THE "PILING UP OF SQUARES" IN ANCIENT CHINA

About the same time Euclid was compiling the Elements, *on the other side of the globe the brilliant work of Chinese scholars was being collected in an ancient text. The solution technique illustrated here, which was developed to a high art in China, should intrigue you and your students.*

By FRANK SWETZ

The Pennsylvania State University
Middletown, PA 17057

THE merits of a mathematics-laboratory approach to problem-solving activities have been widely acclaimed. Indeed, most teachers will readily acknowledge that laboratory learning situations sharpen the appeal to intuition and provide multifaceted learning interactions for the student. It is interesting to note that this modern teaching strategy often employs problem situations and learning devices that are quite historical in their origins. Napier's rods, magic squares, tangrams, the abacus, and the Tower of Hanoi all possess a historical significance that, if known, would enhance their teaching potential. A knowledge of historical origins can aid a teacher in devising a learning task or preparing a teaching strategy, as well as developing students' appreciation for the evolutionary nature of mathematics. The Thirty-first Yearbook of the National Council of Teachers of Mathematics, *Historical Topics for the Mathematics Classroom* (1969), is particularly helpful.

Of the five mathematics-laboratory devices mentioned, three had their historical origins in the Orient, and of these, two—magic squares and tangrams—had their beginnings in China. Yet it is only recently that Western scholars have begun to accord the development of mathematics in China its due attention. Although a variety of mathematical puzzles are ascribed to the Chinese, little acknowledgment has been made of their more noteworthy accomplishments, which include an early familiarity with the use of the Pythagorean theorem; the most accurate approximation for π obtained in the ancient world, 3.1415929293 (Needham 1959, p. 101); an accurate root-extraction process preceding Horner's method by six hundred years (Needham and Wang 1955); and the use of the Pascal triangle for finding coefficients for a binomial expansion three hundred years before Pascal conceived of it (Needham 1959, p. 47). A testimony to the mathematical ability of the early Chinese scholars is provided by the contents of the *Chiu-chang suan-shu* [Nine chapters on the mathematical art]. The *Chiu-chang* is a manual of utilitarian mathematics compiled during the early Han period (ca. 300 B.C.–A.D. 300) and has been described as the most influential of all ancient Chinese texts (Swetz 1972). Although several parts of this book have been translated into English, a completed translation remains to be undertaken.* In studying the contents and methods of the *Chiu-chang*, modern scholars have been intrigued by the depth and resourcefulness of the early Chinese methods

* Several problems of the *Chiu-chang* have been translated in Yoshio Mikami's *Development of Mathematics in China and Japan* (New York: Chelsea Publishing Co., 1913 and in Lam Lay Yong's "Yang Hui's Commentary on the *Ying-nu* of the *Chiu chang suan shu*" (*Historia Mathematica*, February 1974, pp. 47–64). A complete translation of the ninth chapter has been done by Frank Swetz and T. I. Kao in "Right Triangle Computational Techniques of Early China as Revealed by the Kou-ku of the Chiu Chang Suan Shu" (unpublished manuscript, Harrisburg, Pa., 1973). Translations of the complete work are available in Russian (E. I. Berezkina, "Drevnekitajsky Traktat Matematika v devjati Knigach," *Istoriko-matematiceskie issledovaniya* 10[1957]:423–584) and in German (Kurt Vogel, *Chiu Chang Suan Shu: Neun Bucher Arithmetischer Technik* [Brunswick, Germany: Friedrick Vieweg & Sohn, 1968]).

of solution. These methods are empirically based and can easily lend themselves to a mathematics laboratory.

It is the object of this paper to examine one algebraic-geometric solution technique employed in the *Chiu-chang*—a solution technique used in ancient Greece and Babylon but developed to a high art in China. The technique called *chi-chü*, or the "piling up of squares," employs an intuitive geometric approach to solve algebraic problems. Such techniques were the forerunner of algebraic computation. A knowledge of this method can provide insights into the development of algebra as well as become a source for classroom activities.

The "Piling Up of Squares"

"Kou-ku" [right triangles] is the title of the ninth chapter of the *Chiu-chang*. This chapter presents twenty-four problems whose solutions depend on a thorough understanding of the Pythagorean theorem and the basic mathematical properties of right triangles. Problem narratives proceed from very elementary to rather complex and often picturesque situations. The solutions of these problems are obtained by employing one of two techniques, either *chi-chü* or *ch'ung-cha*. The literal translation of *ch'ung-cha* connotes a use of proportions equivalent to the tangent function (Needham 1959, p. 109).

In the consideration of the actual "Kou-ku" problems of the *Chiu-chang*, the original Chinese format will be followed, which includes, in respective order, a statement of the problem; the answer; a brief description of the solution method; and an explanation, usually supplied by a later commentator on the work. Whereas the first three parts of each question will be directly translated from the old Chinese in deference to the modern reading audience, the explanation will be modified by the use of contemporary algebraic symbolism and additional diagrams. The presence of Chinese characters will designate diagrams preserved from antiquity. In their illustrations the Chinese frequently used 3, 4, 5 right triangles, although the actual dimensions of the problem under consideration are quite different. The conjecture exists that early number-theoretic investigations of the well-known 3, 4, 5 triple resulted in the discovery of the Pythagorean equation (Loomis 1968, pp. 3–4), though this is disputed (Neugebauer 1969, p. 36). The apparent inconsistency of using a 3, 4, 5 right triangle is justified for its convenience of illustration and didactical appeal. In Chinese the sides of a right triangle are *ku*, the longer leg; *kou*, the remaining leg; and *shian*, the hypotenuse. In my explanations, I shall represent these sides by k_1, k_2, and s, respectively. The problem numbers indicate the order in which the problems appear in "Kou-ku."

Chinese units of linear measure employed in the problems include these:

1 *chang*	=	10 *ch'ih*
1 *pu* (pace)	=	5 *ch'ih*
1 *ch'ih* (foot)	=	10 *ts'un* (inch)

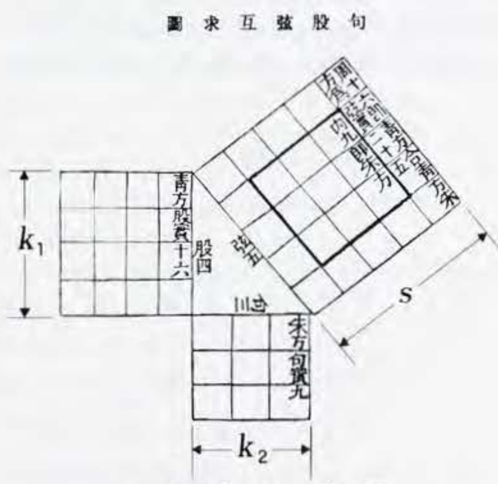
With this perspective established, let us examine some problems involving the "piling up of squares."

(1) *Given: kou* = 3 *ch'ih*, *ku* = 4 *ch'ih*
What is the length of shian?

Answer: shian = 5 *ch'ih*

Method: Add the squares of *kou* and *ku*. The square root of the sum is equal to *shian*.

Explanation: See figure 1.



$$s^2 = (k_1)^2 + (k_2)^2$$

Fig. 1

(6) *Given:* In the center of a square pond whose side measures 10 *ch'ih* grows a cattail whose top reaches 1 *ch'ih* above the water level. If we pull the reed toward the bank, its top becomes even with the water's surface. What is the depth of the pond and the length of the plant?

Answer: The depth of the water is 12 *ch'ih*, and the length of the plant is 13 *ch'ih*.

Method: (See fig. 2.) Find the square of half the pond's width; from it subtract the square of 1 *ch'ih*. The depth of the water will be equal to the difference divided by twice the height of the reed above water. To find the length of the plant we add 1 *ch'ih* to the result.

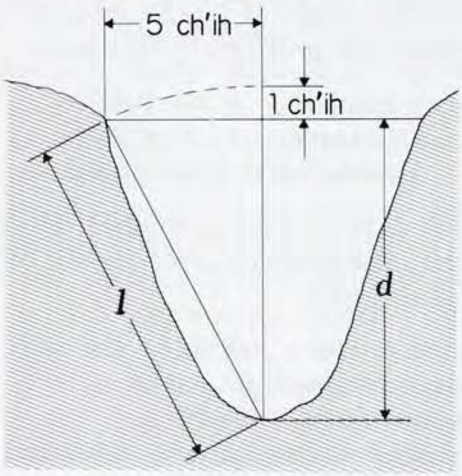


Fig. 2

Explanation: It is known that the sum of the squares erected on the legs of the right triangle (see fig. 3) should equal the area of the square erected on the hypotenuse, that

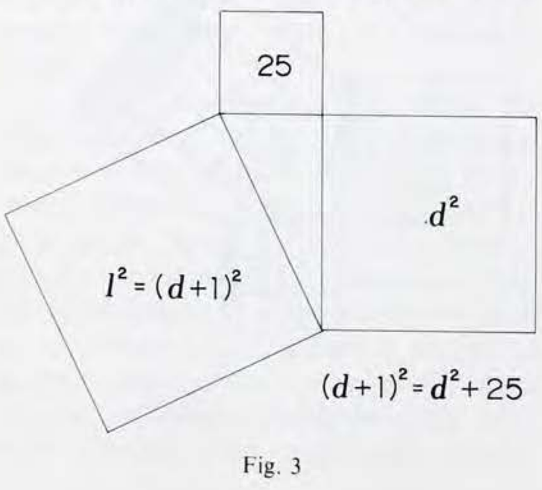


Fig. 3

is, $(d + 1)^2 = d^2 + 25$. Manipulating the resulting squares to perform the indicated operations, we obtain the configuration shown in figure 4. Since the shaded portion of figure 4 is equal in area to d^2, the remaining two rectangles and square must be equal in area to 25.

$$25 = 2d + 1$$

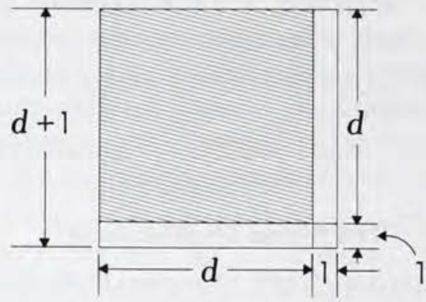


Fig. 4

If we subtract the square of 1 *ch'ih* as instructed, then the sum of the areas of the remaining two rectangles are equal to 24, that is, $2d = 24$. The area of one rectangle, *d*, will then correspond to the depth of the water, 12 *ch'ih*. To obtain the total length of the reed, 1, we add the depth of the water, 12 *ch'ih*, to the length of the reed extending above the surface 1 *ch'ih* and obtain the desired result, 13 *ch'ih*.

A more poetic rendering of this problem is found in the later writing of the Indian mathematician Bhāskara (1114–ca. 1185):

> In a certain lake, swarming with red geese, the tip of a bud of a lotus was seen a span (9 inches) above the surface of the water. Forced by the wind, it gradually advanced and was submerged at a distance of two cubits (approximately 40 inches). Compute quickly, mathematician, the depth of the pond. [Collidge 1940, p. 17]

The similarity of these two problems, even to the ratio of the distances involved, is striking. Such replication gives rise to a controversy concerning the influences of early Chinese mathematics on later Hindu writings (Needham 1959, pp. 146–50).

(11) *Given:* The height of a door is 6 *ch'ih* 8 *ts'un* larger than the width. The diagonal is 10 *ch'ih*. What are the dimensions of the door?

Answer: width = 2 *ch'ih* 8 *ts'un*
height = 9 *ch'ih* 6 *ts'un*

Method: (See fig. 5.) From the square of 10 *ch'ih* subtract twice the square of half of 6 *ch'ih* 8 *ts'un*. Take half of this result and obtain its square root. The width of the door is equal to the difference of this root and half of 6 *ch'ih* 8 *ts'un*, and the height of the door is equal to the sum of the root and half of 6 *ch'ih* 8 *ts'un*.

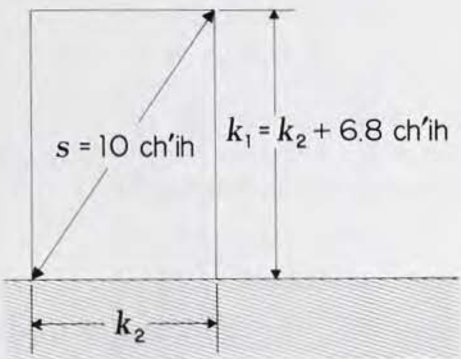


Fig. 5

Explanation: Using the original solution diagram as given in figure 6, we can derive the following equations:

(1) $(k_1 + k_2)^2 = 4(k_1 + k_2) + (k_1 - k_2)^2$;

(2) $s^2 = 2(k_1 \times k_2) + (k_1 - k_2)^2$,

which implies

$$2(k_1 \times k_2) = s^2 - (k_1 - k_2)^2;$$

(3) $(k_1 + k_2)^2 = s^2 + 2(k_1 \times k_2)$.

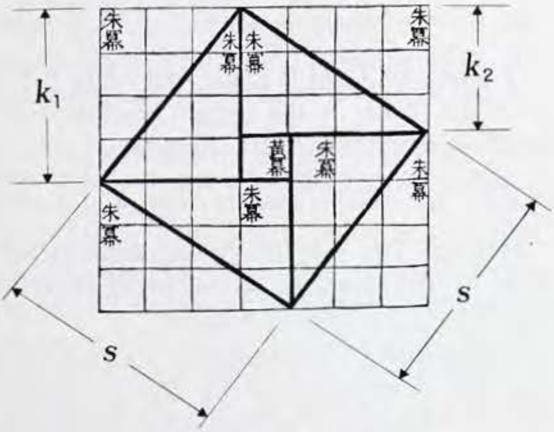


Fig. 6

Combining (2) and (3), we arrive at

(4) $(k_1 + k_2)^2 = 2s^2 - (k_1 - k_2)^2$.

The last equation supplies us with a readily workable expression involving the unknowns and the given.

$$k_1 + k_2 = \sqrt{2s^2 - (k_1 - k_2)^2}\,.$$

$$k_1 + k_2 = \sqrt{2(10)^2 - (6.8)^2}$$

$$= \sqrt{153.76}$$

$$\text{width} = \frac{\sqrt{153.76} - 6.8}{2} = 2.8\ ch'ih$$

$$\text{height} = \frac{\sqrt{153.76} + 6.8}{2} = 9.6\ ch'ih$$

The diagram of figure 5 is known as the *Hsuan-thu* and first appeared in the mathematical classic *Chou pei suan ching* [The arithmetical classic of the gnomon and the circular paths of heaven]. It represents one of the earliest known proofs of the Pythagorean theorem. Admired for its simple elegance, it also found its way into the work of Bhāskara.

The Chinese ability to extract the square root of 153.76 attests to their highly developed computational proficiency. In this particular operation, the methods of the Han mathematicians surpassed those of other contemporary societies (Lam 1970).

(14) *Given:* Two men starting from the same point begin walking in different directions. Their rates of travel are in the ratio 7:3. The slower walks toward the east. His faster companion walks to the south 10 *pu* and then turns toward the northeast and proceeds until both men meet. How many *pu* did each man walk? (See fig. 7.)

Answer: The fast traveler: $24\frac{1}{2}$ *pu*
The slow traveler: $10\frac{1}{2}$ *pu*

Method: (See fig. 8.) The circuit traveled forms a right triangle. The three sides of the triangle are in the ratio given by the following magnitudes:

northeast route, $(7^2 + 3^2) \div 2$

southward route, $\frac{-(7^2 + 3^2)}{2} + 7^2$

eastern route, 3×7

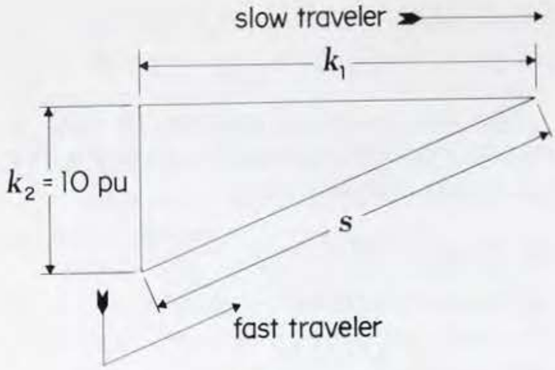


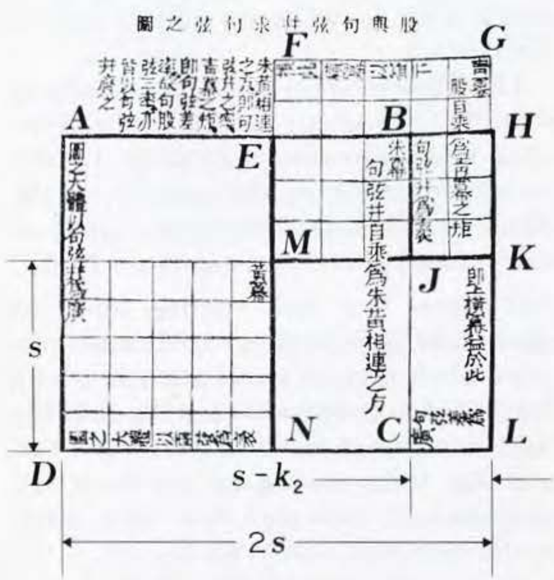
Fig. 7

The distance traveled to the northeast is then found to be

$$10 \times \frac{(7^2 + 3^2)}{2} \div 7^2 - \frac{(7^2 + 3^2)}{2};$$

and to the east,

$$10(3 \times 7) \div 7^2 - \frac{(7^2 + 3^2)}{2}.$$



Explanation: The area of rectangle $ABCD = (k_2 + s)^2$. The area of rectangle $EFGH$ + area of rectangle $BHKJ = (k_1)^2$; this follows, since the area of the square $FGKM = (s)^2$ and the area of the square $EBJM = (k_2)^2$. The area of rectangle $EFGH$ is seen to equal the area of rectangle $JKLC$; therefore, it can be seen that the area of the square $AHLD = (k_2 + s)^2 + k_1{}^2 = (2s)(k_1 + s)$; and therefore one half the area of the square $AHLD = s(k_2 + s)$, which supplies the term of the proportion relative to s. Similarly, the (area of rectangle $EHLN$) = (the area of the rectangle $BHLC$)

$$= (k_2 + s) - (k_1)^2$$
$$= s^2 + (k_2)(s) - (k_1)^2$$
$$= (k_2)^2 + (k_2)(s)$$
$$= k_2(k_2 + s),$$

which supplies a term for the proportion relative to k_2. The term for the proportion relative to k_1 is found to be $k_1(k_2 + s)$; thus

$$\frac{s}{29} = \frac{k_2}{20} = \frac{k_1}{21}.$$

Replacing "k_2" by the distance traveled southward, 10 *pu*, we obtain the distances traveled in the other directions:

$$\frac{s}{29} = \frac{10}{20} = \frac{k_1}{21};$$

northeast distance walked (s)

$$= \frac{29}{2} = 14\frac{1}{2}\ pu;$$

eastward distance traversed (k_1)

$$= \frac{21}{2} = 10\frac{1}{2}\ pu;$$

total distance covered by the faster man

$$= 10 + 14\frac{1}{2} = 24\frac{1}{2}\ pu.$$

(15) *Given:* (A right triangle with) *kou* = 5, *ku* = 12. What is the largest square that could be inscribed in the triangle?

Answer: A square with side 3 and 9/17 *ch'ih*

Method: The side of the square will be given by the quotient of the product of 5 and 12 and the sum of 5 and 12.

Explanation: The product of *ku* (k_1) and *kou* (k_2) is represented by the rectangle $ACKH$ in figure 9. For illustration purposes the dimensions are changed to 12 and 6, respectively. From the diagram the following relations are obvious:

area of triangle ABD

= area of triangle GKJ

area of square $BCED$

= area of square $FGJH$

area of triangle AGF

= area of triangle DEK

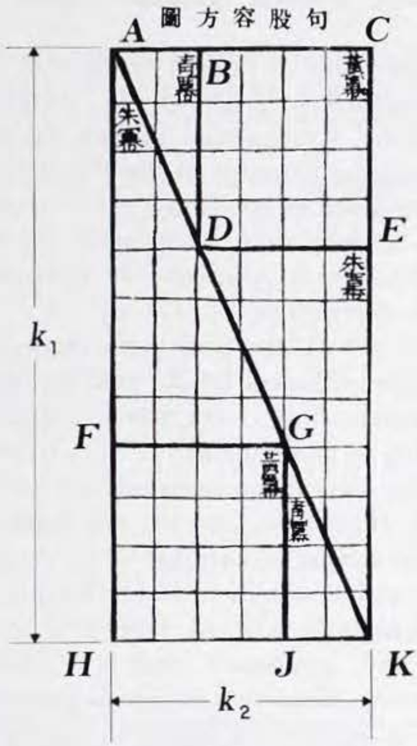


Fig. 9

Now if the rectangle is dissected and the pieces rearranged as indicated in figure 10, it becomes obvious that the side of the desired square is given by $\frac{(k_1)(k_2)}{k_1 + k_2}$. Although the Han authors supply a proof for a particular case, with a little effort this method can be altered to prove the general theorem that the measure of the side of the square inscribed in a right triangle is given by the quotient of the product of the measure of the legs of the triangle and the sum of the measure of the legs of the triangle.

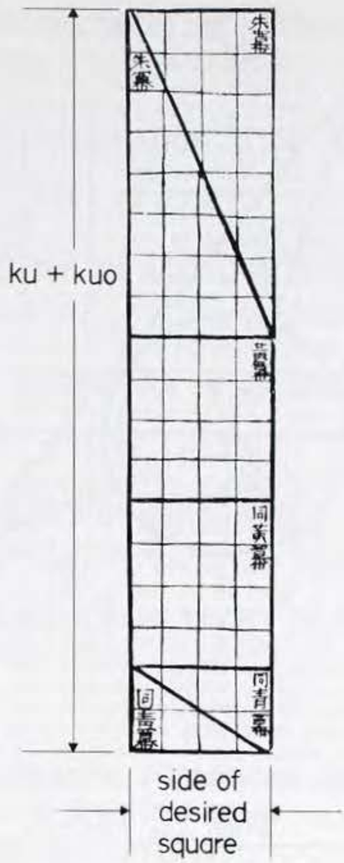


Fig. 10

Given: (A right triangle with) *kou* 8 units long and *ku* 15 units. What is the largest circle that can be inscribed in this triangle?

Answer: A circle with a diameter of six units

Method: Find the length of *shian* from *kou* and *ku*. The diameter will be the quotient of twice the product of *kou* and *ku* and the sum of *kou, ku,* and *shian.*

Explanation: From the diagram given in figure 11, we can obtain the following information: the area of triangle ACD equals the sum of the areas of triangles AFO, AEO, FCO, and GCO, and the area of the rectangle $EOGD$. Since the area of triangle ACD is also equal to $\frac{(k_1)(k_2)}{2}$,

$$2(k_1)(k_2) = 4 \times \text{area of triangle } ACD$$
$$= 4 \times (\text{area of triangle } AFO + \text{area of triangle } AEO) + 4 \times (\text{area of triangle } FCO + \text{area of triangle } CGO) + 4 \times (\text{area of triangle } EOGD).$$

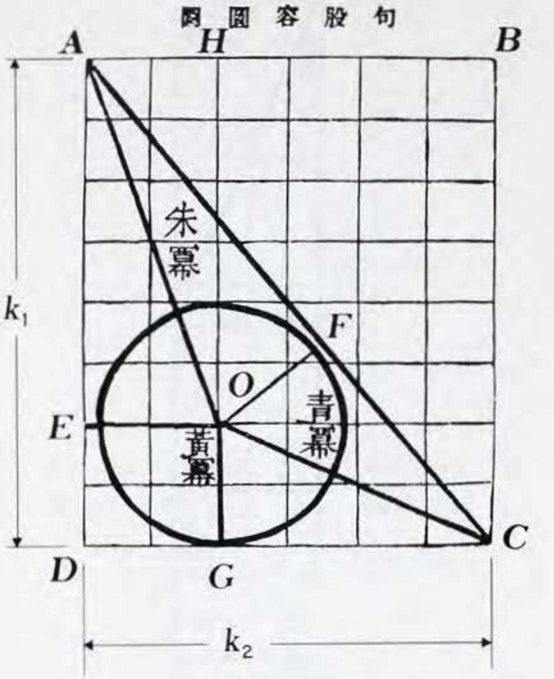


Fig. 11

This in turn equals 4 × (area of rectangle *AHOE*) + 4 × (area of rectangle *OJCG*) + 4 × (area of rectangle *EOGD*) = 2 × (radius) $(k_1 + k_2 + s)$; therefore,

$$2(k_1)(k_2) = 2 \times (\text{radius}) \times (k_1 + k_2 + s),$$

or

$$\text{diameter} = \frac{2k_1k_2}{k_1 + k_2 + s}.$$

Conclusions

The *chi-chü* method of solution is applied to eleven of the twenty-four exercises in "Kou-ku." The traditional Chinese proof of the Pythagorean theorem the *Hsuan-thu* employed in obtaining a solution to problem 11 is admired by Coolidge in his history of geometry for its simple elegance, as is also the solution for the inscribed square in problem 15 (Coolidge 1940, pp. 21–22).

The high degree of facility with which the Han mathematicians could employ such geometrically conceived algebraic solution schemes attests to a keen sense of spatial perception and an appreciation of the additive property of area measure. One cannot help but wonder about the influence of manipulative tangram exercises on the development of this ability. Recent research on tangrams attributes their origin to the Chou dynasty (Li and Morrill 1971), predating the appearance of the *Chiu-chang*, and implies certain psychological and educational designs in their conception. Although the conjecture of tangram exercises' being more than mere children's games is intriguing, its resolution awaits the results of more detailed research on the intent and origin of Chinese tangrams (Gardner 1974).

The methods of "piling up squares" are not completely unknown in American classrooms. Commercial models do exist for the demonstration of the Pythagorean theorem given in problem 1, and for years many teachers have employed concrete models to justify intuitively the expansions of such expressions as $(a + b)^2$, $(a - b)^2$, $(a + b)(a - b)$, and their cubic analogues. Multibase arithmetic blocks used by Dienes or Montessori also employ a technique involving manipulation of squares; however, such demonstrations are usually ends in themselves and do not lead into problem-solving situations. Once students have been introduced to this technique and are given a supply of large-grid graph paper and cardboard, they can explore and devise their own algebraic-geometric solutions for selected problems. It is hoped that this exposition will help in broadening the scope of possible mathematics-laboratory activities.

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Editor's Note: Readers interested in Chinese contributions to mathematics would enjoy the following published version of the text by Swetz and Kao that is mentioned in footnote 2:

Swetz, Frank, and T. I. Kao. *Was Pythagoras Chinese?* University Park, Pa.: Pennsylvania State University Press; Reston, Va.: NCTM, forthcoming.

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