that if we rewrite the quaternions p and q as $p = p_0 + \mathbf{p}$ and $q = q_0 + \mathbf{q}$ where $\mathbf{p} = p_1 i + p_2 j + p_3 k$ and $\mathbf{q} = q_1 i + q_2 j + q_3 k$, we can rewrite the product of p and q as $pq = p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q}$.

We are also able to define a complex conjugate of a quaternion q. Let $q = q_0 + \mathbf{q}$, then the complex conjugate of q, denoted $\overline{q}$, is the quaternion $\overline{q} = q_0 - \mathbf{q}$. We can also define the norm of a quaternion q, denoted $|q|$, to be $\sqrt{q\overline{q}}$. Because we are working in a skew field, it is important to note that $q\overline{q} = \overline{q}q = |q|^2$.

We can now show that the quaternions are indeed a field, which means that every nonzero element has a multiplicative inverse. For a nonzero quaternion q, the inverse of q, denoted q^{-1}, is the quaternion $q^{-1} = \frac{\overline{q}}{|q|^2}$. We can easily verify this by preforming the multiplication $qq^{-1} = \frac{q\overline{q}}{|q|^2} = \frac{|q|^2}{|q|^2} = 1$ and similarly $q^{-1}q = \frac{\overline{q}q}{|q|^2} = \frac{|q|^2}{|q|^2} = 1$.

3 Why dimension 4?

In this section, I will investigate Hamilton's breakthrough concerning the necessity of three distinct imaginary parts. Why was Hamilton unable to create his Theory of Triplets? What made him realize that he would need to add a third imaginary element? This can be understood with a basic understanding of field extensions.

If Hamilton had been able to develop his Theory of Triplets, he would have effectively built a degree three field extension of $\mathbb{R}$ whose vector space forms the basis $\{1, i, j\}$ over $\mathbb{R}$ such that $i^2 = j^2 = -1$.

Let us then call $\mathbb{D}$ the hypothetical field generated by the elements $\{1, i, j\}$ satisfying $i^2 = j^2 = -1$. Because $\mathbb{D}$ is a field, it must be closed under multiplication. Thus the multiplication of i and j must result in some element that is already in the field. In order to find a contradiction, we investigate all of the possibilities of the product ij. Without loss of generality, we may exhaust all possibilities by checking all of the generators and zero.

If $ij = \pm 1$, then $iij = \pm i$, which implies that $j = \mp i$. If j can be written as a multiple of i, then they are not linearly dependent. A basis for $\mathbb{D}$ is then the set $\{1, i\}$, which contradicts our original assumption that the basis is $\{1, i, j\}$.

If $ij = \pm i$, then $iij = \pm i^2 = \mp 1$, which implies that $j = \pm 1$. Again this would imply that $\mathbb{D}$ is a two-dimensional field extension over $\mathbb{R}$.

Analogously, if $ij = \pm j$,then $ijj = \pm j^2 = \mp 1$,which implies that $i = \pm 1$, and we conclude as above.

Suppose finally that $ij = 0$. In this case there would exist a zero-divisor (in fact, both i and j would be zero-divisors), which is incompatible with $\mathbb{H}$ being a field. More directly, if $ij = 0$, then we would have that $ijj = 0$, i.e. this would show that i itself is zero, against the original assumption.

We have exhausted all possibilities and can conclude that there is no third degree field extension over $\mathbb{R}$ that holds the property that $i^2 = j^2 = -1$. Thus it is not possible to create the Theory of Triplets while satisfying the requirements of a field.

Remark 3.1. *While our simple computations show the difficulty of building a three-dimensional field extension over $\mathbb{R}$, this fact is in fact a special case of a much more powerful and general result known as Frobenius theorem, [7]. In 1877, Ferdinand Georg Frobenius characterized the finite dimensional associative division algebras over the real numbers, and in particular proved that the only associative division algebra which is not commutative over the real numbers is the algebra of quaternions. In addition, Frobenius proved that there are only three finite dimensional division algebras over $\mathbb{R}$: $\mathbb{R}$ itself, $\mathbb{C}$, and finally $\mathbb{H}$.*

This quick analysis shows why Hamilton had to consider a four dimensional field extension by adding a new element k that is linearly independent of the generators $1, i$, and j. Thus our question becomes whether it is possible to find a four-dimensional field extension of $\mathbb{R}$ whose basis is $\{1, j, i, k\}$ and such that $i^2 = j^2 = k^2 = -1$. Specifically, the question is to discover (and this must have been the issue that Hamilton was struggling with) what are the relationships between the three units.

Since we have excluded all the previous possibilities on the outcome of ij, we will assume that $ij = k$ and we will investigate the consequences of such an assumption. To begin with, we want to understand what should the product ji be. By an argument analogous to the one above, we can reason that ji yields either k or $-k$. If $ji = k$, then $k^2 = (ij)^2 = i^2 j^2 = (-1)(-1) = 1$. This gives again a contradiction, because the product of two non-zero elements, $1 + k$ and $1 - k$

would then give $(1+k)(1-k) = 1 - k^2 = 1 - 1 = 0$. Once again we have zero-divisors and $\mathbb{H}$ is not a field.

Remark 3.2. *Note that $ij = ji$ is incompatible with the request that $\mathbb{H}$ is a field. However, it is quite possible to consider such a structure, which is of independent interest and is known as the commutative ring of bicomplex numbers [8].*

We (and Hamilton!) are therefore left with the situation in which $ji = -k$. This generates a non-commutative field and it is exactly the structure that Hamilton called with the name of quaternions. It was a rather radical move at the time to study a non-commutative group and this proposal was met by many with skepticism [5]. Most mathematicians, at the time, could not grasp the value of studying an algebraic structure that violated the law of multiplicative commutativity. Modern day mathematics recognize this as a very valuable practice. American mathematician Howard Eves described the significance of Hamilton's new approach by saying that it

> opened the floodgates of modern abstract algebra. By weakening or deleting various postulates of common algebra, or by replacing one of more of the postulates by others, which are consistent with the remaining postulates, an enormous variety of systems can be studied. As some of these systems we have groupoids, quasigroups, loops, semi-groups, monoids, groups, rings, integral domains, lattices, division rings, Boolean rings, Boolean algebras, fields, vector spaces, Jordan algebras, and Lie algebras, the last two being examples of non-associative algebras [5].

4 Holomorphicity

In this section, I will recall the notion of holomorphicity in complex analysis and I will explore how such a notion can be extended to the case of quaternionic functions defined on the space of quaternions. This is a very natural next step, since the power and importance of complex numbers cannot be exploited until a full theory of holomorphic functions is developed. For my discussion of holomorphic functions of a complex variable I will follow [2].

Definition 4.1. *A function $f : \mathbb{C} \to \mathbb{C}$ is said to be complex differentiable at $z_0 \in \mathbb{C}$ if the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, when z approaches z_0 in the complex plane (the limit, therefore, is independent of the curve along which z approaches z_0). This limit will be called the first derivative of f at z_0 and will be denoted by $f'(z_0)$.

Definition 4.2. *A function f is said to be holomorphic in an open set $U \subset \mathbb{C}$, if $f'(z)$ exists for all $z \in U$.*

In real analysis, i.e. for real valued functions of real variables, the property of differentiability is not a very strong property. For example, there are plenty of functions which admit a first derivative, even a continuous first derivative, and yet do not have a second derivative. There are also examples of functions which admit infinitely many derivatives, and yet those derivatives are insufficient to reconstruct the original functions by means of its Taylor series. In other words, there are functions which are C^∞ but not real analytic. For this reason, the next result is a very surprising statement, and one that clearly indicates that the study of complex functions is truly a different topic from real analysis.

Theorem 4.3. *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a function. The following three properties are equivalent:*

1. *f admits a first derivative, in the complex sense, in every point of U.*
2. *f is can be represented, near every point $\alpha \in U$, by a power series $\sum_{n=0}^{\infty} c_n(z-\alpha)^n$.*
3. *f is a solution of the Cauchy-Riemann equation on U. That is*

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0.$$

The first natural question that comes to mind, is whether these three conditions be reformulated in the quaternionic context, and whether they are still equivalent in that setting. In a very well known paper, [9], A. Sudbery investigates and answers these questions.

To begin with, we need a definition of quaternionic differentiability. Because we are working in a skew field, we are not guaranteed commutativity, thus we must define derivatives from the left and right side.

Definition 4.4. *A function is said to be quaternion-differentiable on the left at a point $q \in \mathbb{H}$ if the limit*

$$\frac{df}{dq} = \lim_{h \to 0} \frac{f(q+h) - f(q)}{h}$$

exists, when h converges to zero along any direction in the quaternionic space.

It turns out, [9], that this is not a very good definition, since if this limit exists, then f must necessarily be of the form $f(q) = a + qb$ for $a, b \in \mathbb{H}$. Thus if a function is left differentiable in the quaternionic sense, it must be a linear function.

Given that differentiability is not the appropriate way to generalize the notion of holomorphicity, one can investigate power series for quaternionic functions.

Definition 4.5. *A function $f : \mathbb{H} \to \mathbb{H}$ is said to be a quaternionic monomial if $f(q) = \Pi_{i=1}^{r} a_i q$ where r is a non-negative number and $a_i \in \mathbb{H}$.*

The reason for such a definition is due to the non-commutativity of $\mathbb{H}$, which does not allow the coefficients a_i to move to the same side of the powers of q. It is easy to show, however, that the quaternionic functions that can be represented as a power series of quaternionic monomials are the complex functions that are real analytic, and so once again we have a definition that does not offer any new class of functions.

So, if quaternionic differentiability confines us to linear functions, and expressibility in power series limits us to real analytic functions, the only hope for an interesting theory of holomorphicity seems to be based on the possibility of generalizing the Cauchy-Riemann equations.

It was the Swiss mathematician Fueter, who was able to develop the appropriate generalization of the Cauchy-Riemann equations to the quaternionic case. The Cauchy-Fueter equations (as they are now called in honor of their inventor) were not developed until 1935, almost a century after Hamilton's discovery of quaternions. Because quaternionic derivatives are defined on both the right and left side, there are actually two differential operators, the left Cauchy-Fueter operator, $\frac{\partial_l}{\partial \overline{q}}$, and the right Cauchy-Fueter operator, $\frac{\partial_r}{\partial \overline{q}}$. These operators are defined as follows:

$$\frac{\partial_l f}{\partial \overline{q}} = \frac{\partial f}{\partial q_o} + i\frac{\partial f}{\partial q_1} + j\frac{\partial f}{\partial q_2} + k\frac{\partial f}{\partial q_3}$$

$$\frac{\partial_r f}{\partial \overline{q}} = \frac{\partial f}{\partial q_o} + \frac{\partial f}{\partial q_1} i + \frac{\partial f}{\partial q_2} j + \frac{\partial f}{\partial q_3} k.$$

These operators are used to define the Cauchy-Fueter equations, whose solutions are the appropriate generalization of holomorphic functions to the quaternionic setting. In accordance with the terminology established by Fueter in his early work, and now commonly adopted, these functions are now known as regular functions.

Specifically, we have the following definition.

Definition 4.6. *Let $f : \mathbb{H} \to \mathbb{H}$ be a quaternionic valued function of a quaternionic variable. We say that f is left regular if*

$$\frac{\partial_l f}{\partial \overline{q}} = \frac{\partial f}{\partial q_o} + i\frac{\partial f}{\partial q_1} + j\frac{\partial f}{\partial q_2} + k\frac{\partial f}{\partial q_3} = 0$$

and we say that f is right regular if

$$\frac{\partial_r f}{\partial \overline{q}} = \frac{\partial f}{\partial q_o} + \frac{\partial f}{\partial q_1} i + \frac{\partial f}{\partial q_2} j + \frac{\partial f}{\partial q_3} k = 0.$$

As it is shown in [9], this definition turns out to be very effective, and the theory of regular functions is a fully formed theory, which in many essential ways resembles the theory of holomorphic functions of a complex variable. For example, it is possible to reconstruct (almost with no important changes) the entire Cauchy theory that holds for holomorphic functions (Cauchy theorems, Cauchy representation formulas, etc.) and therefore almost all of its consequences.

An interesting observation is that any quaternionic valued function $f = f_0 + f_1 i + f_2 j + f_3 k$ can be thought of as a vector $\vec{f} = (f_0, f_1, f_2, f_3)$ of four real valued functions, and therefore a regular function corresponds to a vector $\vec{f}$ of real functions satisfying a 4×4 system of linear constant coefficient partial differential equations, whose formal appearance is the same we saw when we represented quaternionic multiplication via a matrix (we leave this simple computation to the reader).

There are, nevertheless, some important differences that emerge between the two theories, and some are substantial enough to justify the emergence of a new approach. I want to conclude this historical review by pointing out the nature of the differences, and one of the new approaches that seem to remedy and offer a different theory.

To begin with, it is an unpleasant discovery that even $f(q) = q = q_0 + q_1 i + q_2 j + q_3 k$ is not regular according to the definition of Fueter. Indeed,

$$\frac{\partial_r q}{\partial \overline{q}} = \frac{\partial q}{\partial q_o} + \frac{\partial q}{\partial q_1} i + \frac{\partial q}{\partial q_2} j + \frac{\partial q}{\partial q_3} k = 1 - 1 - 1 - 1 = -2 \neq 0.$$

Regardless of what one has in mind with this new theory, the fact that the simplest function is not regular, and therefore that no monomial or polynomial is regular is problematic.

There have been several attempts to circumvent this issue, but the most promising, and recent, has just appeared in [4], where the following definition is offered.

Definition 4.7. *Let U be a domain in $\mathbb{H}$. A real differentiable function $f : U \to \mathbb{H}$ is said to be slice-regular if, for every quaternion $I = x_1 i + x_2 j + x_3 k$ such that $x_1^2 + x_2^2 + x_3^2 = 1$, the restriction of f to the complex line $L_I = \mathbb{R} + \mathbb{R}I$ passing through the origin, and containing* 1 *and I is holomorphic on $U \cap L_I$.*

Note that this definition essentially requires that a function be holomorphic on each complex slice of the original domain: this explain the terminology of "slice-regularity". In a sense, this does not seem to be such an interesting idea, but the consequences of this definition are quite far reaching. For example, one can prove that the identity function $f(q) = q$ is slice-regular, but more important all monomials of the form aq^n, with a any quaternion, are slice-regular in this sense. Since regularity respects addition, it is immediate to see that all polynomials of the form $f(q) = \Sigma_{t=0}^{n} a_n q^n$ with $a_n \in \mathbb{H}$ are regular and in fact even power series of the form $f(q) = \Sigma_{t=0}^{\infty} a_n q^n$, with $a_n \in \mathbb{H}$ are regular where convergent, [4].

This theory is only in its infancy at this point (though a cursory check to mathsci.net shows that a significant number of papers have already appeared in this area), but it seems to demonstrate that the history of quaternions, which began in 1835, is not yet concluded, and that new ideas are constantly germinating in this field.

5 Appendix: Letter from Sir W.R.Hamilton to Rev. Archibald H. Hamilton

Letter dated August 5, 1865.

MY DEAR ARCHIBALD - (1) I had been wishing for an occasion of corresponding a little with you on QUATERNIONS: and such now presents itself, by your mentioning in your note of yesterday, received this morning, that you "have been reflecting on several points connected with them" (the quaternions), "particularly on the Multiplication of Vectors." (2) No more important, or indeed fundamental question, in the whole Theory of Quaternions, can be proposed than that which thus inquires What is such MULTIPLICATION? What are its Rules, its Objects, its Results? What Analogies exist between it and other Operations, which have received the same general Name? And finally, what is (if any) its Utility? (3) If I may be allowed to speak of myself in connexion with the subject, I might do so in a way which would bring you in, by referring to an ante-quaternionic time, when you were a mere child, but had caught from me the conception of a Vector, as represented by a Triplet: and indeed I happen to be able to put the finger of memory upon the year and month - October, 1843 - when having recently returned from visits to Cork and Parsonstown, connected with a meeting of the British Association, the desire to discover the laws of the multiplication referred to regained with me a certain strength and earnestness, which had for years been dormant, but was then on the point of being gratified, and was occasionally talked of with you. Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me, "Well, Papa, can you multiply triplets"? Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only add and subtract them." (4) But on the 16th day of the same month - which happened to be a Monday, and a Council day of the Royal Irish Academy - I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; namely, $i^2 = j^2 = k^2 = ijk = -1$ which contains the Solution of the Problem, but of course, as an inscription, has long since mouldered away. A more durable notice remains, however, on the Council Books of the Academy for that day (October 16th, 1843), which records the fact, that I then asked for and obtained leave to read a Paper on Quaternions, at the First General Meeting of the session: which reading took place accordingly, on Monday the 13th of the November following.

With this quaternion of paragraphs I close this letter I.; but I hope to follow it up very shortly with another.

Your affectionate father,

WILLIAM ROWAN HAMILTON.

References

[1] S.L. Altmann, Rotations, Quaternions, and Double Groups. Oxford University Press, Oxford, 1986.

[2] C.A. Berenstein, Complex Variables: An Introduction. Springer, New York, 1997.

[3] J.A. Gallian, Contemporary Abstract Algebra. Houghton Mifflin, Boston, 2006.

[4] G. Gentili and D.C. Struppa, A new theory of regular functions of a quaternionic variable. Adv. Math. **216** (2007), 279-301.

[5] J.B. Kuipers, Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace and Virtual Reality. Princeton University Press, Princeton, 2002.

[6] T. Needham, Visual Complex Analysis. Oxford UIniversity Press, Oxford, 1998.

[7] R.S. Palais, The Classification of Real Division Algebras, American Mathematical Monthly **75** (1968), 366-368.

[8] G.B. Price, An Introduction to Multicomplex Spaces and Functions. Marcel Dekker, New York, 1991.

[9] Sudbery, A. Quaternionic Analysis. Quaternionic Analysis. Aug. 1977. University of York. 19 Mar. 2009.