A Classic from China: The *Nine Chapters*

Randy K. Schwartz (Schoolcraft College)

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Students Explore the *Nine Chapters* from China

We were pleased when the article "A Classic from China: The *Nine Chapters*" (*The Right Angle*, October 2008) was greeted with praise from many different mathematics instructors. They and their students really got into the Reader Challenge: to solve the 10 selected story problems from this ancient Chinese manuscript that were presented in the article.

Thirteen students, enrolled in a variety of mathematics courses at Schoolcraft College, submitted responses to our Challenge. Seven of them are represented in the solutions provided below. (*Editor's note:* In *Convergence,* students are identified only by their first names.)

Below, we provide correct solutions to all 10 challenge problems.

Problem 1. Now pay 5785 coins to purchase 1 *hu* 6 *dou* 7 2/3 *sheng* of lacquer. Tell: how much is 1 *dou*?

Jackie wrote:

5785 coins ↔ 1 *hu* 6 *dou* 7 2/3 *sheng*

5785 coins ↔ 10 *dou* + 6 *dou* + (7 2/3)/10 *dou*

5785 coins ↔ 16.76666… *dou*

5785 ÷ 16.76666… coins ↔ 1 *dou*

Approx. 345 coins ↔ 1 *dou*

[By using fractions, we can get the exact answer, 345 15/503.]

Problem 2. Now given 1 *jin* of silk costs 240 coins. Tell: given 1328 coins, how many *jin* of silk are obtained? (Your answer should be given exactly, in units of *jin*, *liang*, *zhu*, and fractions of a *zhu*.)

Michael wrote:

Since 1 *jin* of silk cost 240 coins, then 1328 coins will buy

1328 ÷ 240 *jin* = 5.5333… *jin*
= 384(5.5333…) *zhu*
= 2124 4/5 *zhu*.

Now subtract 5 *jin* = 384(5) *zhu* = 1920 *zhu*, leaving

2124 4/5 – 1920 *zhu* = 204 4/5 *zhu*
= 204 4/5 ÷ 24 *liang*
= 204 4/5 ÷ 24 *liang*
= 8.5333… *liang*.

Now subtract 8 *liang* = 24(8) *zhu* = 192 *zhu*, leaving

204 4/5 – 192 *zhu* = 12 4/5 *zhu*.

Answer: 5 *jin* 8 *liang* 12 4/5 *zhu* of silk.

Problem 3. Now given a field of 1 *mu*, 6 2/3 *sheng* of millet is collected [as rent]. Tell: given 1 *qing* 26 *mu* 159 [square] *bu* of field how much millet is collected?

Peter wrote:

1 *qing* 26 *mu* 159 bu^2 = 100 *mu* + 26 *mu* + 159 bu^2
= 126 *mu* + 159 bu^2
= 240(126) bu^2 + 159 bu^2
= 30240 bu^2 + 159 bu^2
= 30399 bu^2
= 30399 ÷ 240 *mu*
= 126.6625 *mu*.

So the rent would be 126.6625(6 2/3 *sheng*) = approx. 844.416 *sheng*.

[By using fractions, we can get the exact answer, 844 5/12 *sheng*.]

Problem 4. Now given the task of transporting tax millet is distributed among four counties. County A, 8 days from the tax bureau, has 10,000 households; County B, 10 days from the bureau, has 9,500 households; County C, 13 days from the bureau, has 12,350 households; County D, 20 days from the bureau, has 12,200 households. The total tax millet is 250,000 *hu* needing 10,000 carts. Assume the task is to be distributed in accordance with the distance from the bureau and the number of households. Tell: how much millet should each county transport? How many carts does each county employ?

Allison wrote:

A: 10,000 houses ÷ 8 days = 1250
B: 9,500 houses ÷ 10 days = 950
C: 12,350 houses ÷ 13 days = 950
D: 12,200 houses ÷ 20 days = 610
Total 3760

A: (1250 ÷ 3760)250,000 ≈ 83,111.70 *hu* of millet
B: (950 ÷ 3760)250,000 ≈ 63,164.89 *hu* of millet
C: (950 ÷ 3760)250,000 ≈ 63,164.89 *hu* of millet
D: (610 ÷ 3760)250,000 ≈ 40,558.51 *hu* of millet

Total 250,000 *hu* of millet

250,000 *hu* needs 10,000 carts, so 1 cart holds 250,000 ÷ 10,000 = 25 *hu*.

A: 83,111.70 ÷ 25 ≈3324 carts
B: 63,164.89 ÷ 25 ≈2527 carts
C: 63,164.89 ÷ 25 ≈2527 carts
D: 40,558.51 ÷ 25 ≈1622 carts

Total 10,000 carts.

Problem 5. Now someone transports provisions between two posts. An unloaded cart travels 70 *li* a day and a loaded one 50 *li* a day. Transporting millet from the National Granary to Shanglin. One makes 3 round trips in 5 days, how far is the distance between the two posts?

Let the unknown distance in *li* be called *d*. Since time = distance ÷ speed, the time needed for one round trip is $\frac{d}{70}+\frac{d}{50}$ days. Thus, the time needed for three round trips is $3\left(\frac{d}{70}+\frac{d}{50}\right)$ days. But we were told this is 5 days, so:

$$3\left(\frac{d}{70}+\frac{d}{50}\right)=5$$

$$3\frac{d}{10}\left(\frac{1}{7}+\frac{1}{5}\right)=5$$

$$\frac{3}{10}d\left(\frac{12}{35}\right)=5$$

$$d = 5\left(\frac{35}{12}\right)\left(\frac{10}{3}\right) = \frac{875}{18} = 48\frac{11}{18}li.$$

Problem 6. Now chickens are purchased jointly; everyone contributes 9, the excess is 11; everyone contributes 6, the deficit is 16. Tell: The number of people, the chicken price, what is each?

Ed wrote:

16(9) coins per person ↔ 16 items and 16(11) coins excess
11(6) coins per person ↔ 11 items and 11(16) coins deficit
16(9) + 11(6) coins per person ↔ 27 items

$\frac{16(9)+11(6)}{27}$ coins per person ↔ 1 item

$\frac{70}{9}$ coins per person ↔ 1 item

So the item costs $\frac{70}{9}$ coins per person. But when each person paid $9 = \frac{81}{9}$ coins, the excess was 11 coins. So, when each person overpays $\frac{81}{9} - \frac{70}{9} = \frac{11}{9}$ coins, the excess was 11 coins. Thus, there are 9 persons, and the item price is $\frac{70}{9} \times 9 = 70$ coins.

Problem 7. Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 *dou*. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 *dou*. Tell: how much [*dou*] does one bundle of each grade yield?

Linda and Sindhuja used a matrix method similar to Gaussian Elimination. Jackie and Christie used a matrix method known in Europe as Cramer's Rule of Determinants. Ed and Dena used the algebraic method called Simple Elimination.

Dena wrote:

$$3x + 2y + z = 39 \quad (1)$$
$$2x + 3y + z = 34 \quad (2)$$
$$x + 2y + 3z = 26 \quad (3)$$

Subtract (2) from (1):

$$x - y = 5$$
$$x = y + 5 \quad (4)$$

Triple (2):

$$6x + 9y + 3z = 102 \qquad (5)$$

Subtract (3) from (5):

$$5x + 7y = 76$$
$$5x = 76 - 7y$$
$$x = \frac{76 - 7y}{5} \qquad (6)$$

Equate (4) and (6):

$$y + 5 = \frac{76 - 7y}{5}$$
$$5y + 25 = 76 - 7y$$
$$12y = 51$$
$$y = 4.25 \ \textit{dou}$$
$$x = y + 5 = 9.25 \ \textit{dou}$$
$$z = 39 - (3x + 2y) = 39 - (27.75 + 8.5) = 2.75 \ \textit{dou}.$$

Problem 8. Now given a *qiandu* with a lower breadth of 2 *zhang*, a length of 18 *zhang* 6 *chi* and an altitude of 2 *zhang* 5 *chi*. Tell: what is the volume? (Your answer should be given in cubic *chi*.)

Linda wrote:

1 *zhang* = 10 *chi*
2 *zhang* = 20 *chi*
18 *zhang* 6 *chi* = 186 *chi*
2 *zhang* 5 *chi* = 25 *chi*.

The cross-section is a triangle, whose area is

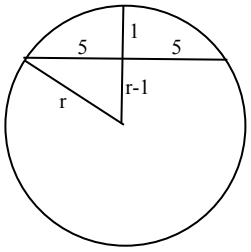
$$A = \tfrac{1}{2}bh$$
$$A = \tfrac{1}{2}(20\,chi)(25\,chi) = 250 \text{ square } chi$$

The volume of the prism is the area times the length,

$$V = AL$$
$$V = (250 \text{ square } chi)(186\,chi)$$
$$V = 46{,}500 \text{ cubic } chi\ .$$

Problem 9. Now given a circular [i.e., cylindrical] log of unknown size buried in a wall. When sawn 1 *cun* deep, it shows a breadth of 1 *chi*. Tell: what is the diameter of the log?

Recall that 1 *chi* = 10 *cun*, so the exposed breadth extends 5 *cun* to each side of the midpoint.



By the Gōugǔ theorem,

$$r^2 = 5^2 + (r-1)^2$$

$$r^2 = 25 + r^2 - 2r + 1$$

$$2r = 26$$

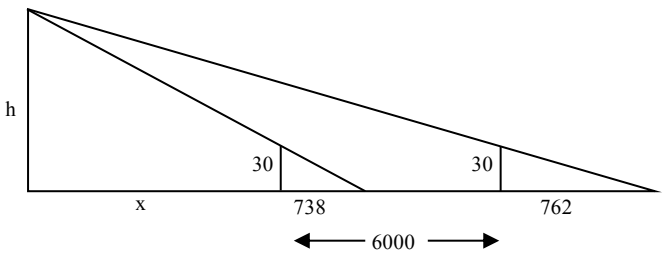
$$r = 13 \text{ } cun$$

diameter = $2r$ = 26 *cun* = 2 *chi* 6 *cun*.

Problem 10. Now survey a sea island. Erect two poles of the same height, 3 *zhang*, so that the front and rear poles are 1000 *bu* apart. They are aligned with the summit of the island. Move backwards 123 *bu* from the front pole, sighting at ground level, and find that the summit of the island coincides with the tip of the pole. Move backwards 127 *bu* from the rear pole, sighting at ground level, and find that the summit of the island also coincides with the tip of the pole. Tell: what are the height of the island and its distance from the [front] pole?

Sindhuja wrote:

Use similar triangles! [She converted all lengths to *chi* in her figure, below.]



By the similarity of the pairs of triangles,

$$\frac{h}{x+738} = \frac{30}{738} \text{ and } \frac{h}{x+6762} = \frac{30}{762}.$$

Thus,

$$30(x+738) = 738h \text{ and } 30(x+6762) = 762h$$

$$30x + 22140 = 738h \text{ and } 30x + 202860 = 762h$$

$$30x = 738h - 22140 \text{ and } 30x = 762h - 202860$$

$$738h - 22140 = 762h - 202860$$

$$202860 - 22140 = 762h - 738h$$

$$180720 = 24h$$

$$h = \frac{180720}{24} = 7530.$$

But $30(x+738) = 738h = 738(7530)$

so $x + 738 = 738(251)$

$x = 738(250) = 184500$

The height of the island is

$$h = 7530 \text{ } chi = 1255 \text{ } bu = 4 \text{ } li \text{ } 55 \text{ } bu.$$

Its distance from the front pole is

$$x = 184500 \text{ } chi = 30750 \text{ } bu = 102 \text{ } li \text{ } 150 \text{ } bu.$$