An Introduction to Symbolic Logic

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1 Introduction

In this project we will study the basics of propositional and predicate logic based on the original historical source *Principia Mathematica* [13] by Russell and Whitehead. Published in three volumes between 1910 and 1913, *Principia* was a culmination of work that had been done in the preceding century on the foundations of mathematics. Since the middle of the nineteenth century, such thinkers as George Boole (1815–1864), Augustus De Morgan (1806–1871), Charles Sanders Peirce (1839–1914), Ernst Schröder (1841–1902), Gottlob Frege (1848–1925), and Giuseppe Peano (1858–1932) had been developing axiomatic bases for logic and the foundations of mathematics. This research program found its culmination in *Principia*, which had a tremendous influence on the development of logic and the foundations of mathematics in the twentieth century.

Logic is a branch of science that studies correct forms of reasoning. It plays a fundamental role in such disciplines as philosophy, mathematics, and computer science. Like philosophy and mathematics, logic has ancient roots. The earliest treatises on the nature of correct reasoning were written over 2000 years ago. Some of the most prominent philosophers of ancient Greece wrote of the nature of deduction more than 2300 years ago, and thinkers in ancient China wrote of logical paradoxes around the same time. However, though its roots may be in the distant past, logic continues to be a vibrant field of study to this day.

Modern logic originated in the work of the great Greek philosopher Aristotle (384–322 BCE), the most famous student of Plato (c.427–c.347 BCE) and one of the most influential thinkers of all time. Further advances were made by the Greek Stoic philosopher Chrysippus of Soli (c.278–c.206 BCE), who developed the basics of what we now call propositional logic.[1]

For many centuries the study of logic was mostly concentrated on different interpretations of the works of Aristotle, and to a much lesser degree of those of Chrysippus, whose work was largely forgotten. However, all the argument forms were written in words, and lacked formal machinery that would create a logical calculus of deduction with which it would be easy to work.

The great German philosopher and mathematician Gottfried Leibniz (1646–1716) was among the first to realize the need to formalize logical argument forms. It was Leibniz's dream to create a universal formal language of science that would reduce all philosophical disputes to a matter of mere calculation by recasting the reasoning in such disputes in such a language.

The first real steps in this direction were taken in the middle of the nineteenth century by the English mathematician George Boole. In 1854 Boole published *An Investigation of the Laws of Thought* [4], in which he developed an algebraic system for discussing logic. Boole's work ushered

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[1]More on the life and work of Chrysippus can be found in [5]. Also, another article in this series [9] offers an historical project on Chrysippus' development of propositional logic.

in a revolution in logic, which was advanced further by Augustus De Morgan, Charles Sanders Peirce, Ernst Schröder, and Giuseppe Peano.[2]

The next key step in this revolution in logic was made by the great German mathematician and philosopher Gottlob Frege. Frege created a powerful and profoundly original symbolic system of logic, as well as suggested that the whole of mathematics can be developed on the basis of formal logic, which resulted in the well-known school of *logicism*.[3]

By the early twentieth century, the stage was set for Russell and Whitehead to give a modern account of logic and the foundations of mathematics in their influential treatise *Principia Mathematica*.

Alfred North Whitehead (1861–1947), the son of a vicar in the Church of England [10, p. 23], was born in Ramsgate, Kent, England, and studied mathematics at Trinity College, Cambridge. In 1884, Whitehead was elected a fellow at Trinity College, and would teach mathematics there until 1910. After his tenure at Trinity College, Whitehead spent time at University College London and Imperial College London, engaging in scholarly work in philosophy. He later emigrated to the United States, and taught philosophy at Harvard University until his retirement in 1937 [8].

While a fellow at Trinity College, Whitehead met Bertrand Russell (1872–1970), who was then a student there [10, p. 223]. Russell was born into an aristocratic family. His grandfather, John Russell, was twice Prime Minister to Queen Victoria [11, p. 5]. Russell graduated from Trinity College in 1893. He went on to become one of the most influential intellectuals of the twentieth century, playing a decisive role in the development of analytic philosophy. Russell was also active in a number of political causes. Notably, he was an anti-war activist and advocated nuclear disarmament. In fact, he served a six-month prison sentence for writing a newspaper editorial protesting World War I [11, p. 521]. Russell was a prolific writer, and in 1950 was awarded the Nobel Prize in Literature "in recognition of his varied and significant writings in which he champions humanitarian ideals and freedom of thought" [12].

Around 1901, Russell and Whitehead began collaborating on a book on logic and the foundations of mathematics [10, p. 254–258]. This resulted in an epochal work, *Principia Mathematica*, which would later be recognized as a significant contribution to logic and the foundations of mathematics. Influenced by the work of Frege, Peano, and Schröder, Russell and Whitehead developed an axiomatic basis for logic and the foundations of mathematics, and tried to free the foundations of mathematics of the existing contradictions.

In what follows, we will introduce the basic principles of contemporary logic through the development of Russell and Whitehead's *Principia Mathematica*.

2   Propositional Logic

In this section we begin our study of propositional logic from *Principia Mathematica*. The chief object of our investigation will be *propositions*—sentences which are either true or false but not both. Thus, we are concerned with sentences such as "Benjamin Franklin was the first president of the United States" and "Two plus two is equal to four." Clearly the first of the two sentences is false and the second one is true. Therefore, both of the sentences are propositions. On the other hand, a sentence such as "Who was the author of *Hamlet*?" is not a proposition because it is neither true nor false. Hence, we will not be concerned with this type of sentence.

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[2]For more information on the life and work of these scientists, see, e.g., [7]. Also, another article in this series [2] offers an historical project on the work of Boole and Peirce, while [3] offers an historical project on the work of Peano.

[3]For more information on Frege's life and work, see, e.g., [6, 7]. Frege's formal treatment of propositional logic is also discussed in another article in this series [9].

To carry out our study of propositions, we introduce the concept of a *propositional variable*, which stands for an arbitrary but undetermined proposition. The letters p, q, r and so forth will be used to denote propositional variables.

Logical connectives

We now turn to the first major topic in propositional logic, the question of how to form complicated propositions out of simpler ones. Russell and Whitehead address this question in the opening pages of *Principia Mathematica*:

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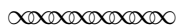
An aggregation of propositions (...) into a single proposition more complex than its constituents, is a function with *propositions as arguments.* [13, Vol. 1, p. 6]

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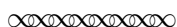
Thus, more complex propositions are formed from simpler propositions by means of functions that take propositions as arguments. What are the functions that yield more complex propositions from simpler ones? Russell and Whitehead employ four fundamental functions.

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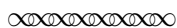
...They are (1) the Contradictory Function, (2) the Logical Sum, or Disjunctive Function, (3) the Logical Product, or Conjunctive Function, (4) the Implicative Function. These functions in the sense in which they are required in this work are not all independent; and if two of them are taken as primitive undefined ideas, the other two can be defined in terms of them. It is to some extent—though not entirely—arbitrary as to which functions are taken as primitive. Simplicity of primitive ideas and symmetry of treatment seem to be gained by taking the first two functions as primitive ideas. [13, Vol. 1, p. 6]

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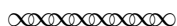
In modern terminology the Contradictory Function of Russell and Whitehead is known as *negation* ("not"), the Logical Sum or Disjunctive Function as *disjunction* ("or"), the Logical Product or Conjunctive Function as *conjunction* ("and"), and the Implicative Function as *implication* ("if, then").

Russell and Whitehead mention that the four functions are not independent of each other. Later on we will see why this is so. For now, let us read how Russell and Whitehead define these four functions.

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The Contradictory Function with argument p, where p is any proposition, is the proposition which is the contradictory of p, that is, the proposition asserting that p is not true. This is denoted by $\sim p$. Thus $\sim p$ is the contradictory function with p as argument and means the negation of the proposition p. It will also be referred to as the proposition not-p. Thus $\sim p$ means not-p, which also means the negation of p.

The Logical Sum is a propositional function with two arguments p and q, and is the proposition asserting p or q disjunctively, that is, asserting that at least one of the two p and q is true. This is denoted by $p \vee q$. Thus $p \vee q$ is the logical sum with p and q as arguments. It is also called the logical sum of p and q. Accordingly $p \vee q$ means that at least p or q is true, not excluding the case in which both are true.

The Logical Product is a propositional function with two arguments p and q, and is the proposition asserting p and q conjunctively, that is, asserting that both p and q are true. This is denoted by $p.q$ (...). Thus $p.q$ is the logical product with p and q as arguments. It is also called the logical product of p and q. Accordingly $p.q$ means that both p and q are true. It is easily seen that this function can be defined in terms of the two preceding functions. For when p and q are both true it must be false that either $\sim p$ or $\sim q$ is true. Hence in this book $p.q$ is merely a shortened form of symbolism for

$$\sim(\sim p \vee \sim q)$$

The Implicative Function is a propositional function with two arguments p and q, and is the proposition that either not-p or q is true, that is, it is the proposition $\sim p \vee q$. Thus if p is true, $\sim p$ is false, and accordingly the only alternative left by the proposition $\sim p \vee q$ is q is true. In other words if p and $\sim p \vee q$ are both true, then q is true. In this sense the proposition $\sim p \vee q$ will be quoted as stating that p implies q. The idea contained in this propositional function is so important that it requires a symbolism which with direct simplicity represents the propositions as connecting p and q without the intervention of $\sim p$. But "implies" as used here expresses nothing else than the connection between p and q also expressed by the disjunction "not-p or q." The symbol employed for "p implies q," *i.e.* for "$\sim p \vee q$," is "$p \supset q$." This symbol may also be read "if p, then q." [13, Vol. 1, pp. 6–7]

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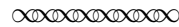
The notation used to denote the four fundamental functions of propositions changed considerably in the decades following the publication of *Principia*. Today the conjunction of the propositions p and q is written as $p \wedge q$ rather than $p.q$, and the symbol $\rightarrow$ is now used in place of Russell and Whitehead's $\supset$.[4] Contemporary logicians also refer to the four fundamental functions of propositions of *Principia Mathematica* simply as *logical connectives*. In what follows, we will adopt modern notation and terminology. We call propositions $p, q, r, \ldots$ *elementary propositions*, and propositions built from elementary propositions by means of logical connectives *compound propositions*. Statements of the form $p \rightarrow q$ are referred to as *conditional statements* or *conditionals*.

Remark 1. There is an apparent ambiguity in reading propositions like $\sim p \vee q$. The proposition can be read as either $(\sim p) \vee q$ (i.e., as the logical sum of $\sim p$ and q) or as $\sim(p \vee q)$ (i.e., as the result of applying the contradictory function to $p \vee q$). This ambiguity is easily resolved by the agreement that $\sim$ binds stronger than any of $\vee, \wedge, \rightarrow$. Thus, $\sim p \vee q$ is read as $(\sim p) \vee q$ rather than $\sim(p \vee q)$. Similarly, we agree that $\vee$ and $\wedge$ bind stronger than $\rightarrow$. For example, the proposition $p \vee q \rightarrow r$ should be read as $(p \vee q) \rightarrow r$, and the proposition $\sim p \wedge q \rightarrow r \vee p$ should be read as $((\sim p) \wedge q) \rightarrow (r \vee p)$.

Our first goal is to obtain a good understanding of propositions and of how the four logical connectives that Russell and Whitehead introduced yield more complex propositions out of simpler ones.

Exercise 2.1. Which of the following sentences are propositions?

(a) The New York Yankees have never won a World Series.

(b) 2 is even.

(c) Please close the door.

(d) The square root of 109.

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[4]Some authors also use $\Rightarrow$ in place of $\supset$.

(e) The sum of two even integers is even.

(f) What is the capital of France?

For the sentences that are propositions, determine whether they are true or false.

Exercise 2.2. Let p denote the proposition "All mammals have four legs," q denote the proposition "All dogs have four legs," and r denote the proposition "All dogs are mammals." Represent each of the following propositions using the four fundamental functions of propositions of *Principia Mathematica*.

(a) Not all dogs have four legs.

(b) All mammals have four legs and all dogs have four legs.

(c) Not all mammals have four legs or not all dogs have four legs.

(d) If all mammals have four legs and all dogs are mammals, then all dogs have four legs.

(e) If not all dogs have four legs, then not all mammals have four legs or not all dogs have four legs.

Which of the above are true and which are false?

Exercise 2.3. Let p denote the proposition "9 is odd," q denote the proposition "81 is the square of 9," and r denote the proposition "81 is odd." Write each of the following propositions verbally in words.

(a) $p \wedge q \rightarrow r$

(b) $q \wedge r \rightarrow p$

(c) $\sim r \rightarrow (\sim p \vee \sim q)$

(d) $p \vee \sim (q \wedge r)$

Determine which of the above are true and which are false.

Truth-values and truth tables

Having developed a language for discussing the logic of propositions, we turn to the task of understanding how the notion of truth relates to our symbolic logic of propositions. In particular, we develop a framework for understanding the truth and falsity of complex propositions based on the truth and falsity of simpler ones. *Principia Mathematica* addresses this issue:

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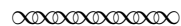
Truth-values. The "truth-value" of a proposition is *truth* if it is true, and *falsehood* if it is false. It will be observed that the truth-values of $p \vee q$, $p.q$, $p \supset q$, $\sim p$ (...) depend only on those of p and q, namely the truth-value of "$p \vee q$" is truth if the truth-value of either p or q is truth, and is falsehood otherwise; that of "$p.q$" is truth if that of both p and q is truth, and is falsehood otherwise; that of "$p \supset q$" is truth if either that of p is falsehood or that of q is truth; that of $\sim p$ is the opposite of that

of p (...). [13, Vol. 1, p. 8]

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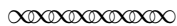
Remark 2. In everyday English, propositions of the form "p or q" are usually intended to mean that either p is true or q is true, but not both. For example, in ordinary English the "or"-statement "the car was red or blue" means that the car was either red or blue, but not both red and blue. The logical connective "or" in such statements is called an *exclusive "or"*. However, according to *Principia Mathematica*, the connective "∨" expresses a different sort of "or"—for "$p \vee q$" is true if at *least one of* p and q is true. The connective ∨ is called an *inclusive "or"* because the truth of "$p \vee q$" allows for the possibility that both p and q are true. In logic and mathematics, "or"-statements always employ the inclusive "or" rather than the exclusive "or".

Remark 3. The truth-values of the implication "$p \rightarrow q$" given in *Principia Mathematica* are somewhat different than one might expect. Russell and Whitehead indicate that "$p \rightarrow q$" is true if either p is false or q is true; so, in particular, they regard statements of the form "if p, then q" as true if p is false. This way of thinking about "if, then" statements may at first seem unusual, but one may understand the motivation for doing so if one thinks of "$p \rightarrow q$" as being analogous to a promise that if p holds true, then q will also hold true. If it so happens that p is not true, then the promise is unbroken regardless of the truth-value of q, and so "$p \rightarrow q$" is true. The promise is broken only if p holds true, but q happens to be false. This is the only situation in which we regard "$p \rightarrow q$" as being false. Note that an implication "if p, then q" which is true because p is false is referred to as *vacuously true*.

Exercise 2.4. If the truth-value of p is truth and the truth-values of q and r are falsehood, compute the truth-values of the following compound propositions.

(a) $(p \vee q) \vee r$

(b) $p \rightarrow (p \wedge q)$

(c) $\sim(p \wedge r) \rightarrow \sim q$

We now understand how to determine the truth-value of a compound proposition from the truth-values of the propositional variables of which it is composed. But we can do better than this. We can design a general method, which will allow us to calculate all possible truth-values of a compound proposition from all possible truth-values of the propositional variables of which it is composed. Such a method was devised independently by the great German philosopher Ludwig Wittgenstein (1889–1951) and the American logician Emil Post (1897–1954). The method employs *truth tables*, tables which list all the possible truth-values of the propositional variables contained in a compound proposition along with the corresponding truth-values of the compound proposition itself. For example, if p is a propositional variable, then the truth table of the compound proposition $\sim p$ is as follows:

p	$\sim p$
T	F
F	T

Here the possible truth-values for the propositional variable p are listed in the column on the left, and the corresponding truth-values of the proposition $\sim p$ are listed in the column on the right. Truth is denoted by T and falsehood is denoted by F.

Compound propositions which contain more propositional variables have more complicated truth tables. For example, if p and q are propositional variables, then the truth table for the proposition $p \vee q$ is:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

When three or more propositional variables occur in a compound proposition, care must be taken to guarantee that every possible combination of truth-values of the propositional variables appears in the truth table. For example, the truth table of the proposition $(p \wedge q) \to r$ looks as follows:

p	q	r	$p \wedge q$	$(p \wedge q) \to r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Notice that the truth-values of the propositional variables p, q, and r are listed systematically in this truth table: the truth-values for p are written in alternating groups of four (four Ts followed by four Fs), those for q are written in alternating groups of two, and those for r alternate with each successive entry. This method of organizing the entries in the truth table guarantees that every possible assignment of truth-values appears in the truth table. A similar organization may be used in constructing the truth table of a compound proposition that contains more than three propositional variables.

Notice also that the truth-values of the compound proposition $p \wedge q$ are listed in the truth table in order to allow us to compute the truth-values of the more complicated proposition $(p \wedge q) \to r$ more easily. When constructing the truth table of a complicated compound proposition, including truth-values for constituent compound propositions (like $p \wedge q$ in this example) is often helpful.

Exercise 2.5. Construct truth tables for each of the following propositions.

  (a) $p \wedge q$

  (b) $p \to q$

  (c) $(p \vee q) \to r$

  (d) $p \to (\sim q \wedge r)$

  (e) $(p \to (q \to r)) \to ((\sim p \vee \sim q) \to r)$

Logical equivalence and biconditionals

In some cases, different propositions are, in some sense, logically the same. For example, the propositions "9 is odd and 81 is the square of 9" and "81 is the square of 9 and 9 is odd" are somehow alike despite having different symbolic representations. More generally, if p and q are propositions, the propositions $p \wedge q$ and $q \wedge p$ apparently have the same meaning. This property of being logically alike, called *logical equivalence*, is one of the most important concepts in propositional logic. *Principia Mathematica* introduces logical equivalence as follows:
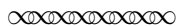
The simplest example of the formation of a more complex function of propositions by the use of these four fundamental forms is furnished by "equivalence." Two propositions p and q are said to be "equivalent" when p implies q and q implies p. This relation between p and q is denoted by "$p \equiv q$." Thus "$p \equiv q$" stands for "$(p \supset q).(q \supset p)$." It is easily seen that two propositions are equivalent when, and only when, they are both true or are both false. [13, Vol. 1, p. 7]
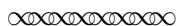
Here Russell and Whitehead have defined the notion of logical equivalence, but they have also introduced a new logical connective. In modern terms, this new logical connective is written using the symbol $\leftrightarrow$ and the proposition "$p \leftrightarrow q$" is taken to stand for "$(p \rightarrow q) \wedge (q \rightarrow p)$". This new connective is known as the *biconditional*, and statements of the form "$p \leftrightarrow q$" are called *biconditional statements*.

It is very important to distinguish between the biconditional statement "$p \leftrightarrow q$" and the logical equivalence of p and q. As the basic logical connectives $\sim$, $\vee$, $\wedge$, and $\rightarrow$, the biconditional $\leftrightarrow$ is also a logical connective. On the other hand, two propositions p and q are *logically equivalent* when p and q have the same truth-values (both are true or both are false). We will see later that two propositions p and q are logically equivalent if and only if the biconditional $p \leftrightarrow q$ is always true, i.e. the biconditional is true irrespective of the truth of p or q.

How can we determine whether two given propositions are logically equivalent? In order to do so, we must be able to guarantee that the propositions have the same truth-value irrespective of the truth-values of the elementary propositions of which they are composed. At first this may seem like a daunting task. However, the use of truth tables makes it a simple matter to determine whether two propositions are logically equivalent. In fact, since the truth tables for the propositions p and q list the truth-values for p and q under all possible assignments of truth-values for the elementary propositions of which they are built, p and q are logically equivalent if and only if they have the same truth tables.

An example may better illustrate this point. We shall use truth tables to establish the logical equivalence of the propositions $(p \vee q) \wedge r$ and $(p \wedge r) \vee (q \wedge r)$:

p	q	r	$(p \vee q) \wedge r$	$(p \wedge r) \vee (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

As the two propositions have identical truth tables, they are logically equivalent.

Exercise 2.6. Use truth tables to show that:

(a) $\sim(p \wedge q)$ is logically equivalent to $\sim p \vee \sim q$.

(b) $\sim(p \vee q)$ is logically equivalent to $\sim p \wedge \sim q$

These equivalences are called *De Morgan's Laws* in honor of Augustus De Morgan.[5]

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[5]For more information on De Morgan's life and work, see, e.g., [1, 7].

Exercise 2.7.

(a) Are $p \to q$ and $\sim q \to \sim p$ logically equivalent? Justify your answer.

(b) Are $p \to q$ and $q \to p$ logically equivalent? Justify your answer.

(c) Are $p \to q$ and $\sim p \to \sim q$ logically equivalent? Justify your answer.

(d) The proposition $\sim q \to \sim p$ is often called the *contrapositive* of the conditional $p \to q$, the proposition $q \to p$ is often called the *converse* of the conditional $p \to q$, and the proposition $\sim p \to \sim q$ is often called the *inverse* of the conditional $p \to q$. Are there any logical equivalences between a conditional statement, its contrapositive, its converse, and its inverse? Justify your answer.

Now that we have introduced the notion of logical equivalence, we will discuss Russell and Whitehead's claim that the four logical connectives are "not all independent; and if two of them are taken as primitive undefined ideas, the other two can be defined in terms of them" [13, Vol. 1, p. 6]. For example, both $\to$ and $\wedge$ can be expressed by means of $\sim$ and $\vee$; namely, $p \to q$ is a shorthand for $\sim p \vee q$ and $p \wedge q$ is a shorthand for $\sim(\sim p \vee \sim q)$.

Exercise 2.8.

(a) Show that $p \vee q$ and $p \to q$ are logically equivalent to propositions only involving $\sim$ and $\wedge$.

(b) Show that $p \vee q$ and $p \wedge q$ are logically equivalent to propositions only involving $\sim$ and $\to$.

(c) What do these results indicate about the logical connectives introduced in *Principia*? Are some connectives redundant? If so, which ones?

Tautologies and contradictions

We have seen that truth tables are a useful tool for determining if two propositions are logically equivalent, but it turns out that truth tables have a variety of other uses. In what follows we will explore two other applications of truth tables. The first of these applications is the identification of certain special propositions. To illustrate what these propositions are and how truth tables can be used to identify them, we consider the truth table of the proposition $p \vee \sim p$:

p	$p \vee \sim p$
T	T
F	T

Notice that the truth-values displayed in the righthand column are always truth; falsehood does not appear. Thus, $p \vee \sim p$ is true under any assignment of truth or falsehood to the proposition p. Propositions with this property—that is, propositions which are true for any assignment of truth-values to the propositional variables of which they are formed—are called *tautologies*. From the truth table above, we can easily see that $p \vee \sim p$ is a tautology. The proposition $p \vee \sim p$ is referred to as *the law of the excluded middle* since it asserts that either p is true or the negation of p is true (so there is no "middle ground" between truth and falsity). In general, one may test to see if a proposition is a tautology by constructing its truth table: if the only possible truth-value of the proposition is truth, then the proposition is a tautology.

Tautologies are special propositions that can be identified by the use of truth tables. Another sort of special propositions that can be identified in the same way are *contradictions*. Whereas

tautologies are propositions which are true under any assignment of truth-values to the propositional variables of which they are formed, contradictions are propositions that are false under any assignment of truth-values to the propositional variables of which they are formed. For example, the proposition $p \wedge \sim p$ is a contradiction. This is illustrated in the following truth table:

p	$p \wedge \sim p$
T	F
F	F

As the only truth-value displayed in the righthand column is falsehood, the proposition $p \wedge \sim p$ is a contradiction.

Exercise 2.9. For each of the following propositions, use a truth table to determine whether the proposition is a tautology, a contradiction, or neither.

(a) $\sim(p \wedge \sim p)$

(b) $p \leftrightarrow \sim(\sim p)$

(c) $p \leftrightarrow \sim p$

(d) $p \vee (q \rightarrow p)$

(e) $(p \vee q) \rightarrow (q \vee p)$

(f) $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

(g) $(p \wedge q) \wedge (\sim p \vee \sim q)$

(h) $p \rightarrow (q \rightarrow p)$

The logical laws expressed in (a) and (b) are called *the law of non-contradiction* and *the law of double negation*, respectively. For any proposition p, the law of non-contradiction asserts that p and $\sim p$ are not both true. We also saw that the law of the excluded middle asserts that at least one of p and $\sim p$ is true. Hence, when taken together, the law of non-contradiction and the law of the excluded middle guarantee that exactly one of p and $\sim p$ is true for each proposition p.

Now, as promised when we defined logical equivalence, we are ready to see that two propositions p and q are logically equivalent if and only if the biconditional $p \leftrightarrow q$ is a tautology.

Exercise 2.10. Show that p and q are logically equivalent if and only if $p \leftrightarrow q$ is a tautology.

It is often helpful to be able to recognize commonly occurring logical equivalences at a glance. For convenience, we summarize some of the most frequently appearing logical equivalences in the table below, where p and q denote arbitrary propositions, $\mathbf{t}$ denotes an arbitrary tautology, and $\mathbf{c}$ denotes an arbitrary contradiction. Following contemporary usage, we write $p \equiv q$ when p and q are logically equivalent. This should not be confused with the biconditional $p \leftrightarrow q$.

Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Associative laws:	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Absorption laws:	$p \wedge (p \vee q) \equiv p$	$p \vee (p \wedge q) \equiv p$
Identity laws:	$p \vee \mathbf{c} \equiv p$	$p \wedge \mathbf{t} \equiv p$
Universal bound laws:	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
De Morgan laws:	$\sim(p \vee q) \equiv \sim p \wedge \sim q$	$\sim(p \wedge q) \equiv \sim p \vee \sim q$
Negation laws:	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
Negations of $\mathbf{t}$ and $\mathbf{c}$:	$\sim\mathbf{t} \equiv \mathbf{c}$	$\sim\mathbf{c} \equiv \mathbf{t}$
Double negation law:	$\sim\sim p \equiv p$	
Other equivalences:	$p \to q \equiv \sim p \vee q$	$\sim(p \to q) \equiv p \wedge \sim q$
	$p \to q \equiv \sim q \to \sim p$	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Exercise 2.11. Use the equivalences listed in the table above to write each of the following propositions as a simpler, logically equivalent proposition.

(a) $p \wedge (p \vee (q \vee p))$

(b) $p \vee \sim(p \vee \sim p)$

(c) $\sim(\sim(p \vee \sim q) \wedge r)$

(d) $(p \wedge \sim p) \to (p \vee (q \vee p))$

Exercise 2.12. Use the equivalences listed in the table above to determine which of the following logical equivalences hold. Justify your answers, but do not use a truth table.

(a) $(p \vee (q \vee r)) \wedge \sim(p \vee (q \vee r)) \equiv q \wedge \sim q$

(b) $(p \wedge r) \vee ((\sim p \to \sim q) \to (q \to p)) \equiv p \wedge r$

(c) $\sim(p \to q) \leftrightarrow (p \wedge \sim q) \equiv p \to (q \to p)$

Inference rules

Up until now we have been concerned exclusively with propositions and their properties. However, the central concern of logic is not just the study of propositions. Rather the object of study in logic is *inference*—the process of drawing correct conclusions from premises. Our study of inference begins with a simple rule which allows us to deduce the proposition q from the propositions p and $p \to q$. This rule is referred to as *Modus Ponens*. *Principia Mathematica* describes the rule as follows:

∞∞∞∞∞∞∞∞∞∞
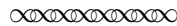
Inference. The process of inference is as follows: a proposition "p" is asserted, and a proposition "p implies q" is asserted, and then as a sequel the proposition "q" is asserted. The trust in inference is the belief that if the two former assertions are not in error, the final assertion is not in error. [13, Vol. 1, p. 9]

∞∞∞∞∞∞∞∞∞∞
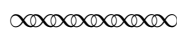
Russell and Whitehead claim that Modus Ponens is a *valid* logical rule; that is, if the premises p and $p \to q$ are both true, then the conclusion q is guaranteed to be true. While this fact is intuitively obvious, a skeptical reader may wonder how we really know this is the case. Fortunately, the skeptic's concerns may easily be allayed: we can verify that Modus Ponens is valid. For this we need to be able to guarantee the truth of q based on the truth of the propositions p and $p \to q$. We thus need only to verify that q is true in every circumstance in which both p and $p \to q$ are true. To do so, we construct a truth table listing the possible truth-values of the propositions p, q, and $p \to q$:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The only row of the truth table in which both p and $p \to q$ are true is the first. Since in this case q is also true, we know that if p and $p \to q$ are both true, then q must be true as well. Thus, Modus Ponens is a valid logical rule.

So far we have focused exclusively on Modus Ponens, but it is worth noting that we might just as well have discussed any one of several valid logical rules. There are many valid logical rules (called *rules of inference*), and Modus Ponens is distinguished only by being the simplest of these. In the exercises, we will encounter several other historically important rules of inference and establish their validity.

Exercise 2.13. Use truth tables to verify the validity of the following rule of inference: if $\sim q$ and $p \to q$, then infer that $\sim p$. (This rule of inference is referred to as *Modus Tollens*.)

Exercise 2.14. Use truth tables to verify the validity of the following rule of inference: if $\sim p$ and $p \vee q$, then infer that q. (This rule of inference is referred to as the *disjunctive syllogism*.)

Exercise 2.15. Use truth tables to verify the validity of the following rule of inference: if $p \to q$ and $q \to r$, then infer that $p \to r$. (This rule of inference is referred to as the *hypothetical syllogism*.)

Exercise 2.16. Consider the following rule of inference: if $p \to q$ and q, then infer that p. An application of this rule of inference is referred to as *affirming the consequent*. Is affirming the consequent a valid rule of inference? If so, use truth tables to establish its validity. If not, give examples of propositions p and q for which $p \to q$ and q are true and p is not.

Exercise 2.17. Late Saturday night the police are called to Crickwell Manor. That evening Mr. and Mrs. Crickwell had been holding a dinner party, and at the party Mr. Crickwell had been murdered. Other than the deceased, there had been four people in attendance: Ms. Anderson, Mr. Boalt, Mrs. Crickwell, and Mr. Dunham. The police interview each of these witnesses, and they offer the following statements:

(a) Ms. Anderson says that the murderer must be Mr. Boalt or Mr. Dunham.

(b) Mr. Boalt says that neither Mr. Dunham nor Mrs. Crickwell could have killed Mr. Crickwell.

(c) Mrs. Crickwell says that she certainly did not kill her husband, and that if Mr. Boalt were not the murderer then it must have been Ms. Anderson.

(d) Mr. Dunham says that he himself had far more motive to kill Mr. Crickwell than did Mr. Boalt, so if he did not kill Mr. Crickwell then neither did Mr. Boalt.

Assuming that only one of the people attending the party killed Mr. Crickwell and that everyone except the murderer was truthful with the police, who killed Mr. Crickwell?

3 Predicate Logic

While the propositional logic developed in the previous section allows us to address a number of significant issues in logic, it turns out that it is not capable of answering all of the logical questions which are important to mathematicians. For example, recall that a positive integer a is a prime number if $a \neq 1$ and the only divisors of a are 1 and a. A mathematician may be interested whether the proposition "every prime number greater than 2 is odd" is true. Propositional logic is not very helpful in this regard because it is not possible to view this proposition as being composed from several elementary propositions by logical connectives. Thus, from the perspective of propositional logic it must itself be an elementary proposition. However, if "every prime number greater than 2 is odd" is an elementary proposition, then propositional logic does not provide us with any information about when this proposition is true—the propositional logic developed in the previous section says only that elementary propositions are either true or false. In order to provide a more informative analysis of the proposition "every prime number greater than 2 is odd," we must introduce ideas which allow us to discuss the internal structure of propositions that we previously regarded as elementary.

Individual variables and predicates

We start by introducing new kind of variables, called *individual variables* to distinguish them from the propositional variables of the previous section. Individual variables range over objects in some domain we are presently concerned with rather than over propositions. For example, the domain for the proposition "every prime number greater than 2 is odd" should range over all positive integers. We will use the letters x, y, z and so forth to denote individual variables.

With the notion of an individual variable at hand, we can now give an account of how "every prime number greater than 2 is odd" may be analyzed in terms of simpler propositions. Suppose that x is an individual variable, and that we take the expression $P(x)$ to mean that "x is a prime number greater than 2" and the expression $O(x)$ to mean that "x is odd." Then it is clear that the proposition "for every x, $P(x) \to O(x)$" has the same meaning as "every prime number greater than 2 is odd." Here the domain the individual variable x ranges over is all positive integers. Thus, if we can formalize the meanings of $P(x)$, $O(x)$, and "for every x," then we can give an informative analysis of the proposition "every prime number greater than 2 is odd."

We will first formalize the meanings of the expressions $P(x)$ and $O(x)$. These expressions are examples of what Russell and Whitehead referred to as "propositional functions," which *Principia Mathematica* describes as follows:

<center>∞∞∞∞∞∞∞∞∞∞</center>
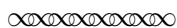
Propositional Functions. Let ϕx be a statement containing a variable x and such that it becomes a proposition when x is given any fixed determined meaning. Then ϕx is called a "propositional function"; it is not a proposition, since owing to the ambiguity of x it really makes no assertion at all. Thus "x is hurt" really makes no assertion at all, till we have settled who x is. Yet owing to the individuality retained by the ambiguous variable x, it is an ambiguous example from the collection of propositions arrived at by giving all possible determinations to x in "x is hurt" which yield a proposition, true or false. Also if "x is hurt" and "y is hurt" occur *in the same context*, where y is another variable, then according to the determinations given to x and y, they can be settled to be (possibly) the same propositions or (possibly) different propositions. But apart from some determination given to x and y, they retain in that context their ambiguous differentiation. Thus "x is hurt" is an ambiguous "value" of a propositio-

⚭⚭⚭⚭⚭⚭⚭⚭⚭
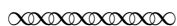
Today it is more common to denote the expression ϕx by $\phi(x)$. As the statements "x is a prime number greater than 2" and "x is odd" become propositions when x is assigned a particular integer value, $P(x)$ and $O(x)$ as described above are propositional functions. Today the propositional functions of Russell and Whitehead are referred to as *predicates*.

Let $\phi(x)$ be a predicate and D be some specified domain. Then the collection of objects a in D for which $\phi(a)$ is true is called the *truth set* of the predicate $\phi(x)$ with respect to the domain D. In other words, the truth set of $\phi(x)$ with respect to the domain D is the collection of those objects in D that make the predicate true.

We denote by $\mathbb{N}$ the collection of nonnegative integers

$$0, 1, 2, \ldots$$

As usual, we will refer to them as *natural numbers*. We also denote the collection of integers

$$\ldots, -2, -1, 0, 1, 2, \ldots$$

by $\mathbb{Z}$, the collection of rational numbers (ratios of two integers, where the denominator is nonzero) by $\mathbb{Q}$, and the collection of real numbers by $\mathbb{R}$. The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are among the most common domains of the predicates employed in mathematics. For example, it is most natural to consider the predicates $P(x)$ and $O(x)$ discussed above as having the domain $\mathbb{N}$. On the other hand, if one were considering the predicate $Q(x)$ defined by "x is greater than π," then one is most likely thinking of $\mathbb{R}$ as the domain of $Q(x)$. However, note that if instead we consider $\mathbb{Z}$ to be the domain of $Q(x)$, then we have specified a much different predicate than if we consider the domain to be $\mathbb{R}$. For example, if we consider $\mathbb{R}$ to be the domain of $Q(x)$, then propositions such as $Q(\pi + 1)$ and $Q(4\pi)$ are true (because $\pi + 1$ and 4π are both real numbers greater than π). However, if the domain of $Q(x)$ is $\mathbb{Z}$, then $Q(\pi + 1)$ and $Q(4\pi)$ do not even make sense because $\pi + 1$ and 4π are not integers. From this it is clear that in order to completely specify a predicate we must specify its domain. The domains for the predicates we will discuss will usually be clear from the context.

Exercise 3.1. As above, let $P(x)$ be the predicate "x is a prime number greater than 2" and $O(x)$ be the predicate "x is odd." State each of the following propositions verbally in words. Also determine whether each proposition is true or false.

(a) $P(4) \vee O(7)$

(b) $(P(3) \wedge P(13)) \wedge O(12)$

(c) $P(2) \rightarrow P(23)$

Exercise 3.2. Identify the truth sets of the following predicates.

(a) $A(x)$ defined by "x is a real number satisfying the equation $x^2 - 5x + 6 = 0$."

(b) $B(x)$ defined by "x is a rational number satisfying the inequality $x^2 \leq 1$."

Exercise 3.3. For each of the following sets, give an example of a predicate with the specified domain.

(a) $\mathbb{Z}$

(b) The set of all letters in the English alphabet.

(c) The set of all continuous functions defined on the real numbers.

(d) The set of all college students in the United States.

Universal and existential quantifiers

Now that we have introduced predicates, we are almost able to analyze the proposition "every prime number greater than 2 is odd." Recall that if we let $P(x)$ be the predicate "x is a prime number greater than 2" and $O(x)$ be the predicate "x is odd," then the proposition "every prime number greater than 2 is odd" can be expressed as "for every x, $P(x) \rightarrow O(x)$." In order to fully understand the latter proposition, we only need to clarify what it means for $P(x) \rightarrow Q(x)$ to hold "for every x." To do so, we will introduce the notion of a *quantifier*, a logical operator that specifies whether a predicate $\phi(x)$ holds for all values of x or for some value of x. In *Principia Mathematica*, a predicate quantified in the former way is written as $(x).\phi(x)$ and a predicate quantified in the latter way is written as $(\exists x).\phi(x)$:

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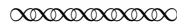
Thus corresponding to any propositional function...there is a range, or collection, of values, consisting of all the propositions (true or false) which can be obtained by giving every possible determination to x in ϕx...in respect to the truth or falsehood of propositions of this range three important cases must be noted and symbolised. These cases are given by three propositions of which one at least must be true. Either (1) all propositions of the range are true, or (2) some propositions of the range are true, or (3) no proposition of the range is true. The statement (1) is symbolised by "$(x).\phi x$," and (2) is symbolised by "$(\exists x).\phi x$." No definition is given of these two symbols, which accordingly embody two new primitive ideas in our system. The symbol "$(x).\phi x$" may be read "ϕx always," or "ϕx is always true," or "ϕx is true for all possible values of x." The symbol "$(\exists x).\phi x$" may be read "there exists an x for which ϕx is true"...and thus conforms to the natural form of the expression of thought. [13, Vol. 1, pp. 15–16]

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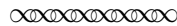
In the rest of the project we will follow the common practice of denoting $(x).\phi x$ by $(\forall x)\phi(x)$ and $(\exists x).\phi x$ by $(\exists x)\phi(x)$. With quantifiers, we may now see that "every prime number greater than 2 is odd" may be represented symbolically as "$(\forall x)(P(x) \rightarrow O(x))$," where $P(x)$ is the predicate "x is a prime number greater than 2" and $O(x)$ is the predicate "x is odd."

When we refer to propositions of the form $(\forall x)\phi(x)$ and $(\exists x)\phi(x)$, we will always have in mind some fixed domain of values over which x may vary. This domain is sometimes called a *domain of discourse*. Then $(\forall x)\phi(x)$ means that $\phi(x)$ holds for each value of the individual x in our domain of discourse D. In other words, $(\forall x)\phi(x)$ asserts that, for each a in D, the proposition $\phi(a)$ holds. Thus, the proposition $(\forall x)\phi(x)$ is true if, for each a in D, the proposition $\phi(a)$ is true. On the other hand, $(\forall x)\phi(x)$ is false if this condition fails to hold; that is, if for some a in D, the proposition $\phi(a)$ is false.

Similarly, $(\exists x)\phi(x)$ means that there exists a value of x in our domain of discourse D for which $\phi(x)$ holds. Thus, $(\exists x)\phi(x)$ asserts that there is some a in D such that $\phi(a)$ holds. From this it is clear that $(\exists x)\phi(x)$ is true if there is at least one a in D such that $\phi(a)$ holds, and that $(\exists x)\phi(x)$ is false if $\phi(a)$ happens to be false for every a in D.

This indicates that there is a close connection between $(\forall x)$ and $(\exists x)$. We will discuss this in greater detail later in the project.

The symbol $(\forall x)$ is referred to as a *universal quantifier* since $(\forall x)\phi(x)$ asserts that $\phi(x)$ holds universally, i.e., for all values of x, and the symbol $(\exists x)$ is referred to as an *existential quantifier* since $(\exists x)\phi(x)$ asserts that there exists a value of x for which $\phi(x)$ holds.

Exercise 3.4. Let $E(x)$ be the predicate "x is an even natural number" and $O(x)$ be the predicate "x is an odd natural number." Express each of the following statements symbolically using quantifiers and the predicates $E(x)$ and $O(x)$.

 (a) There is at least one even natural number.

 (b) Every natural number is either even or odd.

 (c) Some natural number is both even and odd.

 (d) No natural number is both even and odd.

 (e) Every natural number that is not even is odd.

 (f) The square of every even natural number is even.

Which of the above statements are true and which are false? Explain why.

Exercise 3.5. Let $R(x)$ denote the predicate "x is a real number," $Z(x)$ denote the predicate "x is an integer," $O(x)$ denote the predicate "x is an odd integer," and $P(x)$ denote the predicate "x is a prime number." Translate each of the following statements into English.

 (a) $(\forall x)(Z(x) \rightarrow R(x))$

 (b) $(\forall x)(P(x) \rightarrow O(x))$

 (c) $(\exists x)(P(x) \wedge R(x))$

Which of the above statements are true and which are false? Explain why.

Exercise 3.6. For each of the following statements, define appropriate predicates and write the statement using predicates and quantifiers.

 (a) Every square is a rectangle.

 (b) Every real number is either positive, negative, or equal to zero.

 (c) Every animal that has a heart has kidneys, and every animal that has kidneys has a heart.

 (d) Some natural number is not a prime number.

Unary and binary predicates

Our logical analysis of the proposition "every prime number greater than 2 is odd" relied on our ability to attribute certain properties to an individual variable x, and the introduction of predicates was what allowed us to accomplish this. Predicates are very versatile: they can provide symbolic representations for a multitude of interesting properties. For example, such properties as "being a rectangle," "being a prime number," "being greater than 2" are represented by the predicates "x is a rectangle," "x is a prime number," and "x is greater than 2," respectively. However, not every property that is of mathematical interest can be readily expressed using such predicates. For

example, one may wish to express the fact that there exists a natural number n such that $n \leq m$ holds for all natural numbers m (i.e., the fact that the natural numbers have a least member).

The most obvious way to formalize this statement is to introduce a symbol $P(x, y)$ that expresses "x and y are natural numbers such that $x \leq y$." We may then express the existence of a least natural number by the proposition "$(\exists x)(\forall y)P(x, y)$." In this expression, the role of the symbol $P(x, y)$ is similar to the role that predicates played in our previous discussion. Indeed, $P(x, y)$ is a sort of predicate, but, unlike the predicates previously discussed, $P(x, y)$ accepts two variables as arguments rather than one. Such a predicate is referred to as a *binary predicate*. Those predicates that attribute a property to only a single variable (such as those discussed previously) are referred to as *unary predicates*.

Exercise 3.7. Let $N(x)$ denote the unary predicate "x is a natural number," $Z(x)$ denote the unary predicate "x is an integer," and $P(x, y)$ denote the binary predicate "$x \leq y$." Represent each of the following propositions symbolically.

(a) There is a least natural number.

(b) There is a greatest natural number.

(c) There is a least integer.

(d) There is no greatest integer.

Which of the above propositions are true? Explain why.

Exercise 3.8. Let $I(x, y)$ denote the binary predicate "the point x lies on the line y" (if a point x lies on a line y then x is said to be *incident* to y). Write each of the following propositions verbally in words.

(a) $(\forall x)(\forall y)(\exists z)(I(x, z) \land I(y, z))$

(b) $(\forall x)(\exists y)(I(y, x))$

What is the mathematical meaning of these two statements? Are they true or false? Explain why.

Exercise 3.9. Allowing the variables x and y to range over the domain of all lines in the Cartesian plane, let $P(x, y)$ denote the binary predicate "x is parallel to y." Write each of the following propositions verbally in words.

(a) $(\forall x)(P(x, x))$

(b) $(\forall x)(\forall y)(P(x, y) \rightarrow P(y, x))$

(c) $(\forall x)(\forall y)(\forall z)((P(x, y) \land P(y, z)) \rightarrow P(x, z))$

When the three propositions above hold for a binary predicate $P(x, y)$, then the relationship defined between x and y by the predicate $P(x, y)$ is said to be an *equivalence relation*. Is "x is parallel to y" an equivalence relation?

Exercise 3.10. Recall that a natural number x is a multiple of a natural number y if there exists a natural number n such that $x = ny$. Allowing the variables x and y to range over the domain of all natural numbers, let $P(x, y)$ denote the binary predicate "y is a multiple of x." Is $P(x, y)$ an equivalence relation? Justify your answer.

Exercise 3.11. For natural numbers x and y, define $x \bmod y$ to be the remainder obtained upon dividing x by y. Allowing the variables x and y to range over the domain of natural numbers, let $P(x, y)$ denote the binary predicate "$x \bmod 2 = y \bmod 2$" and let $Q(x, y)$ denote the binary predicate "$x \bmod 3 = y \bmod 3$."

(a) Is $P(x, y)$ an equivalence relation? Justify your answer.

(b) Is $Q(x, y)$ an equivalence relation? Justify your answer.

(c) Fix a natural number n.

    1. If $P(n, 2)$ holds, what must be true of n?

    2. What about if $P(n, 1)$ holds?

    3. For a fixed natural number n, must at least one of $P(n, 2)$ or $P(n, 1)$ hold?

    4. Can both $P(n, 2)$ and $P(n, 1)$ hold for a single natural number n?

(d)   1. List the natural numbers n for which $Q(n, 1)$ holds, list the natural numbers n for which $Q(n, 2)$ holds, and list the natural numbers n for which $Q(n, 3)$ holds.

    2. Do the three lists have any natural numbers in common? Does every natural number appear in at least one of the three lists?

    3. For each of the lists, the collection of numbers appearing in that list is said to be an *equivalence class* of the equivalence relation $Q(x, y)$. What are the equivalence classes of the equivalence relation $P(x, y)$?

Logical equivalence of quantified statements

Precisely characterizing when two quantified statements are logically equivalent turns out to be rather technical, and a thorough treatment of that topic is beyond the scope of this project. Nevertheless, from the comments above one can see that the proposition $(\forall x)\phi(x)$ is true exactly when there is no value of x in our domain of discourse for which $\phi(x)$ is false. That is, $(\forall x)\phi(x)$ is true exactly when $\sim(\exists x)\sim\phi(x)$ is true. One may likewise note that $(\exists x)\phi(x)$ is true exactly when $\phi(x)$ fails to be false for all x. In other words, $(\exists x)\phi(x)$ is true exactly when $\sim(\forall x)\sim\phi(x)$ is true. As one may expect from this discussion, it so happens that $(\forall x)\phi(x)$ is logically equivalent to $\sim(\exists x)\sim\phi(x)$ and that $(\exists x)\phi(x)$ is logically equivalent to $\sim(\forall x)\sim\phi(x)$.

Exercise 3.12. Explain in your own words why

$$(\forall x)\phi(x) \equiv \sim(\exists x)\sim\phi(x)$$

and

$$(\exists x)\phi(x) \equiv \sim(\forall x)\sim\phi(x).$$

One may derive many useful facts about the logical equivalence of quantified statements from these two laws.

Exercise 3.13. Use the laws $(\forall x)\phi(x) \equiv \sim(\exists x)\sim\phi(x)$ and $(\exists x)\phi(x) \equiv \sim(\forall x)\sim\phi(x)$ to rewrite each of the following statements. Make sure that in your solution all quantifiers precede any instances of negation.

(a) $\sim(\exists x)(\forall y)(x < y)$

(b) $\sim(\forall x)(\exists y)(x < y)$

Which of the above two statements is true? Explain why.

Exercise 3.14. Allowing x and y to range over the domain $\mathbb{N}$, let $E(x)$ denote the unary predicate "x is even," $O(x)$ denote the unary predicate "x is odd," and $L(x, y)$ denote the binary predicate "$x < y$."

(a) Express the proposition "Every odd natural number is less than some even natural number" using quantifiers and the predicates $E(x)$, $O(x)$, and $L(x, y)$.

(b) Use the laws $(\forall x)\phi(x) \equiv \sim(\exists x)\sim\phi(x)$ and $(\exists x)\phi(x) \equiv \sim(\forall x)\sim\phi(x)$ to find the negation of the proposition obtained in (a). Make sure that in your solution all quantifiers precede any instances of negation.

(c) Which of the two propositions is true, the one obtained in (a) or its negation? Justify your answer.

The laws $(\forall x)\phi(x) \equiv \sim(\exists x)\sim\phi(x)$ and $(\exists x)\phi(x) \equiv \sim(\forall x)\sim\phi(x)$ are of central importance in studying the logical equivalence of quantified statements, but a number of other simple equivalences are also of tremendous importance. For example, it is readily seen that if $P(x, y)$ is a binary predicate, then $(\forall x)(\forall y)P(x, y)$ is logically equivalent to $(\forall y)(\forall x)P(x, y)$ and that $(\exists x)(\exists y)P(x, y)$ is logically equivalent to $(\exists y)(\exists x)P(x, y)$. We describe these simple equivalences by saying that universal quantifiers (respectively, existential quantifiers) *commute* with one another.

A far more interesting question is whether universal and existential quantifiers commute with each other, i.e., whether statements of the forms $(\forall x)(\exists y)P(x, y)$ and $(\exists y)(\forall x)P(x, y)$ are in general logically equivalent. The next three exercises explore this question.

Exercise 3.15. Allowing the variables x and y to range over the domain of all people, let $L(x, y)$ denote the binary predicate "x likes y."

(a) Translate the two statements $(\forall x)(\exists y)L(x, y)$ and $(\exists y)(\forall x)L(x, y)$ into English.

(b) In your opinion which of the above two statements is true and which is false?

(c) Do the above two statements have the same meaning?

(d) Are the two statements $(\forall x)(\exists y)L(x, y)$ and $(\exists y)(\forall x)L(x, y)$ logically equivalent? Justify your answer.

(e) What can you conclude about the relationship between universal quantifiers and existential quantifiers from the above example?

Exercise 3.16. Allowing the variables x and y to range over the domain $\mathbb{R}$, let $E(x, y)$ denote the binary predicate "$x = y$." Write each of the following statements verbally in words.

(a) $(\forall x)(\exists y)E(x, y)$

(b) $(\exists x)(\forall y)E(x, y)$

Which of the above propositions are true? Are the propositions $(\forall x)(\exists y)E(x, y)$ and $(\exists x)(\forall y)E(x, y)$ logically equivalent?

Exercise 3.17. Give your own example of a binary predicate $P(x, y)$ such that $(\forall x)(\exists y)P(x, y)$ and $(\exists x)(\forall y)P(x, y)$ are not logically equivalent. Justify your example.

We conclude our discussion of the logical equivalence of quantified statements by discussing several equivalences that hold between quantified conditional statements. We have seen that in propositional logic a conditional statement $p \rightarrow q$ is logically equivalent to its contrapositive $\sim q \rightarrow \sim p$, and that the converse $q \rightarrow p$ of this conditional is equivalent to its inverse $\sim p \rightarrow \sim q$. It turns out that similar equivalence laws hold for quantified statements.

Let $\phi(x)$ and $\psi(x)$ be predicates. We call propositions of the form $(\forall x)(\phi(x) \rightarrow \psi(x))$ *universal conditional statements*. If $(\forall x)(\phi(x) \rightarrow \psi(x))$ is a universal conditional statement, we define its *contrapositive* to be the statement $(\forall x)(\sim\psi(x) \rightarrow \sim\phi(x))$, its *converse* to be the statement $(\forall x)(\psi(x) \rightarrow \phi(x))$, and its *inverse* to be the statement $(\forall x)(\sim\phi(x) \rightarrow \sim\psi(x))$. Just as the conditional studied in propositional logic is logically equivalent to the contrapositive, so too is the universal conditional $(\forall x)(\phi(x) \rightarrow \psi(x))$ logically equivalent to its contrapositive $(\forall x)(\sim\psi(x) \rightarrow \sim\phi(x))$. Similarly, the converse $(\forall x)(\psi(x) \rightarrow \phi(x))$ is logically equivalent to the inverse $(\forall x)(\sim\phi(x) \rightarrow \sim\psi(x))$.

Exercise 3.18. Let $\phi(x)$ and $\psi(x)$ be unary predicates, and let $\chi(x, y)$ be a binary predicate. Determine which of the following logical equivalences hold. Justify your answers.

(a) $(\exists x)(\phi(x) \wedge \psi(x)) \equiv (\exists x)\phi(x) \wedge (\exists x)\psi(x)$

(b) $(\forall x)(\phi(x) \rightarrow (\exists y)\chi(x, y)) \equiv \sim(\exists x)(\phi(x) \wedge (\forall y)\sim\chi(x, y))$

(c) $(\exists x)(\exists y)\chi(x, y) \rightarrow (\exists y)(\exists x)\chi(x, y) \equiv (\forall y)(\forall x)\chi(x, y) \rightarrow (\forall x)(\forall y)\chi(x, y)$

Inference rules in predicate logic

Our development of individual variables, predicates, and quantifiers has provided us with a language for discussing propositions that is much more expressive than the propositional language discussed in the previous section. In particular, predicate logic is capable of addressing questions about complex propositions such as "every prime number greater than 2 is odd" that the propositional logic of the previous section cannot. However, we have not yet discussed the issue of logical inference in predicate logic. This is the topic to which we now turn.

Recall that if p and q are propositions, then one can deduce the proposition q from the propositions p and $p \rightarrow q$. We called this rule of inference Modus Ponens, and in the previous section we proved that Modus Ponens is a valid rule of inference (i.e., if p and $p \rightarrow q$ are true, then the deduced proposition q must also be true). As it turns out, there is an analogous rule of inference for predicate logic.

Suppose that we fix some domain D over which our individual variables may range, that a is some element in D, and that $P(x)$ and $Q(x)$ are predicates. Then if both $(\forall x)(P(x) \rightarrow Q(x))$ and $P(a)$ hold, we may deduce that $Q(a)$ holds as well. This gives a valid rule of inference, which we call *Universal Modus Ponens*.

A mathematically rigorous verification of the validity of Universal Modus Ponens requires an analogue of truth tables for predicate logic, and is beyond the scope of this project. However, we can give an informal argument for the validity of Universal Modus Ponens as follows.

Suppose that both $(\forall x)(P(x) \rightarrow Q(x))$ and $P(a)$ hold; we must show that $Q(a)$ holds as well. Since $(\forall x)(P(x) \rightarrow Q(x))$ holds, we have from the definition of the universal quantifier that the proposition $P(b) \rightarrow Q(b)$ holds for each member b of our specified domain D. In particular, as a is an element of D, we have that $P(a) \rightarrow Q(a)$ holds. Now notice that both $P(a) \rightarrow Q(a)$ and $P(a)$ are propositions. Since Modus Ponens is a valid rule of inference of propositional logic, we have that as $P(a) \rightarrow Q(a)$ and $P(a)$ hold, it follows that $Q(a)$ holds as well, completing our argument.

Exercise 3.19. Fix some domain D over which we allow individual variables to range, suppose a is an element of D, and further suppose that $P(x)$ and $Q(x)$ are predicates. Provide an informal verification that the following rule of inference is valid: If both $(\forall x)(P(x) \to Q(x))$ and $\sim Q(a)$, infer that $\sim P(a)$. (This rule is referred to as *Universal Modus Tollens*.)

Exercise 3.20. Fix some domain D over which we allow individual variables to range, suppose a is an element of D, and further suppose that $P(x)$ and $Q(x)$ are predicates. Consider the following rule of inference: if both $(\forall x)(P(x) \to Q(x))$ and $Q(a)$, then infer $P(a)$. Provide an informal argument for whether this is a valid rule of inference.

Exercise 3.21. A certain city is governed by a council consisting of five voting members: Fay, Gould, Haber, Ishikawa, and Jacobs. The council frequently votes on issues pertaining to city government, and several facts about the voting patterns of the council members are known:

(a) On every proposal, if Fey votes in favor of the proposal then so does Gould.

(b) Ishikawa and Jacobs never both vote in favor of the same proposal.

(c) On every proposal, if Haber does not vote in favor of the proposal then neither does Gould.

Suppose that a new budget for the city is going to come up for a vote before the council, and that Fay will vote in favor of the budget. Formalize this situation using predicate logic, and deduce whether the budget will pass. Justify every step in your deduction. (In order for the budget to pass, at least three voting members of the council must vote in favor of it.)

4   Looking Forward

The logical system we have studied in this project is today known as *classical logic*. The publication of *Principia Mathematica* led to a considerable amount of research in classical logic in the first half of the 20th century, and the work of such logicians as David Hilbert (1862–1943), Alfred Tarski (1901–1983), and Kurt Gödel (1906–1978) during this period led to a thorough understanding of classical logic. However, original research in logic continues today. Contemporary research in logic focuses on a variety of so-called *non-classical logics*, each of which is in some way a modification or expansion of classical logic. The classical logic developed in this project hence forms the core of contemporary research, and thus remains of relevance almost a century after the publication of *Principia Mathematica*.

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Notes to the instructor

This project is a self-contained treatment of the topics from propositional and predicate logic typically covered in a first course in discrete mathematics. It may be used as a text for the logic unit of a standard one semester course at the freshman or sophomore level, and should require approximately three/four weeks of class time to complete. Very few prerequisites are assumed. Students with a year course in calculus are more than prepared for the material contained here, and for the majority of exercises no more background than college algebra is required.

The exercises form a particularly important part of the project, and were designed to simultaneously provide students with practice applying the ideas discussed and to extend the discussion to new material. Several exercises introduce concepts from logic, number theory, the theory of relations, and other topics that are not normally covered until later in a discrete mathematics course. All such exercises are elementary, and may be taken as a stand-alone opportunity to study the primary material, or as an invitation to explore more advanced concepts. Because it is customary to cover logic at the beginning of a discrete mathematics course, the instructor may wish to begin with the material here, and use these exercises as a way of connecting logic to the material covered later in the course.

Several exercises (e.g., Exercise 2.8(c), the last question of Exercise 3.8, and Exercise 3.15(e)) are slightly open-ended. In our opinion, this stimulates independent thinking, as well as provides an opportunity for further in-class discussion. In our experience, such discussions enhance students' understanding of the material.

Developing the logical skills necessary to read and write mathematical proofs is emphasized throughout. The instructor may wish to discuss the material covered here and proceed to introduce basic proofs using number theory or naive set theory. The instructor may even use exercises that discuss concepts from number theory or the theory of relations as a jumping off point to assign outside exercises on reading or writing proofs before the material here is completed. This provides an extremely quick way of exposing students to proofs.