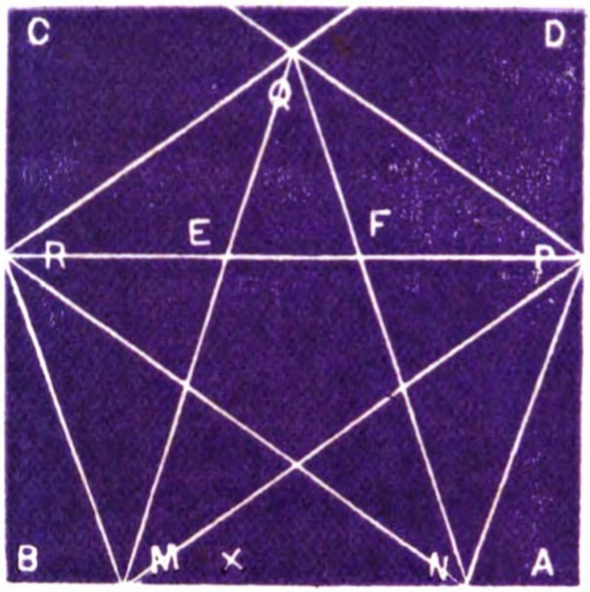
T. Sundara Row [Rao], *Geometrical Exercises in Paper Folding* (Madras: Addison & Co., 1893), pp. 20–23. Public domain.

CHAPTER IV.

THE PENTAGON.



To cut off a regular pentagon from the square ABCD.

Divide AB in X in medial section and take M the mid point of XB.

Then AB . BX = AX^2, BM=MX.

Take AN=BM or MX.

Then MN=AX.

Lay NP and MR equal to MN, so that P and R may lie on AD and BC respectively.

Lay RQ and PQ=MR and NP.

MNPQR is the pentagon required.

In fig. in para. 18, Chap. III., AN which is equal to AB, has the point N on the perpendicular MO. If A be moved on AB over the distance MB, then it is evident that N will be moved on to BC, and X to M.

Therefore in the present figure NR=AB. Similarly MP= AB. PR is also equal to AB and parallel to it.

$\angle$BMR is $\frac{4}{5}$ of a right angle. Therefore the angle NMR= $\frac{6}{5}$ of a right angle. Similarly $\angle$MNP is $\frac{6}{5}$ of a right angle.

From the triangles NMR and RQP, $\angle$NMR= $\angle$RQP=$\frac{6}{5}$ of a rt. angle.

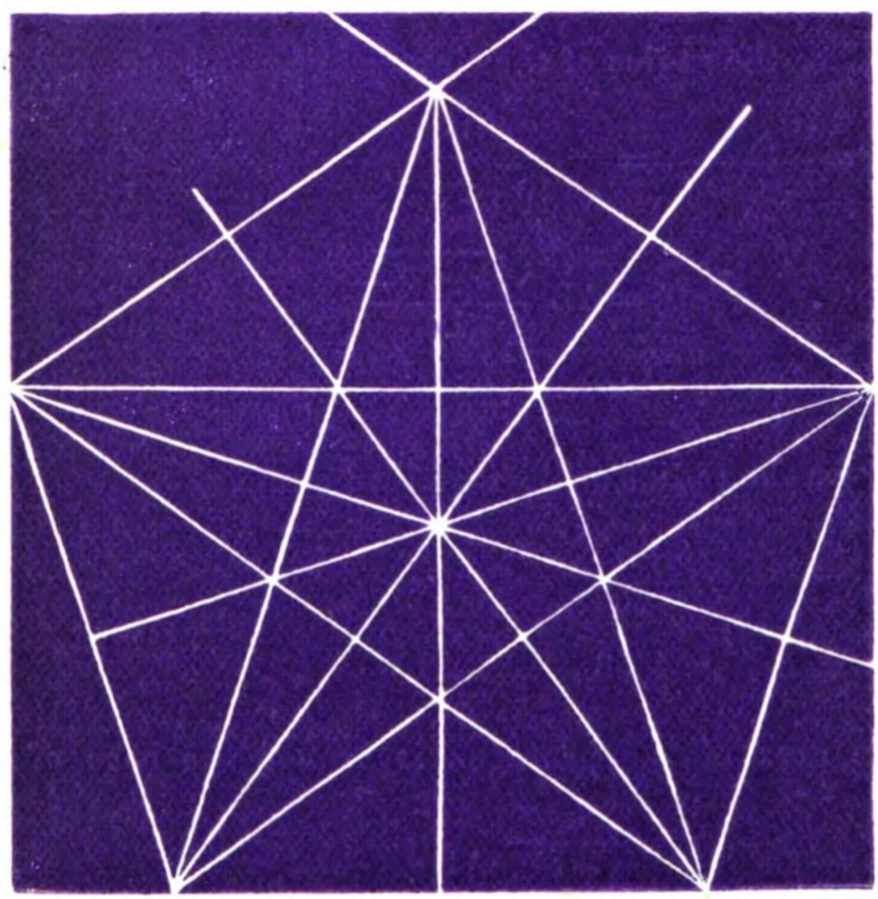
The three angles at M, N and Q of the pentagon being each equal to $\frac{6}{5}$ of a rt. $\angle$, the remaining 2 angles are together equal to $\frac{12}{5}$ right angles, and they are equal. Therefore each of them is $\frac{6}{5}$ of a rt. angle.

Therefore all the angles of the pentagon are equal.

It is also equilateral from the construction.

2. The base MN of the pentagon is equal to AX, *i.e.*, to $\frac{AB}{2}(\sqrt{5}-1) = AB \times \cdot 6180\ldots\ldots$

3. The greatest breadth of the pentagon is AB.



4. If p be the altitude,

$$AB^2 = p^2 + \left\{\frac{AB}{4}(\sqrt{5}-1)\right\}^2$$

$$=p^2+AB^2.\frac{3-\sqrt{5}}{8}.$$

$$p^2=AB^2\left\{1-\frac{3-\sqrt{5}}{8}\right\}$$

$$=AB^2.\frac{5+\sqrt{5}}{8}$$

$$p=AB.\frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$=AB\times\cdot9510\ldots\ldots=AB\ \mathrm{Cos}\ 18^\circ$$

5. If R be the radius of the circumscribing circle,

$$R=\frac{AB}{2\cos 18^\circ}.=\frac{2AB}{\sqrt{10+2\sqrt{5}}}$$

$$=AB\frac{\sqrt{5-\sqrt{5}}}{10}$$

$$=AB\times\cdot5257\ldots\ldots$$

6. If r be the radius of the inscribed circle,

$$r=p-R=AB.\sqrt{\frac{5+\sqrt{5}}{40}}$$

$$=AB\times\cdot4253\ldots\ldots$$

7. The area of the pentagon is $5r\times\frac{1}{2}$ the base of the pentagon,

$$i.e.,\ 5AB.\sqrt{\frac{5+\sqrt{5}}{40}}.\frac{AB}{4}(\sqrt{5}-1)$$

$$=AB^2.\frac{5}{4}.\sqrt{\frac{5-\sqrt{5}}{10}}=AB^2\times\cdot6571\ldots\ldots$$

8. In fig. in para. 1, Chap. IV., let PR be divided by MQ and NQ in E and F.

Then $RE=FP=\frac{MN}{2}\cdot\frac{1}{\cos 36^\circ}=AB\cdot\frac{\sqrt{5}-1}{\sqrt{5}+1}$

$=AB.\ \frac{3-\sqrt{5}}{2}$. ....(1)

$EF=AB-2\ RE=AB-AB(3-\sqrt{5})=AB\ (\sqrt{5}-2)$......(2)

$RF=MN.$

$RF:RE::RE:EF$...(3)

$\sqrt{5}-1:3-\sqrt{5}::3-\sqrt{5}:2\sqrt{5}-4$........................(4)

The area of the inner pentagon

$=EF^2.\ \frac{5}{4}\sqrt{\frac{5-\sqrt{5}}{10}}$

$=AB^2.\ (\sqrt{5}-2)^2.\ \frac{5}{4}\cdot\sqrt{\frac{5-\sqrt{5}}{10}}$

$=AB^2.\ (9-4\sqrt{5}).\ \frac{5}{4}\cdot\sqrt{\frac{5-\sqrt{5}}{10}}$........................(5)

The larger pentagon : the smaller $::1:(\sqrt{5}-2)^2$

$::1:\ \cdot05569$......

9. If in the figure in Art. 1, Chapter IV, angles QEK and QFL are made equal to EQR or FQP, K, L being points on the sides QR and QP respectively, then EFLQK will be a regular pentagon equal to the inner pentagon. Pentagons can be similarly described on the remaining sides of the inner pentagon. The resulting figure consisting of six pentagons is very elegant.