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Consequences of the Varignon Parallelogram Theorem

This article is the second of a pair appearing in this journal. The first article, on pages 316–19 of the April 2001 issue, discussed the life of Varignon, as well as his parallelogram theorem.

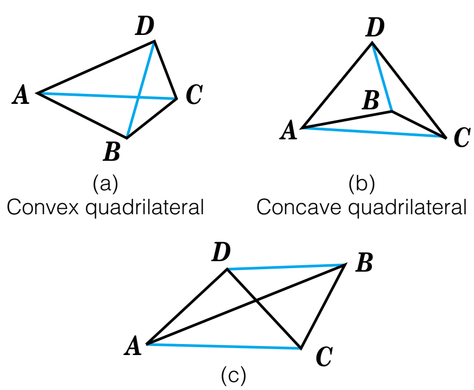
GEOMETRY STUDENTS have long been encouraged to investigate the following problem: What figure is formed when the midpoints of the sides of a quadrilateral are joined in order?

Because a four-sided polygon is not a rigid figure, students might examine a variety of quadrilaterals, both convex and concave. They could consider crossed, or nonsimple, quadrilaterals with one pair of opposite sides intersecting. See **figure 1.** They might even envision a skew quadrilateral, the sides of which are not all contained in any one plane, which could be modeled as a piece of construction paper folded upward along a diagonal.

The problem soon becomes a doodler's delight, easily lending itself to exploration with such interactive geometry software as Cabri Geometry or The Geometer's Sketchpad.

The problem soon becomes a doodler's delight

Students discover that howsoever the original quadrilateral is configured, the figure formed by joining the midpoints of its sides is a parallelogram, which is sometimes called a *Varignon parallelogram.* This figure is named after Pierre Varignon (1654–1722), who first proved that such a parallelogram fits each quadrilateral. A biography of Varignon appeared in the April 2001 issue of the *Mathematics Teacher.* This article suggests several approaches that allow students to explore the implications of the parallelogram theorem.



(a) Convex quadrilateral

(b) Concave quadrilateral

(c) Crossed quadrilateral

Fig. 1
Three categories of quadrilaterals

DISCOVERY ACTIVITIES

Elementary school students

Elementary school students can assemble model polygons of three, four, and more sides that have fully flexible hinges, such as paper fasteners, as their vertices and then investigate their rigidity. The students can experiment with convex, concave, and crossed quadrilaterals. They learn that a diagonal divides a quadrilateral into two triangles and stiffens it into a rigid form. After they have learned to classify quadrilaterals, they might tentatively explore the Varignon parallelogram.

Middle school students

Middle school students can continue the investigation. For crossed polygons, they can tabulate and graph the relation between the number of intersections and the regions in the plane. They will note that quadrilateral *ABCD* can be reconfigured into three categories of shapes.

To distinguish the three distinct categories of shapes formed by moving the sides, students can draw the two diagonals, *AC* and *BD,* of each quadrilateral, as shown in **figure 1.** In the convex quadrilateral in **figure 1a,** both diagonals are in its interior. The concave quadrilateral in **figure 1b,** sometimes called a *re-entrant quadrilateral,* has one diagonal inside and one outside. **Figure 1c** shows a crossed quadrilateral with both diagonals

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lying outside the figure. For each of these three configurations, students could be asked to characterize the new quadrilateral that would be formed by joining the midpoints of the sides.

High school students

After high school geometry students have mastered the triangle-congruence theorems and the relations of angles formed by a transversal intersecting parallel lines, the teacher can encourage them to discover the Varignon parallelogram for themselves, either by sketching it on graph paper or by using interactive geometry software.

The teacher can next ask students to consider the special cases of parallelograms that are formed from various types of quadrilaterals. A series of related questions was posed as the December 1995 Math Forum "Geometry Project of the Month" from the Swarthmore College Math Forum Project:

> What figure is formed when the consecutive midpoints of the sides of a quadrilateral are joined? What if the original quadrilateral were a rectangle? A kite? An isosceles trapezoid? A square? A rhombus? Other shapes? Explain why you think your answer is true.

Students can justify their answers with geometric proofs similar to Varignon's proof of the parallelogram theorem, given in Oliver (2001). They can also apply the techniques of analytic geometry, first using the midpoint formula and then the distance formula. The midpoint formula reads as follows:

> The midpoint of the segment joining the points (x_1, y_1) and (x_2, y_2) is the point
> $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The distance formula is as follows:

> The distance between two points (x_1, y_1) and (x_2, y_2) is
> $$d = \sqrt{(x_1 - x_2)^2 + (y_1 + y_2)^2}.$$

Varignon showed that the connected midpoints of the quadrilateral form a parallelogram with sides parallel to the diagonals of the original quadrilateral. By examining special cases of the original quadrilateral, geometry students can show that (*a*) the midpoints of the sides of a rectangle can be connected in a rhombus, (*b*) the midpoints of the sides of a square can be connected to form another square, (*c*) the midpoints of the sides of a rhombus can be connected to form a rectangle, and (*d*) the midpoints of the sides of an isosceles trapezoid can be connected to form a rhombus. For the Varignon parallelogram to be a rectangle, we can generalize that the diagonals of the original quadrilateral have to be perpendicular.

AREA OF THE VARIGNON PARALLELOGRAM

Geometry students are sometimes asked to demonstrate that the Varignon parallelogram has half the area of the quadrilateral from which it is formed.

An examination of the area problem is set forth in *Geometry Revisited,* in which Coxeter and Greitzer (1967, pp. 51–53) consider the area relationship to be of such importance that they extended the parallelogram theorem with an additional provision:

THEOREM. *The figure formed when the midpoints of the sides of a quadrilateral are joined in order is a parallelogram, and its area is half that of the quadrilateral.*

Geometry students can discover the Varignon parallelogram for themselves

In discussing the restated theorem, Coxeter and Greitzer use only a few lines to prove that the new figure is a parallelogram but closely scrutinize the area relationship. They begin by examining a general quadrilateral's three distinct configurations, which have been illustrated previously. They express the area of each quadrilateral in terms of the areas of the triangles formed when its diagonals are drawn.

By using the same technique of adding and subtracting areas, after the Varignon parallelogram has been drawn within a convex quadrilateral, its area can be derived by removing from the original quadrilateral the triangles formed between the sides of the parallelogram and the vertices of the original quadrilateral. See **figure 2.** This decomposition is illustrated in the following, where $(PQRS)$ represents the area of $PQRS$:

$$\begin{aligned}(PQRS) &= (ABCD) - (PBQ) - (RDS) - (QCR) - (SAP)\\ &= (ABCD) - \frac{1}{4}(ABC) - \frac{1}{4}(CDA) - \frac{1}{4}(BCD)\\ &\qquad - \frac{1}{4}(DAB)\\ &= (ABCD) - \frac{1}{4}(ABCD) - \frac{1}{4}(ABCD)\\ &= \frac{1}{2}(ABCD)\end{aligned}$$

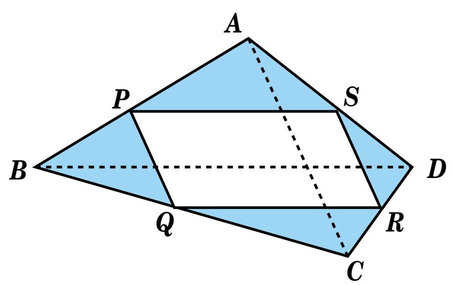

Fig. 2
The area of quadrilateral *PQRS* is equal to the area of quadrilateral *ABCD* minus the areas of the shaded triangles.

Students can also draw the Varignon parallelograms of a concave quadrilateral and a crossed quadrilateral. They can next verify that a series of area decompositions would work in these cases, as well (Coxeter and Greitzer 1967, pp. 51–53).

RELATED PROBLEMS

Geometers have devised many challenging exercises, at different levels of difficulty, that relate to the Varignon parallelogram. Students should begin each problem by illustrating the relationship described.

The five regular solids are well-suited for constructing a three-dimensional analogy

1. The diagonals of quadrilateral *ABCD* are six inches long and ten inches long. The midpoints of *AB, BC, CD,* and *DA* are connected in that order. Find the length of each side of the new quadrilateral.

   Hint: Refer to Varignon's proof, given in Oliver (2001), which draws on Euclid, VI.2: If a line is parallel to one side of a triangle and intersects the other two sides, it divides them proportionally.

2. Prove: If the midpoints of two adjacent sides of a parallelogram are joined, the area of the triangle so formed equals one-eighth the area of the parallelogram.

   Hint: Varignon showed that the segment connecting consecutive midpoints is parallel to a diagonal of the parallelogram. The interior of the parallelogram can be divided into a tessellation of eight congruent triangles.

3. If *ABCD* is a quadrilateral with the property that the midpoints of the sides are the four corners of a rectangle, show that

   $$AB \cdot AB + CD \cdot CD = BC \cdot BC + DA \cdot DA,$$

   that is, the sums of the squares of the opposite sides are equal.

   Hint: The Varignon parallelogram is a rectangle when the original quadrilateral is a rhombus, which by definition has sides of equal length. If each side measures x units, the equation in this problem can be simplified to $2x^2 = 2x^2$.

4. Prove: The perimeter of a Varignon parallelogram equals the sum of the diagonals of the original quadrilateral.

   Hint: This result is a generalization of problem 1.

5. Coxeter and Greitzer (1967) use the characteristics of the Varignon parallelogram as the starting point for their examination of projective geometry. The proof of the following theorem, as developed in *Geometry Revisited,* depends on the Varignon parallelogram relationship.

   THEOREM. *The segments joining the midpoints of pairs of opposite sides of a quadrangle and the segment joining the midpoints of the diagonals are concurrent and bisect one another.*

   Hint: Envision both a convex quadrilateral and a crossed quadrilateral that share the same vertices. When given a worksheet with three identical pictures of four noncollinear points, students could use these points as vertices for illustrating a convex quadrilateral, a crossed quadrilateral, and the figure that results when one of the quadrilaterals is superimposed on the other. The Varignon parallelograms of these quadrilaterals have one diagonal in common.

6. In solid geometry, a three-dimensional analogy to connecting the midpoints of a quadrilateral is constructing a polyhedron within another polyhedron such that a vertex of the inner figure is located at the center of each face of the outer polyhedron. The five regular solids, the Platonic solids, are well suited for this exercise, since their faces are regular polygons. Finding their center points by construction is convenient. Polyhedra that nestle within each other in this manner have the characteristic of duality. See Naylor (1999). Why are the cube and octahedron duals of each other? What about the dodecahedron and the icosahedron? What figure is the dual of the tetrahedron?

7. A parallelogram is said to have the midpoint property if it is similar to its Varignon parallelogram. Prove that for each r, where

   $$\sqrt{2} - 1 < r < \sqrt{2} + 1,$$

   a parallelogram exists whose sides have ratio r and which has the midpoint property.

   Hint: Envision a parallelogram with adjacent sides having lengths of 1 unit and r units. Position the figure on a coordinate plane with a vertex at the origin and a side lying on one of the axes.

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