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The Pascal-Fermat Correspondence

How Mathematics Is Really Done

A letter records how two of the greatest mathematicians of all time struggled for several weeks to solve a probability problem.

Keith Devlin

According to an old saying, there are two things to avoid seeing made—laws and sausages. See either process, and you will no longer like the product. Mathematics is the opposite. Few people ever see new mathematics being made, and yet, if they did, they might well like the product a whole lot more.

The mathematics our students see presented in their textbooks is highly polished. The steps required to solve a problem are all clearly laid out, the methods having been honed to perfection by many generations of teachers and authors.

The result is that students are denied what could be a valuable learning experience. Often when students meet a problem that differs only slightly from the ones in the book, they are unable to proceed, afraid to "play with" the problem for a few minutes to

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see whether they can find a way to do it, convinced that they simply do not have what it takes to do mathematics. No matter that the teacher makes suggestions—after all, mathematics teachers get their jobs precisely because they are among those rare people who are born miraculously able to see how to do it, right? But if students could see examples of the false starts and the erroneous attempts of the experts, they might be more inclined to persevere themselves.

The same type of reaction occurs at the college level, when students encounter advanced mathematics. They see Euclid's proof that there are infinitely many prime numbers or the classic ancient Greek demonstration that $\sqrt{2}$ is irrational, and they think they could never come up with the clever tricks those arguments use.

A lot of the mystique about what it takes to do mathematics might be dispelled—surely to the benefit of mathematics education—if our students occasionally saw how professional mathematicians measure up when they try a problem for the first time. Unfortunately, when mathematicians finally manage to solve a problem, they generally throw away the reams of false starts and failed attempts and show the world only the final, polished, and sanitized solution. Such sanitizing can give the impression that doing mathematics requires a highly unusual mind.

One exception is a letter, never intended for publication, sent by one of the best mathematicians the world has ever seen, to a colleague of even greater stature. On Monday August 24, 1654, the French mathematician Blaise Pascal (of Pascal's triangle) sent a letter to his countryman Pierre de Fermat (of Fermat's last theorem), outlining the solution to a problem that had puzzled gamblers and mathematicians alike for decades.

THE UNFINISHED GAME

Known as the Unfinished Game problem, the puzzle asked how the pot should be divided when a game of dice has to be abandoned before it has been completed. The challenge is to find a division that is fair according to how many rounds each player has won by that stage.

Today the (polished!) solution to the problem can be explained to high school students in a few minutes (see the **sidebar**), but when you read Pascal's letter you realize that it didn't seem at all obvious to him how to solve it.

Pascal begins his letter hesitantly:

> I was not able to tell you my entire thoughts regarding the problem of the points by the last post, and at the same time, I have a certain reluctance at doing it for fear lest this admirable harmony which obtains between us and which is so dear to me should begin to flag, for I am afraid that we may have different opinions on this subject. I wish to lay my whole reasoning before you, and to have you do me the favor to set me straight if I am in error or to indorse [sic] me if I am correct. I ask you this in all faith and sincerity for I am not certain even that you will be on my side.

Just think about that. The two have already exchanged several previous letters on the subject, and still one of the greatest mathematicians of all time is not sure whether he has got it right. It turns out he hasn't (at least not fully), although an alternative and much simpler approach suggested by Fermat, which Pascal also summarizes in the letter, does work. (See the **sidebar** for Fermat's simple solution.)

PASCAL'S SOLUTION

Much of the nearly three-thousand-word letter is devoted to Pascal's attempts to get his own approach to work. His approach is so convoluted that it is difficult to follow. But that is precisely why I think it would be valuable to show students this historical document. It provides a close-up view of mathematical sausage in the making, in all its messy detail.

In summary, Pascal approaches the problem by looking at the quantity $e(a, b)$, which represents the share of the stake that player 1 should be given if player 1 requires a winning throws to win and player 2 requires b winning throws. Clearly, if the numbers a and b are equal, then $e(a, b) = 1/2$. The idea is to see how $e(a, b)$ changes when each player wins one more throw. This idea leads to an algebraic expression for $e(a, b)$ in terms of $e(a - 1, b)$ and $e(a, b - 1)$, and Pascal solves the problem of the points by using recursion to calculate $e(2, 3)$, the desired share in the particular game they considered. This solution requires some complicated algebra dependent on the theory of combinations Pascal worked out in connection with his famous triangle.

Students can read Pascal's entire account of his argument in the August 24, 1654, letter on the Web (www.york.ac.uk/depts/maths/histstat/pascal.pdf). The argument has also been reproduced in its entirety, together with a commentary (Devlin 2008).

Pascal's actual mathematics doesn't really matter. He went off in a direction that doesn't work well, and he became confused. He ends his letter with a plea for help:

> Consequently, as you did not have my method when you sent me [your solution], [hence] I fear that we hold different views on the subject.

> I beg you to inform me how you would proceed in your research on this problem. I shall receive your reply with respect and joy, even if your opinions should be contrary to mine.

The moral of the tale is clear: Even professional mathematicians don't necessarily get it right the first time, or even the second, or the third. The secret—but it really should not be a secret—is to just keep trying. Successful mathematicians learn from their mistakes, sometimes work with someone else, and occasionally ask for help.

THE LETTER AS A CLASSROOM RESOURCE

The educational benefit of students examining false starts and failed attempts to solve problems is well known and discussed in, for example, Brown and Walter (2004).

The August 24, 1654, letter from Pascal to Fermat is particularly well suited for exposing high school and college-level students to the process of actual mathematical discovery and problem solving for a number of reasons. First, the problem itself is a simple one that requires no mathematical knowledge to understand. Second, many accomplished mathematicians failed to solve this problem over several hundred years, some of whom went as far as to conclude that it was unsolvable.

Yet when the problem was finally solved, the solution was an extremely short and simple one that required no mathematical techniques beyond counting. Moreover, the solution to the problem turned out to be pivotal in the development of modern society, leading directly to the development of modern probability theory, risk management, futures prediction, and the insurance industry (Devlin 2008).

How often can a scientific advance of such magnitude be made the focus of a middle school or high school mathematics class?

A class can begin by carrying out a practical exploration of the problem. Students can obtain an empirical solution by repeatedly tossing a coin or rolling a pair of dice. An obvious way is to have students work in pairs, with one student in the role of the player who has won two games, the other in the role of the player who has won one game. The student pairs then play out the remaining ("unplayed") two rounds, recording which player wins three out of five. They should find that the player who starts out having won two rounds wins the imaginary five-game tournament three times as frequently as the other player.

To gain some insight into one of the issues that challenged Pascal, students can repeat the exercise with the amended rule that they stop playing as soon as one player has won three rounds. They will

Fermat's Solution to the Unfinished Game Problem

Two gamblers, Blaise and Pierre, place equal bets on who will win the best of five tosses of a fair coin. On each round, Blaise chooses heads, Pierre tails. But they have to abandon the game after three tosses, with Blaise ahead, 2 to 1. How do they divide the pot?

The idea is to look at all possible ways the game might have turned out had Blaise and Pierre played all five rounds. Since Blaise is ahead 2 to 1 after round three, the first three rounds must have yielded two heads and one tail. The remaining two throws can yield these combinations:

H H   H T   T H   T T

Each of these outcomes is equally likely. In the first, the final outcome is four heads and one tail, so Blaise wins; in the second and the third, the outcome is three heads and two tails, so again Blaise wins; in the final outcome, the result is two heads and three tails, so Pierre wins. This means that in three of the four possible ways the game could have ended, Blaise wins; in only one possible play does Pierre win. Blaise has a 3-to-1 advantage over Pierre when they abandon the game. Therefore, Blaise should receive 3/4 of the winnings, and Pierre should receive 1/4.

This solution may seem "simple" today, but it definitely did not seem that way to the two mathematicians who worked it out. In fact, several world-class mathematicians had tried to solve the problem earlier and had failed completely. Some of them even went so far as to declare that the problem could not be solved. Now where have you heard that before?

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again find that the player who starts ahead wins roughly three-fourths of the time.

With the practical exploration behind them, students should have no trouble following Fermat's argument. Even better, they can be asked to try to find a solution themselves, either singly or in groups.

Students can then be asked to try to follow Pascal's own attempted solution. They can read Pascal's own words as he tries to grasp the simple solution Fermat has sent him—the very solution the students have just discovered for themselves. The teacher should make it clear that the aim is not to fully understand Pascal's intricate reasoning but to see just how much more complicated it is than Fermat's.

Students can also be asked to speculate exactly why a renowned mathematician like Pascal had such trouble following Fermat's reasoning. (No one knows for sure. Most likely part of the problem was that the very idea of counting hypothetical futures was entirely novel, although other factors are possible [Devlin 2008].)

Teachers may want to show the class an excellent video treatment of Fermat's solution, a five-minute segment from program 6 ("Chances of a Lifetime") in the PBS television series *Life by the Numbers*, first broadcast in 1998 (available on DVD at www.montereymedia.com/science/). Program 6 also has other highly informative segments about probability theory. (The other five programs also provide valuable classroom resources, although so rapid has been the progress in real-world applications of mathematics that many are already quite dated.)

FINAL REMARKS

The Pascal-Fermat correspondence is an excellent teaching resource. It shows students that mathematics does not come easily, even to the world's best mathematicians; that it can take time and effort even to understand a problem, let alone solve it; that the experts make mistakes; and that the principal requirement for doing mathematics is perseverance. Moreover, this resource does all this with a problem that is not only real but also one whose solution was seminal in the development of modern society. There are, to be sure, other such examples, but they lack one feature that makes the story of the unfinished game such a valuable educational resource: The mathematics is short, simple (to today's reader), and totally accessible to a middle school student.

I stumbled on this superb educational example by accident. Like many of my colleagues, I knew about the role Pascal and Fermat played in the establishment of modern probability theory, but until I researched the history, I never realized just how dramatic was the change in society their correspondence brought about, leading from a widely accepted belief that mathematics could not be used to predict the outcome of future events, to the establishment of modern predictive probability theory, risk management, actuarial science, and the insurance industry, *all within a single lifetime*. Nor did I appreciate just how great was Pascal's confusion nor how fully he displayed it in his letter.

I wrote this article to make this wonderful example more widely known among the mathematics education community.

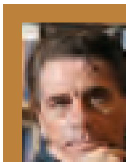
BIBLIOGRAPHY

Brown, Stephen I., and Marion I. Walter. *The Art of Problem Posing.* 3rd ed. Hillsdale, NJ: Lawrence Erlbaum Associates, 2004.

de Fermat, Pierre, and Blaise Pascal [the complete extant 1654 correspondence]. "Fermat and Pascal on Probability." Translated into English. www.york.ac.uk/depts/maths/histstat/pascal.pdf.

Devlin, Keith. *The Unfinished Game: Pascal, Fermat, and the Seventeenth-Century Letter that Made the World Modern.* New York: Basic Books, 2008.

Life by the Numbers. [PBS television series, WQED-TV]. www.montereymedia.com/science/.



KEITH DEVLIN, devlin@stanford.edu, is a mathematician at Stanford University in Palo Alto, California, whose educational interests focus on using different media to teach mathematics and to raise the general awareness of mathematics. He lectures frequently at schools, colleges, and public venues, has written a number of mathematics books for general readers, was an adviser to CBS television for the first season of the NUMB3RS fictional crime series, and appears regularly on National Public Radio as "the Math Guy." RICHARD RESSMAN