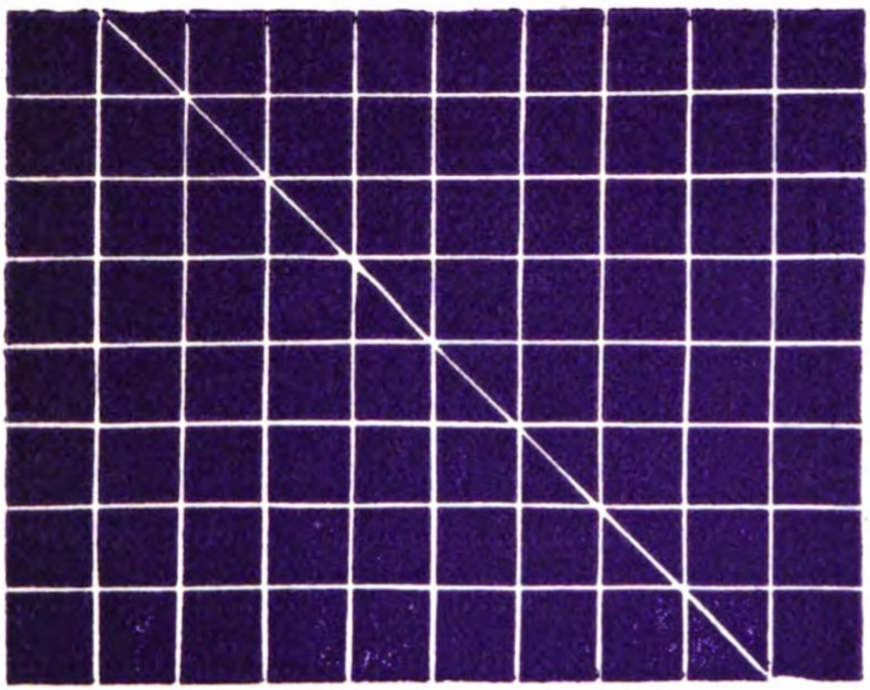
T. Sundara Row [Rao], *Geometrical Exercises in Paper Folding* (Madras: Addison & Co., 1893), pp. 39–42. Public domain.

CHAPTER X.

THE PROGRESSIONS.

Arithmetical Progression.

The annexed diagram exemplifies *Arithmetical Progression.*



The horizontal lines to the left of the diagonal, including the upper and lower edges are in A.P. The initial line being a and b the common difference, the series is a, $a+b$, $a+2b$, $a+3b$, &c.

2. The portions of the horizontal lines to the right of the diagonal are also in A.P., but are in reverse order and decrease with a common difference.

3. If, generally, l be the last term, and S the sum of the series, the above diagram graphically proves the formula

$$S=\frac{n}{2}(l+a).$$

4. If a and c are two alternate terms, the middle term is

$$\frac{1}{2}(a+c).$$

5. To insert n means between a and l, the vertical line has to be folded into $n + 1$ equal parts. The common difference will be $\frac{l - a}{n+1}$.

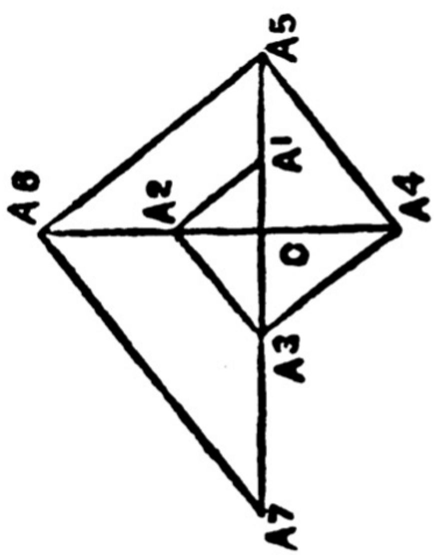
6. Considering the reverse series and interchanging a and l, the series becomes

$$a, a - b, a - 2b, \quad . \quad . \quad . \quad l.$$

The terms will be positive so long as $a > (n - 1)b$, and thereafter they will be negative.

Geometrical Progression.

7. In a right-angled triangle, the perpendicular from the vertex on the hypotenuse is a geometric mean between the segments of the hypotenuse. Hence, if two alternate or consecutive terms of a G.P. be given in length, the series can be determined as in the accompanying figure. Here CA_1, CA_2, CA_3, CA_4, and CA_5, are in G.P., the common ratio being $\frac{CA_2}{CA_1}$.



8. If CA_1 be the unit of length, the series consists of the natural powers of the common ratio.

9. Representing the series by $a, ar, ar^2, \ldots\ldots\ldots\ldots$

$$A_1 A_2 = a \sqrt{1 + r^2}.$$
$$A_2 A_3 = ar \sqrt{1 + r^2}.$$
$$A_3 A_4 = ar^2 \sqrt{1 + r^2}.$$
$$. \quad . \quad . \quad . \quad .$$

These lines also form a G.P., with the common ratio r.

10. The terms can also be found backwards, in which case the common ratio will be a proper fraction. If CA_5 be the unit, CA_4 is the common ratio. The sum of the series to infinity is $\frac{CA_5}{CA_5 - CA_4}$.

11. In the manner described above, one Geometrical mean can be found between two given lines, and by continuing the process, 3, 7, 15, &c., means can be found. In general, $2^n - 1$ means can be found, n being any positive integer.

12. It is not possible to find two Geometrical means between two given lines, merely by folding through known points. In the above figure, CA_1 and CA_4 being given, it is required to find A_2 and A_3. Take two rectangular pieces of paper, and so arrange them, that their outer edges lie on A_1 and A_4, and a corner of each lies on the straight lines CA_2 and CA_3, while at the same time the other edges ending in the said corners coincide. The positions of the corners determine CA_2 and CA_3.

13. This process gives the cube root of a given number, for if CA_1 be the unit, the series is 1, r, r^2, r^3.

14. There is a very interesting legend in connection with this problem. "The Athenians when suffering from the great plague of eruptive typhoid fever in 430 B.C., consulted the oracle at Delos as to how they could stop it. Apollo replied that they must *double* the *size* of his altar which was in the form of a *cube*. Nothing seemed more easy, and a new altar was constructed having each of its *edges* double that of the old one. The God, not unnaturally indignant, made the pestilence worse than before. A fresh deputation was accordingly sent to Delos, whom he informed that it was useless to trifle with him, as he must have his altar exactly doubled. Suspecting a mystery, they applied to the Geometricians. Plato, the most illustrious of them, declined the task, but referred them to Euclid, who had made a special study of the problem." Euclid's name is an interpellation for that of Hippocrates. Hippocrates reduced the question to that of finding two means between two straight lines, one of which is twice as long as the other. If a, x, y and $2a$ be the terms of the series $x^3 = 2a^3$. He did not, however, succeed in finding the means. Menæchmus, a pupil of Plato, who lived between 375 and 325 B.C., gave the following two solutions :

$$a : x :: x : y :: y : 2a.$$

From this relation we obtain the following three equations:

$$x^2 = ay \quad \ldots\ldots\ldots\ldots (1)$$
$$y^2 = 2ax \quad \ldots\ldots\ldots\ldots (2)$$
$$xy = 2a^2 \quad \ldots\ldots\ldots\ldots (3)$$

(1) and (2) are equations of *parabolas* and (3) is the equation of a *rectangular hyperbola*. Equations (1) and (2) as well as (1) and (3) give $x^3 = 2a^3$. The problem was solved by taking the intersection (α) of the two parabolas (1) and (2) and (β) of the parabola (1) with the rectangular hyperbola (3).