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The Evolutionary Character of Mathematics

In her article "The Changing Concept of Change: The Derivative from Fermat to Weierstrass," Grabiner (1983) notes the following:

> Historically speaking, there were four steps in the development of today's concept of the derivative, which I list here in chronological order. The derivative was first *used;* it was then *discovered;* it was then *explored and developed;* and it was finally *defined*. That is, examples of what we now recognize as derivatives first were used on an ad hoc basis in solving particular problems; then the general concept lying behind these uses was identified (as part of the invention of calculus); then many properties of the derivative were explained and developed in applications to mathematics and to physics; and finally, a rigorous definition was given and the concept of derivative was embedded in a rigorous theory.

As Grabiner observes, the historical order of the development of the derivative is exactly the reverse of the usual order of textbook exposition, which tends to be formally deductive rather than intuitive and inductive. Grabiner's article contains a number of other well-articulated historical and pedagogical messages, and I strongly encourage every mathematics instructor to read it in its entirety. However, this article emphasizes only her use-discover-explore/develop-define (UDED) paradigm to describe the derivative's evolution. This model is extremely useful for constructing accounts of the evolution of numerous mathematical concepts and theories in addition to the derivative. In various courses that I teach, I often ask my students to use UDED to compile their own accounts of the evolution of mathematical entities. Occasionally, I have also required students to report their findings to the class, but the final, structured account is usually intended for the individual student's benefit alone.

The historical order of the derivative's development reverses the usual order of textbook exposition

Such assignments have many advantages. By encouraging my students to refer to such reputable histories of mathematics as those cited in the **bibliography** in constructing their accounts, I introduce them to the history of mathematics in a manner that is not overwhelming. This same exercise helps students understand that because most historical accounts are somewhat subjective, students need to justify their historical claims by citing reliable sources. For example, by using the UDED paradigm, students can learn to appreciate the basis that an author uses to assert that Isaac Newton and G. W. Leibniz invented calculus, that Girolamo Cardano was the first to solve the general cubic equation, that Carl F. Gauss, János Bolyai, and Nikolai Lobachevsky invented non-Euclidean (hyperbolic) geometry, and the like. Furthermore, as Grabiner observes, students learn that creating mathematics is often incremental, inductive, and exciting and that our modern versions of mathematical theories are polished diamonds that started off as rough pieces of carbon.

When I heard a colleague in the physics department describe the scientific method as "the development of knowledge from observation of specifics to conjecture to experiment to theory," it dawned on me that the UDED paradigm is essentially nothing more than using the scientific, or experimental, method to describe how mathematical theories and concepts evolve. Fuzzy foreshadowings, false starts, and dead ends have occurred in developing scientific models before such modern theories as those of the atom, light, heat, electricity, evolution, and the cosmos have crystallized and have been accepted as legitimate scientific theories. Students need to see this connection of shared modi operandi in the evolution of both mathematics and the natural sciences.

The accounts that teachers and students write using UDED can be detailed, brief, or anywhere in between. At times, the "big picture" is precisely what students should absorb; at other times, a mini–term paper might be appropriate. In assigning the UDED account as a student project, the instructor can easily set the parameters for the UDED project.

One of my favorite abridged applications of the UDED model is using it to construct a brief chronicle of the acceptance of the principle of mathematical induction as a valid method of proof in mathematics. In the sixth century B.C.E., the Pythagoreans certainly *used* the ideas underlying this principle when, proceeding geometrically, they conjectured and

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accepted as "true" such number-theoretic patterns as theorem S, which states that the sum of the first n odd integers is equal to the nth square number (Burton 1999, pp. 91–93). Francesco Maurolico gave the first formal inductive proof in the history of mathematics when he proved theorem S by induction; his proof (*discovery*) can be found in his work *Arithmeticorum Libri Duo,* published in 1575, the year of his death (Burton 1999, p. 426). In the next century, Blaise Pascal *explored and developed* the technique of mathematical induction in connection with his work on the arithmetic triangle and its applications (Burton 1999, pp. 418–28). Although John Wallis and Augustus De Morgan helped name this procedure *induction,* only in the latter part of the nineteenth century did Richard Dedekind—and then Gottlob Frege and Giuseppe Peano—*define* it mathematically. When formulating their sets of categorical properties for the natural numbers, each included the principle of mathematical induction or one of its logical equivalents as an axiom (Katz 1998, pp. 735–37).

USING "UDED" TO DESCRIBE THE EVOLUTION OF COMPLEX NUMBERS

The UDED model can also be used to describe the evolution of the complex numbers, a more commonplace high school mathematical topic than induction. Girolamo Cardano and other sixteenth-century Italian algebraists reluctantly began to *use* complex numbers when they saw that negative values appearing under the radical sign in the Cardano-Tartaglia formulas for solving specific cubic equations sometimes corresponded to recognizable real roots and when Cardano attempted to solve the problem of dividing 10 into two parts such that the product is 40. In *Ars Magna,* his famous algebra text of 1545, Cardano showed by "completing the square" that the two parts must be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Although he checked that these answers formally satisfied the conditions of the problem, he still regarded them as being "fictitious" and useless; he was only halfheartedly *using* complex numbers.

A generation later, Raphael Bombelli *discovered* the complex numbers in analyzing the "irreducible case" of the cubic equation when all three roots are real and nonzero and yet negative values always appear under the radical when a Cardano-Tartaglia type formula is used. When he published his treatise *Algebra* in 1572, he became the first mathematician bold enough to accept the existence of "imaginary," or complex, numbers and to present an algebra for working with such numbers. He assumed that they behaved like other numbers in calculation and proceeded to manipulate them formally, with $\sqrt{-a} \cdot \sqrt{-a} = -a$ for $a > 0$ being his key observation.

During the next three centuries, many mathematicians *explored and developed* various aspects of the complex, that is, imaginary, numbers. For example, in conjunction with their formative work in analytic geometry, calculus, and algebra, such mathematicians as René Descartes, Isaac Newton, G. W. Leibniz, Leonhard Euler, Jean d'Alembert, Carl F. Gauss, and Bernhard Riemann all employed complex numbers in describing their theories of equations, formulating the general logarithmic and exponential functions, and devising analytic tools for modeling and solving real-world problems. Casper Wessel, Jean Argand, and Carl F. Gauss contributed a crucial *development* to accepting and understanding the nature of complex numbers when they began to represent them geometrically in the real plane, much as we do today.

Finally, William Rowan Hamilton established the theory of complex numbers on a firm mathematical footing when he *defined* them in terms of ordered pairs of real numbers in almost the same way that modern textbooks define them. This *definition* and his rules for performing arithmetical calculations with his ordered pairs can be found in his 1837 paper "The Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time." Additional details concerning this UDED account of the evolution of the complex numbers can be found in Burton (1999) and Katz (1998).

The UDED paradigm can be used to construct brief accounts of the evolution of branches of mathematics

"UDED" AND THE EVOLUTION OF BRANCHES OF MATHEMATICS

The UDED paradigm can also be used to construct brief accounts of the evolution of such entire branches of mathematics as Euclidean geometry. Most ancient peoples *used* formulas to calculate the areas of simple rectilinear figures and to approximate the circumference and areas of circles. For example, the early Egyptians, Babylonians, and Chinese used algorithms to compute the volumes of rectangular blocks, cylinders, and pyramids. Furthermore, the latter two civilizations discovered the general Pythagorean theorem and used it in geometrical and astronomical applications. These civilizations had no real notion of an axiomatic system on which they could base "proofs" of their geometric formulas and theorems. As most students do today, they accepted their geometrical results on the basis of diagrams and intuition and often did not even distinguish between exact and approximate answers.

From the sixth century B.C.E. to the beginning of the third century B.C.E., Thales, Pythagoras, Eudoxus, Plato, Aristotle, and other Greek mathematicians and philosophers shaped mathematics into a deductive, axiomatic science and *discovered* Euclidean geometry. Around 300 B.C.E., Euclid compiled their accumulated discoveries in geometry and number theory and presented them axiomatically in his famous book, the *Elements.*

Over the next two millennia, Euclidean geometry was *explored and developed* by mathematicians from virtually every society that learned of the *Elements.* Such additional mathematical advances occurred as Archimedes' replacement of the Euclidean theorem "The areas of circles are to one another as the squares on their diameters" with a proof of the precise Babylonian formula "The area of any circle is equal to the area of a right triangle in which one of the legs is equal to the radius and the other to the circumference" (equivalent to the modern formula area = πr^2). However, the principal *explorations and developments* did involve repeated attempts to prove that Euclid's fifth, or parallel, postulate followed as a theorem from his other four more self-evident postulates and his common notions. The celebrated attempts of Proclus, ibn al-Haytham, John Wallis, Girolamo Saccheri, Adrien-Marie Legendre, Johann Lambert, and untold others were doomed to failure because—as we now know from the work of János Bolyai, Carl F. Gauss, and Nikolai Lobachevsky in the early nineteenth century—Euclid was indeed on sound logical ground when he made his parallel postulate an axiom for his geometry. It is logically independent of his other four.

Classroom teachers can use the UDED paradigm as a pedagogical strategem

Finally, at the very end of the nineteenth century, David Hilbert completely and logically *defined* Euclidean geometry in his classic monograph *Foundations of Geometry* (1899). Hilbert began his treatment of Euclidean geometry by postulating three undefined terms (*point, line,* and *plane*) connected by three undefined relations—*incidence* (on), *order* (betweenness), and *congruence*. He then offered a set of twenty-one axioms on which a logically consistent and complete treatment of Euclidean geometry could be based. In axiomatic studies of Euclidean geometry today, authors often distill Hilbert's collection of twenty-one axioms down to a set of fifteen logically independent axioms by combining related ones and deleting those that are implied by the others.

The principal pedagogical message here is that anyone purporting to offer high school geometry students a complete, deductive study of Euclidean geometry will fail. NCTM's curricular standards and recommendations indicate that a school geometry course should emphasize discovery, applications, and a representative sample of truly accessible proofs of such theorems as the Pythagorean theorem. Additional details concerning this UDED account of the evolution of Euclidean geometry can be found in Burton (1999) and Katz (1998).

CONCLUSION

Topics in addition to those already noted to which the UDED paradigm can be applied without unduly forcing the issue include the evolution of the concept and theory of a function, limit, infinite series, the integral, the number zero, negative numbers, real numbers, the theory of equations, and numerical procedures. It can be applied to describing the evolution of such entire branches of mathematics as non-Euclidean geometry, analytical geometry, and algebra (both manipulative and structural); such subareas of modern algebra as group theory; and trigonometry.

I encourage classroom teachers of mathematics to use Grabiner's generic paradigm both as a tool for their own acquisition of authentic historical accounts of the evolution of mathematical topics and as a pedagogical stratagem for their students to do the same.

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The author thanks Joanne Peeples for her assistance in revising an early version of this article.