Elementary Soroban Arithmetic Techniques in Edo Period Japan

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In this article, we introduce methods for doing basic arithmetic on the Japanese abacus – known as the *soroban* – as used during the Edo period (1603 – 1868 CE). This introduction is informed by books one and two of the *Taisei Sankei* 大成算経 'Great Accomplishment of Mathematics', a mathematical work compiled by Seki Takakazu, Takebe Katahiro, and Takebe Kataakira between 1683 and 1711 CE.

I Introduction

During the Japanese Edo period (1603 – 1868 CE), the most common tool for calculation was the Japanese bead abacus known as the *soroban* 算盤. This device was used by merchants, farmers, and mathematicians alike to calculate everything from the cost of exchanging silver into gold to calculating the value of π. In this article, we introduce the common techniques for doing basic arithmetic on the *soroban* in the traditional style, so modern students, mathematicians, and historians alike can see how this device was used for over 250 years in Japan. We also provide a series of exercises in Section III for those who desire to try applying the techniques of the *soroban* themselves to traditional Edo period problems. Before introducing the techniques for basic calculation on the *soroban*, we first briefly detail the history and layout of this device, as well as the nature of the number system used in the Edo period.

I.1 Soroban

I.1.1 History

Since ancient times, the Japanese have performed arithmetic using calculation tools originating from China. The earliest form of calculation in Japan was done using counting rods known as *sangi* 算木 which were adapted from Chinese rods called *chóu* 籌. However, in the eras prior to the Edo period, an altered form of the Chinese bead abacus – the *suanpan* 算盤 (see Figure 1) – found its way to Japan and began to be distributed amongst citizens. The Japanese named this abacus the *soroban.*

The Chinese *suanpan* is thought to have found its way to Japan from mainland

China in the late 1500s CE. Though its exact date of origin is unknown, the *suanpan* is believed to have become popular in China in the fourteenth century. Upon its entry into Japan, the *suanpan* was altered to have a smaller frame and contain smaller grooved beads (which greatly aided in the speed and ease with which one could do calculations on the device). During the Edo period, it was common for *soroban* to consist of 2 upper beads and 5 lower beads. However, particularly from the Meiji era onwards, the Japanese opted for one bead instead of two in the upper section. Later the abacus was altered again to have four beads in the lower section instead of five.

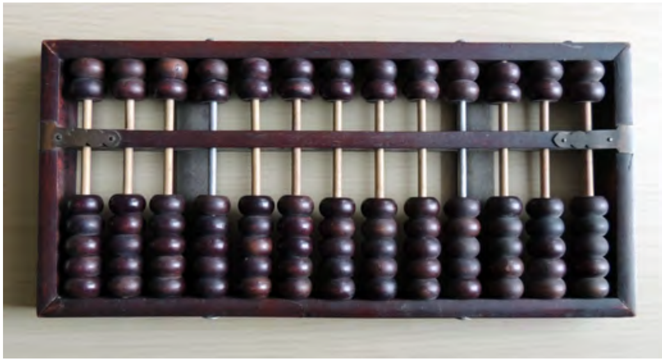


Figure 1: Chinese *suanpan* from Beijing (36 x 16.5 cm)

The earliest known *soroban* in existence in Japan is the *Shibei Shigekatsu Hairyo soroban* 四兵衛重勝拝領算盤 dating back to 1591 CE. This *soroban* was given to the *samurai* Shibei Shigekatsu by the warlord Toyotomi Hideyoshi 豊臣秀吉 (1537 - 1598 CE). Each individual bead and the frame were hand-crafted from the wood of plum trees, and silver was also used for decoration. An image of the *Shibei Shigekatsu Hairyo soroban* is shown in Figure 2.

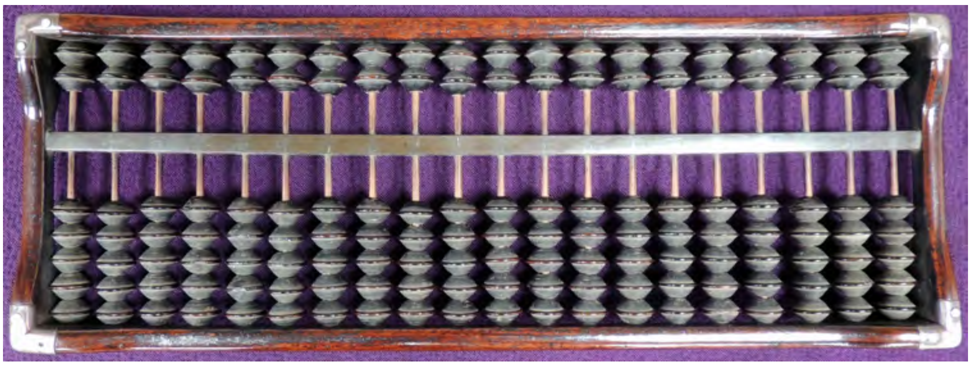


Figure 2: Oldest known *soroban* from 1591 CE (33.2 x 6.5 cm)

The *soroban* was first popularised in Japan by Mōri Shigeyoshi 毛利重能, who published the first Japanese text on the *soroban* – the *Warizan-sho* 割算書 – in 1622 CE. Following this, the *Jinkōki* of Yoshida Mitsuyoshi 吉田光由 (1598 – 1672) was published in 1627 CE. This work detailed how to use the *soroban* for various practical purposes in everyday Japanese life.[1] This work enabled people from all classes of society to learn and use the device, dramatically increasing its popularity.

I.1.2 Presentation

Soroban physically consist of a series of columns containing beads. These beads are separated into upper and lower sections by a bar known as the reckoning bar. The number of columns was usually an odd number, although there are rare examples such as the *Shibei Shigekatsu Hairyo soroban* where an even number are present.

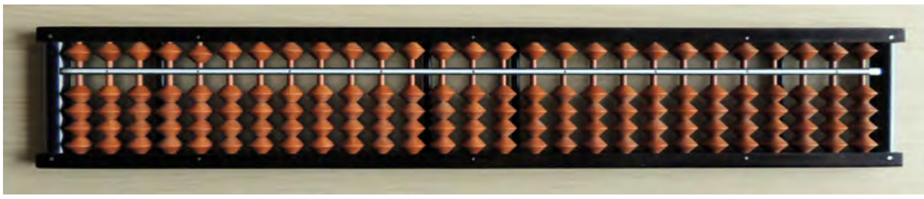


Figure 3: Starting position on modern *soroban* (39 x 6.3 cm)

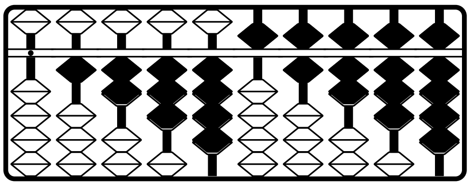
The bead(s) in the upper section – known as 'heaven' – have a numerical value of 5 while beads in the lower section – referred to as 'earth' – each have a value of 1. The starting position of the device consists of all heaven beads raised and all earth beads lowered (shown in Figure 3). Numbers are then represented by lowering heaven beads and raising earth beads as required. On the modern *soroban*, each column can represent a maximum total value of 9 by lowering 1 upper heaven bead and raising 4 earth beads. However, as mentioned, *soroban* during the Edo period contained 2 heaven beads and 5 earth, while in the Meiji period (1868 - 1912 CE) just 1 heaven bead began to be used (see Figure 4).

Each column represents a place value, though the exact value of each place is not physically set meaning numbers may be placed in any column. For example, the number 21 can be put in the first two columns or the last two columns and retain this value. However, this means that the values 2100, 210, 21, 2.1, 0.21, etc are all represented the same way on the *soroban*, and it is only by context that their specific value is known.

[1]During the Edo period, many mathematical textbooks were written in *kanbun* 漢文, an academic language of traditional Chinese with Japanese readings. Only those part of the *samurai* and upper classes were taught to read this language.

Figure 4: Meiji/Taishō Era Japanese *soroban* (28 x 8.5 cm)

Although traditionally the Japanese read from right to left (and top to bottom), numbers are placed and read from left to right on *soroban*. For example, when representing the value 21, the number 2 would be placed to the left of the 1. This allows for numbers to be easily manipulated while they are being read out to students. The following diagram shows how the numbers 0 to 9 can be represented by manipulating the beads.



0 1 2 3 4 5 6 7 8 9

I.2 Taisei Sankei

The examples of basic Edo period arithmetic in this article are drawn from the *Taisei Sankei* 大成算経 'Great Accomplishment of Mathematics', compiled between 1683 and 1711 CE by Seki Takakazu 関孝和 (? – December 5, 1708), Takebe Katahiro 建部賢弘 (1664 – August 24, 1739), and Takebe Kataakira 建部賢明 (1661 – 1716). The work – over 900 pages in length – was intended to be a reference on all mathematical knowledge of the time. Within its pages are discoveries such as a general theory of elimination and calculation of π using the Aitken delta-squared method for series acceleration – both made independently of and prior to Western mathematicians.

To make this large work easier to examine, modern historians have classified it into an introduction and 20 separate books. These are as follows: introduction, book 1 – basic arithmetic, book 2 – advanced arithmetic and extraction of roots, book 3 – discriminant of algebraic equations, book 4 – discussion of flow and ebb, book 5 – multiplication, book 6 – methods of fractions, book 7 – magic squares, books 8 and 9 – daily mathematics, book 10 – squares, rectangles and polygons, book 11 – regular polygons, book 12 – circle theory, book 13 – measurement, books 14 and 15 – techniques of figure, books 16 to 20 – theory of equations.

Regarding its authors, Seki Takakazu – also known as Seki Kōwa – is considered to be perhaps the greatest mathematician of the Edo period due to his work on elimination theory, Bernoulli numbers, the calculation of π, and his development of a method for symbolic manipulation which operated similar to modern algebra. The two other authors, Takebe Katahiro and Takebe Kataakira were brothers and direct students of Seki. In the *Takebe-shi Denki* 建部氏伝記 'Biography of the Takebe' – a family history of the Takebe clan – the following section details the creation of this work:

> Thus three gentlemen, under the leadership of Katahiro, consulted and started in summer of the third year of Ten'a Period (1683) the compilation of a book to describe all the details of the exquisite theory newly invented, and expose all the theories descended from antiquity. (Morimoto (2013))

According to the *Takebe-shi Denki*, due to the failing health of Seki and the busy schedule of Takebe Katahiro as a government official, the book was completed by Takebe Kataakira in 1711 CE.

I.3 Number System in Edo Period

The number system of the Japanese during the Edo period was base 10 and derived from China. In this system the largest value is placed before the smaller. For instance, 15 is written *jū go* 十五, or literally 'ten five'. A value such as 20 is expressed *ni jū* 二十 'two ten'. For values in the hundreds and thousands, *hyaku* 百 and *sen* 千 are used in a similar manner. For example, 221 is *ni hyaku ni jū ichi* 二百二十一 and 2221 is *ni sen ni hyaku ni jū ichi* 二千二百二十一. Numbers from 1 to 20 are represented below in Table 1.

Table 1: Japanese Number System

1	2	3	4	5	6	7	8	9	10
一	二	三	四	五	六	七	八	九	十
11	12	13	14	15	16	17	18	19	20
十一	十二	十三	十四	十五	十六	十七	十八	十九	二十

Numbers from 1 to 10 were sometimes also represented using other characters, shown in Table 2. These numbers are referred to as *daiji* 大字 or 'large characters' in modern Japanese, and are often used in legal and financial situations to write monetary values. In the *Taisei Sankei*, both ways of representing numbers are used.

Table 2: *Daiji* number system

1	2	3	4	5	6	7	8	9	10
壹	貳	参	肆	伍	陸	漆	捌	玖	十

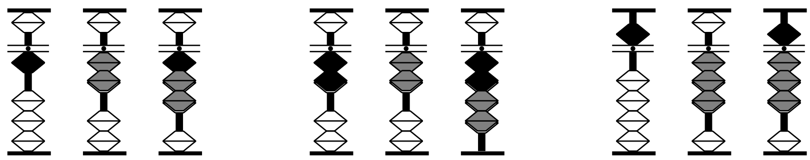
II Elementary Arithmetic on the Soroban

This section details how to add, subtract, multiply, and divide using traditional *soroban* techniques of the Edo period. Each part describes the technique itself and provides a traditional example and solution from book 1 of the *Taisei Sankei.*

Although *soroban* in the Edo period differed physically to modern forms, most Edo period mathematical problems can still be solved using the modern *soroban.* For this reason, we visually depict calculations using the form of a modern *soroban.* If a *soroban* is difficult to obtain, there are a number of websites and mobile phone applications which can be used to work through the problems in the following sections. Alternatively, problems can be worked through using pen and paper.

II.1 Addition

To add two numbers on the *soroban*, the number of beads are combined. For example, the following images depict how $1 + 2 = 3$, $2 + 2 = 4$ and $5 + 3 = 8$ can be calculated. For instance, in the first image one lower earth bead is raised to represent the number 1. Then to add 2, another 2 beads are raised to give the answer.



1 + 2 = 3   2 + 2 = 4   5 + 3 = 8

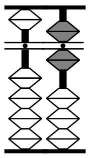
To add a number which is greater than 5, there are various methods. One method is to first split the addend into two parts – the first being 5 and the second the remainder. Next add the remainder to the augend, and then subtract the 5 from the sum. Finally, add 10. For example, with $6 + 6$, 5 is removed from the addend, leaving a remainder of 1 (as $6 - 5 = 1$). This remainder is added to the augend – 6 – to produce $6 + 1$. Then the 5 from the addend is subtracted giving $(6 + 1) - 5$. Lastly, 10 is added producing $((6 + 1) - 5) + 10$, which equals 12. For example, using the above method, adding the numbers 1 through 9 to the number 6 can be calculated in the following two ways:

$6 + 1 = 7$
$6 + 2 = 8$
$6 + 3 = 9$
$6 + 4 = 10$
$6 + 5 = ((6 + 0) - 5) + 10 = 11$
$6 + 6 = ((6 + 1) - 5) + 10 = 12$
$6 + 7 = ((6 + 2) - 5) + 10 = 13$
$6 + 8 = ((6 + 3) - 5) + 10 = 14$
$6 + 9 = ((6 + 4) - 5) + 10 = 15$

Note that in the last calculation, on a modern *soroban* there are not enough beads in the column to add 4 beads to the 6 beads already present. Due to this, the user first adds 3 of the 4 beads to the column. Seeing no more can be added, and knowing that $9 + 1 = 10$, the value 9 is removed from the column and one bead is added to the column to the left (representing the tens place). Lastly 5 is added to the column on the right, giving a total of 15. As discussed in section I.1.2, *soroban* of the Edo and Meiji periods contained 5 single 1 beads, allowing the value of 10 to be represented in a single column.

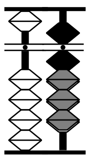
Example: 6 + 8

1. Place 6 on the *soroban*.



0 6

2. Split 8 into 5 and 3. Add 3 to the *soroban*.



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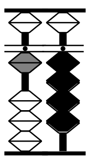
3. Lastly subtract 5 by removing the upper bead from the 9. Then add 10 by placing 1 bead in the tens place, obtaining 14.



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II.1.1 Traditional Problem

Problem 1 [Book 1: Problem 1]

The following problem comes from book 1 of the *Taisei Sankei*. It provides a basic addition problem on the *soroban* and instructions for how to perform the calculation.

For example, there is 342.5 *ko*.[2] Add 819.5 *ko*. Find the total.

Answer: 1162 *ko*.

Solution: Put the first number 342.5 and add the next number 819.5 to obtain the total.

3	**4**	**2**	**5**
Raise 3	Raise 4	Raise 2	Lower 5
8	1	9	5
[1]	[2]	[3]	[4]
1	1	6	2

[1] Withdraw 2 and advance 10.

[2] Lower 5 and subtract 4.

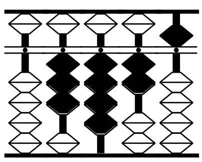
[3] Withdraw 1 and advance 10.

[4] Withdraw 5 and advance 10.

Solution Using Modern Soroban

The table above provides instructions to the reader on physically performing the calculation. In this section we illustrate how this calculation can be done using these instructions on a modern Japanese *soroban*.

First, raise 3, 4, and 2 beads respectively and lower an upper bead to present the number 342.5.

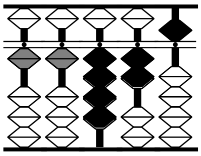


0 3 4 2 5

[2]The term *ko* 個 refers to a Japanese counter. This particular counter is used to count a variety of different objects.

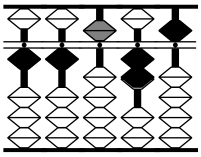
Rosalie Joan Hosking, Tsukane Ogawa, and Mitsuo Morimoto, "Elementary *Soroban* Arithmetic Techniques in Edo Period Japan," *MAA Convergence* (June 2018):
www.maa.org/press/periodicals/convergence/elementary-soroban-arithmetic-techniques-in-edo-period-japan

1 Subtract 2 beads from the hundreds place (the 3 of **3**425) and add 1 bead to the thousands place.



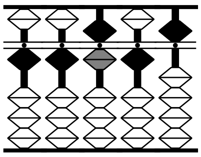
1 1 4 2 5

2 Lower the upper bead while lowering the 4 beads in the tens place.[3]



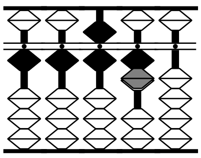
1 1 5 2 5

3 Subtract 1 bead from the ones place and add 1 bead to the tens place.



1 1 6 1 5

4 Lower the upper bead from the tenths place and add 1 bead to the ones place. This produces the solution 1162.0.

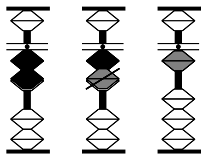


1 1 6 2 0

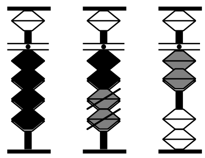
II.2 Subtraction

To subtract one number from another, the number of beads in the subtrahend are removed from the minuend. For example, the following images depict how to calculate $2-1=1$ and $4-2=2$.

[3]On the *soroban*, this constitutes one action. While lowering the upper bead, in the same stroke the 4 single beads are lowered.



2 - 1 = 1



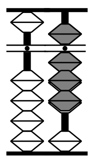
4 - 2 = 2

When the minuend is greater than 5 but less than 10 and the subtrahend equal to or less than 5, then 5 should be subtracted from the minuend and the subtrahend put in terms of 5 and treated as '[5 − *subtrahend*]' and the remainders combined. For example, $8-4$ would become $(8-5)+(5-4)$ which is $(3)+(1)=4$. To do this calculation, the upper bead is removed from 8 and a single bead added to the remaining 3 to produce the answer 4. Example 1 illustrates this process.

When the subtrahend and minuend are greater than 5, subtract 5 from both the minuend and subtrahend and then combine the remainders. For example, $10-6$ is treated as $(10-5)-(6-5)$ which is equal to $5-1$ which produces 4. When the minuend is greater than 10 this method can also be used. For example, $14-6$ would be changed to $(14-5)-(6-5)=(9)-(1)$, as shown in Example 2.

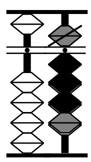
Example 1: $8-4$

1. Place 8 on the *soroban*.



0 8

2. Find the result of $8-5$ which is 3. Then calculate $5-4$ to obtain 1. Raise the upper bead from the initial number, and while doing this raise 1 bead to produce 4.[4]



0 4

[4] This is because, where possible, it is preferable to complete two steps in one action on the *soroban* for simplicity and to save time

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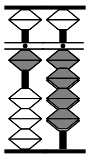
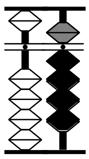
Example 2: $14-6$

1. Place 14 on the *soroban*.



1 4

2. Find the result of $14-5$ which is 9. As $-5=-10+5$, lower the bead in the tens place and lower the upper bead in the ones place to make 9.



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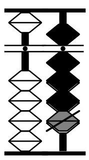
3. Find the result of $6-5$, which is 1. Subtract 1 lower bead to produce 8.



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II.2.1 Traditional Problem

Problem 2 [Book 1: Problem 4]

The following problem from the *Taisei Sankei* deals with basic subtraction.

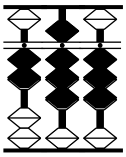
> For example, there is 283 *ko*. Subtract 147. What is the remainder?
>
> Answer: 136 *ko*.
>
> Solution: Put the first number 283 and subtract the number 147 to obtain the resulting number.

2	**8**	**3**
Raise 2	Raise 8	Raise 3
1	4	7
$\boxed{1}$	$\boxed{2}$	$\boxed{3}$
1	3	6

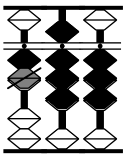
1 Withdraw 1.
2 Add 1 and remove 5.
3 Withdraw 10 and add 3. Lower 5 and subtract 2.

Solution Using Modern Soroban

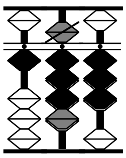
Place 2, 8, and 3 beads on the *soroban* to produce 283.



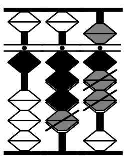
1 As $2 - 1 = 1$, remove 1 bead from the 2 in the hundreds place.



2 Next $8 - 4$ becomes $(8 - 5) + (5 - 4)$, which is $3 + 1 = 4$. Raise the upper bead in the tens place while adding 1 single bead.



3 As $3 < 7$, make $43 - 7$ the next subtraction. Because $7 = 10 - 3$ remove 10 from 43. As $5 - 2 = 3$, lower the upper bead in the ones place (to add 5 to this place) and subtract 2 from this place. This leaves 36 in the tens and one places and 136 in all three.



II.3 Multiplication

II.3.1 Single Digit Numbers

Multiplication of single digit numbers was typically done using a nine times nine table known as *ku ku* 九九 in the Edo period. This table was usually memorised in the form of a chant which would be recited. In Table 3, the version of the *ku ku* from the *Taisei Sankei* is given.[5]

Table 3: *Ku Ku* table from *Taisei Sankei*

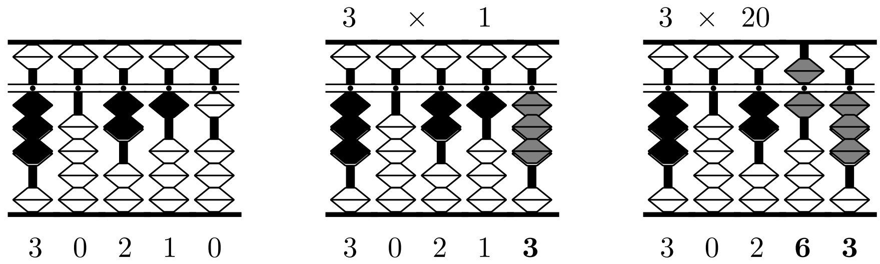
一一如[6]一 1 [times] 1 is 1	一二如二 1 [times] 2 is 2	一三如三 1 [times] 3 is 3	一四如四 1 [times] 4 is 4
一五如五 1 [times] 5 is 5	一六如六 1 [times] 6 is 6	一七如七 1 [times] 7 is 7	一八如八 1 [times] 8 is 8
一九如九 1 [times] 9 is 9	二二如四 2 [times] 2 is 4	二三如六 2 [times] 3 is 6	二四如八 2 [times] 4 is 8
二五一[7]十 2 [times] 5 [is] 10	二六一十二 2 [times] 6 [is] 12	二七一十四 2 [times] 7 [is] 14	二八一十六 2 [times] 8 [is] 16
二九一十八 2 [times] 9 is 18	三三如九 3 [times] 3 is 9	三四一十二 3 [times] 4 [is] 12	三五一十五 3 [times] 5 [is] 15
三六一十八 3 [times] 6 [is] 18	三七二十一 3 [times] 7 [is] 21	三八二十四 3 [times] 8 [is] 24	三九二十七 3 [times] 9 [is] 27
四四一十六 4 [times] 4 [is] 16	四五二十 4 [times] 5 [is] 20	四六二十四 4 [times] 6 [is] 24	四七二十八 4 [times] 7 [is] 28
四八三十二 4 [times] 8 [is] 32	四九三十六 4 [times] 9 [is] 36	五五二十五 5 [times] 5 [is] 25	五六三十 5 [times] 6 [is] 30
五七三十五 5 [times] 7 [is] 35	五八四十 5 [times] 8 [is] 40	五九四十五 5 [times] 9 [is] 45	六六三十六 6 [times] 6 [is] 36
六七四十二 6 [times] 7 [is] 42	六八四十八 6 [times] 8 [is] 48	六九五十四 6 [times] 9 [is] 54	七七四十九 7 [times] 7 [is] 49
七八五十六 7 [times] 8 [is] 56	七九六十三 7 [times] 9 [is] 63	八八六十四 8 [times] 8 [is] 64	八九七十二 8 [times] 9 [is] 72
九九八十一 9 [times] 9 [is] 81			

[5]The version of *ku ku* in the *Taisei Sankei* does not include all values. For example, the twos times table begins at '2 × 2 is 4' rather than '2 × 1 is 2'.

II.3.2 Multiple Digit Numbers

The basic method of multiplying multiple digit numbers is the same as used globally today, where multiplication is done by multiplying the last digit the smaller number by the ones place, tens place, hundreds place, and so on of the larger. This is then repeated for the other digits of the smaller number until all digits have been multiplied by the larger number.

On the *soroban*, usually the smallest number will be the multiplier on the left-hand and the larger the multiplicand on the right. It is also common practice to leave empty columns to the right of the multiplicand. These are used for the answer, and their number will usually be equal to the number of digits of the left-hand number. For example, if multiplying 67×4621, the multiplier has two digits so two empty columns should be placed after the multiplicand. As numbers of the multiplicand are multiplied, they are removed and their columns used to represent the answer as it grows in size. The diagrams below illustrate how $3 \times 21 = 63$ is be performed.



II.3.3 Additional Multiplication Methods

Book 2 of the *Taisei Sankei* also contains a series of additional methods of multiplication that can be used depending on the numbers being multiplied. Below we briefly describe these techniques, though we do not provide examples due to the scope of this article being elementary techniques.

1. **Multiple Multiplication (*jūjō* 重乗)**
   Uses factorisation and multiple instances of multiplication to reduce the size of numbers if they are too large to calculate on the *soroban*.

[6]The character *jo/ga* 如 is used as an empty place to indicate the answer is single digit. It also has an aesthetic function and helps the chant to flow when it is verbalised.

[7]For numbers from 10 to 19, the character 一 *ichi* which means 1 is not pronounced when reciting the multiplication chant. For example, 一十二 '12' is spoken 'ju ni' rather than 'ichi ju ni'.

2. **Changing Multiplication (*kōjō* 更乗)**
   Allows for the swapping of the left-hand and right-hand numbers if the original left-hand number is large.

3. **Splitting Multiplication (*setsujō* 截乗)**
   When there are two values such as $A = a_1 + a_2$ and B $= b_1 + b_2$ being multiplied, this technique is used to convert them into $AB = a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$.

4. **One Number Multiplication (*kojō* 孤乗)**
   When there is a squared number, this method transforms the number into $(a + b)^2$ by splitting the number into the last digit and the remaining digits. For example, 1384^2 would be transformed into $(1380 + 4)^2$. Then this is expanded as $a^2 + ab + b^2$, and the value of $ab + b^2$ found. The process is then repeated with the next to last digit (in the case of our example, the 8 of 13**8**4).

5. **Breaking Top Multiplication (*hatōjō* 破頭乗)**
   This method multiplies from the first digit (rather than last) of each number to the last and gradually replaces the digits of the right-hand number from the first to last with digits from the answer.

6. **Last Place Multiplication (*chōbijō/tōbijō* 掉尾乗)**
   Similar to *Breaking Top Multiplication*, except in this case multiplication occurs from the last digits to the first. Also, instead of placing the answer in empty columns, the answer is placed within the right-hand side number from its last digit.

7. **Skipping Place Multiplication (*kakuijō* 隔位乗)**
   When multiplying, the second to last number of the larger number is 'skipped' and multiplied last.

8. **Inserting Multiplication (*senjō* 穿乗)**
   This technique creates a new multiplication table based on the left-hand number and then uses this table to find the answer.

9. **Minus Multiplication (*sonjō* 損乗)**
   This technique subtracts the left-hand number from 1, 10, 100, etc based on how many digits it has and uses the remainder for the calculation.

10. **Add Next Digit (*shingaika* 身外加)**
    If the first digit of the left-hand number begins with 1, then this number can be removed from the multiplication and the remaining numbers used.

11. **Add First Digit (*shinzenka* 身前加)**
    Similar to *Add Next Digit.* If the last digit of the left-hand number is 1 then this can be removed and multiplication done with the remaining numbers.

II.3.4 Traditional Problems

Problem 3 [Book 1: Problem 8]

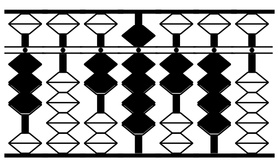
The following problem from the *Taisei Sankei* deals with single digit multiplication.

> For example, there is 2924 *ko*. Multiply it by 3. Find the product.
>
> Answer: 8772 *ko*.
>
> Solution: Put the number 2924 and make it the multiplicand. Make 3 the multiplier and multiply by this.

Multiplier	Multiplicand				
3	**2**	**9**	**2**	**4**	
	2 [×] 3	3 [×] 9	2 [×] 3	3 [×] 4	
	6	27	6	12	
	[4]	[3]	[2]	[1]	
	Obtain	8	7	7	2

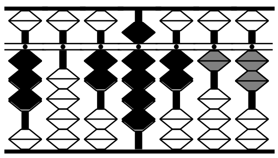
[1] Break this column and make 1. To the right column add 2.
[2] Break and clear this column. To the right column add 6.
[3] Break this column and make 2. To the right column add 7.
[4] Break and clear this column. To the right column add 6.

Solution Using Modern Soroban



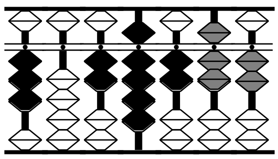
3 0 2 9 2 4 0

[1] Find the result of 4×3 (12) in the nine times nine table. Place 2 beads in the right-most empty column. Remove 4 (the last digit of the multiplicand) and use its place for the tens place of the answer. Add 1 to this.



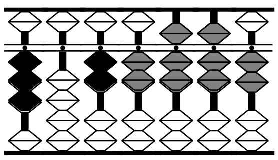
3 0 2 9 2 **1** **2**

[2] Find the result of 20×3 (60). Add 6 to the tens place of the answer.



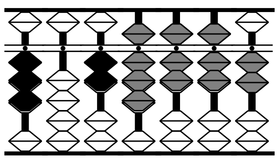
3 0 2 9 2 **7** **2**

[3] Find the result of 900×3 (2700). Remove the 2 from 29**2** – as this has already been multiplied by 3 – and replace it with 7. In a similar manner remove 9 from **29**2 and replace it with 2.



3 0 2 **2** **7** **7** **2**

[4] Find the result of 2000×3 (6000). Add 6 to the thousands place. This produces 8772.



3 0 2 **8** **7** **7** **2**

Problem 4 [Book 1: Problem 17]

The following problem from the *Taisei Sankei* deals with double digit multiplication.

> For example, there is 83 *ko*. Multiply it by 37. Find the product.
>
> Answer: 3071 *ko*.
>
> Solution: Put the number 83 and make it the multiplicand. Make 37 the multiplier and multiply by this.

Multiplier		Multiplicand			
3	**7**	**8**	**3**		
			3 [×] 3	3 [×] 7	
			9	21	
			[2]	[1]	
		3 [×] 8	7 [×] 8		
		24	56		
		[4]	[3]		
Obtain		3	0	7	1

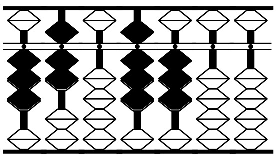
[1] At this column make 2. To the right column add 1.

[2] Break and clear this column. To the right column add 9.

[3] To this column add 5. To the right column add 6.

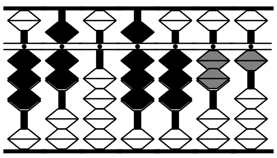
[4] Break this column and make 2. To the right column add 4.

Solution Using Modern Soroban



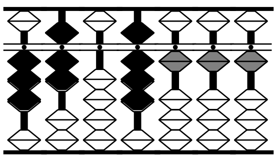
3 7 0 8 3 0 0

[1] Find the result of 3×7 (21) in the nine times nine table. Place 1 in the second empty column, which will be the ones place of the answer. Place 2 in the tens place.



3 7 0 8 3 **2** **1**

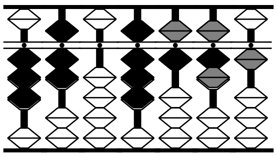
[2] Find the result of 3×30 (90). Add 9 to the tens place. As this results in 110, remove 1 bead from the tens place leaving 1. Remove the 3 of 8**3** and make this the hundreds place of the answer. Add 1 bead to this.



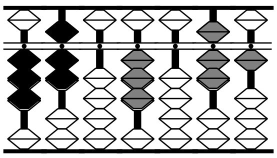
3 7 0 8 **1** **1** **1**

Rosalie Joan Hosking, Tsukane Ogawa, and Mitsuo Morimoto, "Elementary *Soroban* Arithmetic Techniques in Edo Period Japan," *MAA Convergence* (June 2018):
www.maa.org/press/periodicals/convergence/elementary-soroban-arithmetic-techniques-in-edo-period-japan

[3] Find the result of 7×80 (560). Lower the upper bead in the hundreds place. Add 6 to the tens place.



[4] Find the result of 30×80 (2400). Add 4 to the hundreds place. As this makes the value 1000, remove 8 from the thousands place and add 1 bead. Then add 2 additional beads to the thousands column. This gives 3071.



II.4 Division

II.4.1 Single Digit Numbers

As well as having a multiplication table, a single digit division table was also known in the Edo period which could be memorised and applied in a similar manner. A version of this table from the *Taisei Sankei* is shown in Table 4.

At first, when examining this table, the answers appear incorrect. For example, in division by 3 in Table 4, $3\backslash 1 = 0.31$ is given instead of $0.3333\cdots$.[8] From the first digit (of 0.31) – namely 3 – we get the quotient 0.3, and from the second digit, 1, the remainder 0.01. The rule can be applied again to the remainder, producing 0.03 as the new quotient and 0.001 the new remainder. Combined in total, the quotient becomes 0.33 and the remainder 0.001. This process can be repeated recursively to obtain the quotient $0.3333\cdots$ for as many digits as the *soroban* permits. Solutions containing repeating decimals were expressed in this manner.

II.4.2 Multiple Digit Numbers

When dividing numbers with multiple digits, two methods could be used depending on the numbers being divided. We refer to these as *Case 1* and *Case 2* division.

[8]Here n\m is equivalent to $\frac{m}{n}$ and is used to represent 'Divide *m* by *n*'.

Table 4: Single Digit Division Table

二歸
Division by 2

二一添作五	逢二進一十
2\1, make 5	2\2, advance 10

三歸
Division by 3

三一三十一	三二六十二	逢三進一十
3\1, make 31	3\2, make 62	3\3, advance 10

四歸
Division by 4

四一二十二	四二添作五	四三七十二
4\1, make 22	4\2, make 5	4\3, make 72
逢四進一十		
By 4\4, advance 10		

五歸
Division by 5

五一倍作二	五二倍作四
5\1, double 1 and get 2	5\2, double 2 and get 4
五三倍作六	五四倍作八
5\3, double 3 and get 6	5\4, double 4 and get 8
逢五進一十	
5\5, advance 10	

六歸
Division by 6

六一下加四	六二三十二	六三添作五
6\1, make 14	6\2, make 32	6\3, make 5
六四六十四	六五八十二	逢六進一十
6\4, make 64	6\5, make 82	6\6, advance 10

七歸
Division by 7

七一下加三	七二下加六	七三四十二	七四五十五
7\1, make 13 4	7\2, make 26	7\3, make 42	7\4, make 55
七五七十一	七六八十四	逢七進一十	
7\5, make 71	7\6, make 84	7\7, advance 10	

八歸
Division by 8

八一下加二	八二下加四	八三下加六
By 8\1, make 12	By 8\2, make 24	By 8\3, make 36
八四添作五	八五六十二	八六七十四
8\4, make 5	8\5, make 62	8\6, make 74
八七八十六	逢八進一十	
8\7, make 86	8\8, advance 10	

九歸
Division by 9

九一下加一	九二下加二	九三下加三
9\1, make 11	9\2, make 22	9\3, make 33
九四下加四	九五下加五	九六下加六
9\4, make 44	9\5, make 55	9\6, make 66
九七下加七	九八下加八	逢九進一十
9\7, make 77	9\8, make 88	9\9, advance 10

Case 1

Case 1 division was used for instances when the following conditions were found:

- The two digits were equal. Examples are $1 \div 1$, $2 \div 2$, and so on.
- The dividend was smaller than the divisor. Examples are $2 \div 5$, $3 \div 9$, and so on.

For these cases, a separate division table was available. The table from the *Taisei Sankei* is depicted in Table 5.

Case 1 division worked by replacing numbers using 'See' rules. For example, with $266 \div 28$, the 26 of **26**6 is less than the divisor 28. In this case, we cannot divide the number and obtain a positive integer. The 'See 2 cannot [divide], convert into 9 [add] 2' rule can then be used to convert 26 into 98, making the division now $986 \div 28$.

After a 'See' rule has been applied, the 9 which replaced the original first digit is multiplied by the second digit of the divisor. The result is then subtracted from the second (and third if necessary) number(s) of the dividend. In the case of $986 \div 28$, the 9 of **9**86 and 8 of 2**8** are multiplied and the answer (72) subtracted from the 86 of 9**86**. After this, we then have $914 \div 28$.

Division then begins again on the next number in the dividend. In our example, this will be the 1 of 9**1**4 and 2 of **2**8. In this case, the single digit multiplication table can be used. Using the '2\1, make 5' rule, the 1 is changed to 5 to produce $954 \div 28$.

Table 5: *Case 1* Division

一歸 見一無歸作九一 無除起一還一 Division by 1 See 1 cannot [divide], make 9, [add] 1 No division, minus 1, add 1	二歸 見二無歸作九二 無除起一還二 Division by 2 See 2 cannot [divide], make 9, [add] 2 No division, minus 1, add 2
三歸 見三無歸作九三 無除起一還三 Division by 3 See 3 cannot [divide], make 9, [add] 3 No division, minus 1, add 3	四歸 見四無歸作九四 無除起一還四 Division by 4 See 4 cannot [divide], make 9, [add] 4 No division, minus 1, add 4
五歸 見五無歸作九五 無除起一還五 Division by 5 See 5 cannot [divide], make 9, [add] 5 No division, minus 1, add 5	六歸 見六無歸作九六 無除起一還六 Division by 6 See 6 cannot [divide], make 9, [add] 6 No division, minus 1, add 6
七歸 見七無歸作九七 無除起一還七 Division by 7 See 7 cannot [divide], make 9 [add] 7 No division, minus 1, add 7	八歸 見八無歸作九八 無除起一還八 Division by 8 See 8 cannot [divide], make 9 [add] 8 No division, minus 1, add 8
九歸 見九無歸作九九 無除起一還九 Division by 9 See 9 cannot [divide], make 9, [add] 9 No division, minus 1, add 9	

Next the 5 in 9**5**4 is multiplied by the 8 in 2**8** and the product (40) subtracted to give 950. As the divisor is a two digit number, a decimal point is placed after the last 2 digits in the answer, giving the solution 9.5.

In cases where the product of multiplying the digit from the dividend and divisor cannot be subtracted, the 'Cannot [complete] division, minus 1 add [*number*]' rule is used. This decreases the quotient's value by 1 and adds the original value of the quotient to the following number. The new value of the first digit is then multiplied by the number in the divisor. This process is repeated until a result is found that can be subtracted from the dividend.

There are instances where this rule causes the second digit to become equal to or greater than 10. For example, say we want to calculate $420 \div 49$. First the 'See 4, cannot [divide], convert into 9 [add] 4' rule is used to change the 4 into 9, and to add 4 to the next digit giving 960. Next, multiplying the quotient by the second digit in the divisor gives $9 \times 9 = 81$, which cannot be subtracted from the 60 of 9**60**. In this case the 'Cannot [complete] division, minus 1 add 4' rule can be applied. However, $6 + 4 = 10$, meaning the result is **810**0.

As mentioned in section I.1, *soroban* had two upper beads and five lower beads in the Edo period. This meant values between 10 and 15 could be represented within one column by utilising two upper beads. Unfortunately this process cannot be replicated on the modern Japanese *soroban*. Due to this, we use examples from the *Taisei Sankei* which can be reproduced without this issue.

Case 2

Case 2 division is used for cases when *Case 1* division does not apply. For *Case 2*, the single digit division table is used to complete division. Both *Case 1* and *Case 2* division can be used in the same problem as necessary.

II.4.3 Traditional Problems

Problem 5 [Book 1: Problem 24]

For example, there is 2586 *ko*. Divide this by 2. Find the quotient.

Answer: 1293 *ko*.

Solution: Put the number 2586 and make it the dividend. Make 2 the divisor and divide by this.

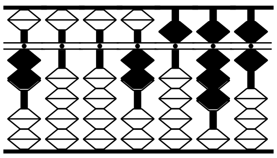
Divisor	Dividend				
2		**2**	**5**	**8**	**6**
		2\2, advance 10	2\4, advance 20	2\8, advance 40	2\6, advance 30
		[1]	[2]	[4]	[5]
			2\1, make 5		
			[3]		
	Obtain				
	1	2	9	3	

[1] This column is cleared. To the left column add 1.

[2] Clear 4 from this column. To the left column add 2.

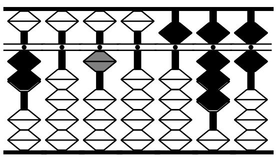
3 In this column lower 5 and remove 1.
4 Clear this column. To the left column add 4.
5 Clear this column. To the left column add 3.

Solution Using Modern Soroban



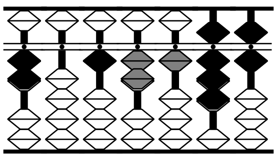
2 0 0 2 5 8 6

1 Find the result of $2 \div 2$ (1). Remove the 2 from the thousands place and place 1 in the ten thousands column.



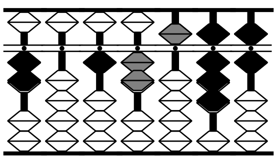
2 0 **1** 0 5 8 6

2 Instead of dividing 5 by 2, the number 4 is used. The result of $4 \div 2$ is 2. Remove 4 from the hundreds place and add 2 to the thousands place to make 1**21**86.



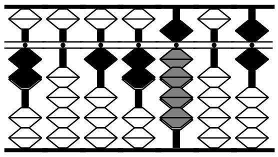
2 0 **1** **2** **1** 8 6

3 In 2 the number 4 was used instead of 5, which gives a remainder of 1 (as $5 - 4 = 1$). Next this remainder is divided by 2, which – from the division table – produces 0.5. Lower the upper bead in the hundreds column and remove one lower bead to make 1**25**86.



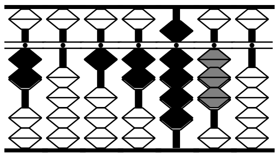
2 0 **1** **2** **5** 8 6

[4] Find the result of $8 \div 2$ (4). Remove 8 from the tens column and add 4 to the hundreds column to make 12**906**.



2 0 **1** **2** **9** **0** 6

[5] Find the result of $6 \div 2$ (3). Remove 6 from the ones column and add 3 to the tens column to create 12930. Remove the last 0 to produce 1293.



2 0 **1** **2** **9** **3** 0

Problem 6 [Book 1: Problem 32]

For example, there is 12720 *ko*. Divide this by 16. Find the quotient.

Answer: 795 *ko*.

Solution: Put the number 12720 and make it the dividend. Make 16 the divisor and divide by this.

Divisor		Dividend				
1	**6**	**1**	**2**	**7**	**2**	**0**
		See 1 make 9, add 1	7 [×] 6			
			42			
		[1]	[3]	9 [×] 6		
				54		
		Minus 2, add 2	See 1 make 9, add 1	[5]		
		[2]	[4]	1\5, advance 50	6 [×] 5	
					30	
				[6]	[7]	
		Obtain				
		7	9	5		

[1] In this column add 8 to make 9. To the right column add 1.

[2] In this column clear 2. To the right column add 2.

[3] In this column clear 4. In the right column clear 2.

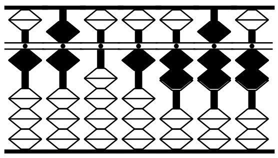
[4] In this column add 8. In the right column add 1.

5 In this column clear 5. In the right column clear 4. Remove 10 and return 6.

6 In this column add 5. In the right column clear 5.

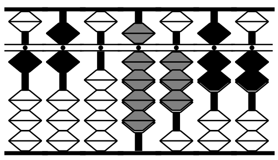
7 Clear this column.

Solution Using Modern Soroban



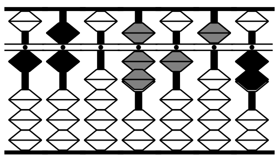
1 6 0 1 2 7 2

1 Since the first two digits of each number are equal, apply the 'See 1 make 9, add 1' rule from the *Case 1* division table.



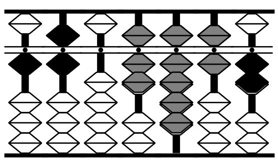
1 6 0 **9** **3** 7 2

2 Multiply 9 and the 6 of 1**6**. The result (54) is too large to subtract from the 37 of 9**37**2. Apply the 'Cannot [complete] division, minus 1 add 1' rule, which gives 8472. As $8 \times 6 = 48$, which also cannot be removed (from 47), apply the rule again giving 7572. This time $7 \times 6 = 42$, which can be removed from 57.



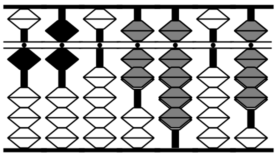
1 6 0 **7** **1** **5** 2

3 With the second digit of 7152, the dividend and divisor are both equal (being 1). Apply the 'See 1 make 9, add 1' rule to produce 7962.



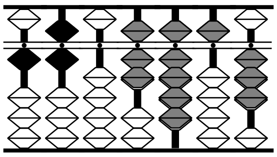
1 6 0 **7** **9** **6** 2

4 Multiply the 9 of 7**9**62 with the 6 of 1**6**. The result (54) is subtracted.



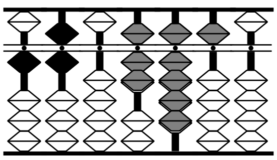
1 6 0 **7 9 0 8**

5 As $8 = (0.5 \times 10) + 3$, and $3 = (0.5 \times 6)$, converting the 08 of 79**08** into 79**53** allows for division to be completed. To do this, trial and error can be used to determine that adding 5 to the first number and subtracting 5 from the second will complete the division. Lower the upper bead and add 5 to the 0 and raise the upper bead to subtract 5 from the 8. This produces 7953.



1 6 0 **7 9 5 3**

6 Now as $5 \times 6 = 30$, the 3 from 795**3** can be removed, finishing the division. Moving the decimal place forward two places from the last digit of 7950 gives an answer of 79.5.



1 6 0 **7 9 5 0**

Problem 7 [Book 1: Problem 33]

> For example, there is 14514 *ko*. Divide this by 246. Find the quotient.
>
> Answer: 59 *ko*.
>
> Solution: Put the number 14514 and make it the dividend. Make 246 the divisor and divide by this.

Divisor			Dividend				
2	**4**	**6**	**1**	**4**	**5**	**1**	**4**
			2\1, make 5	5 [×] 4	6 [×] 5		
			[1]	20	30		
				[2]	[3]		
				See 2, make 9, add 2	9 [×] 4	9 [×] 6	
				[4]	36	54	
					[5]	[6]	
			Obtain				
			5	9			

[1] In this column lower 5 and remove 1.

[2] In this column clear 2.

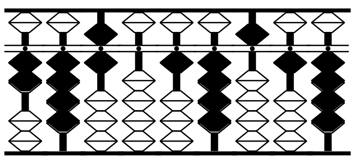
[3] In this column raise 2 and clear 5.

[4] In this column add 7. To the right column add 2.

[5] In this column clear 3. In the right column, clear 6.

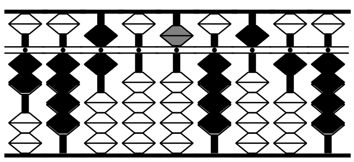
[6] In this column clear 5. In the right column clear 4.

Solution Using Modern Soroban



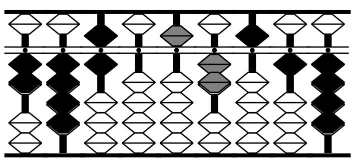
2 4 6 0 1 4 5 1 4

[1] Find the result of $1 \div 2$ in the division table (0.5). Subtract 1 from **1**4514 and replace it with the upper bead.



2 4 6 0 **5** 4 5 1 4

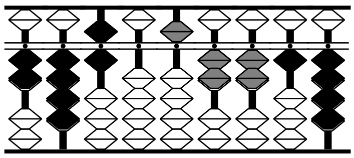
[2] Multiply the 5 of the ten thousands place by the 4 in 2**4**6. Subtract the result (20).



2 4 6 0 **5** **2** 5 1 4

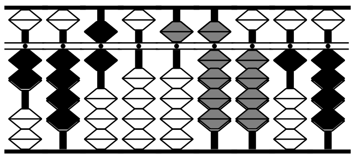
Rosalie Joan Hosking, Tsukane Ogawa, and Mitsuo Morimoto, “Elementary *Soroban* Arithmetic Techniques in Edo Period Japan,” *MAA Convergence* (June 2018):
www.maa.org/press/periodicals/convergence/elementary-soroban-arithmetic-techniques-in-edo-period-japan

3 Multiply the 5 in the ten thousands place by the 6 in 24**6**. Subtract the result (30) from the number.



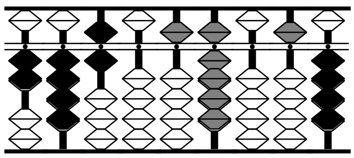
2 4 6 0 **5** **2** **2** 1 4

4 From the *Case 1* division table, when seeing the 2 of 5**2**214, convert it into 9 and add 2 to the following number.



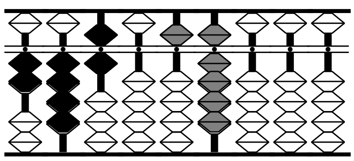
2 4 6 0 **5** **9** **4** 1 4

5 Multiply the 9 in the thousands place by the 4 in 2**4**6. Subtract the result (36).



2 4 6 0 **5** **9** **0** **5** 4

6 Multiply the 9 in the thousands place by the 6 in 24**6**. Subtract the result (54). As no numbers are left in the original number, this concludes the division producing 59.



2 4 6 0 **5** **9** **0** **0** **0**

III Exercises

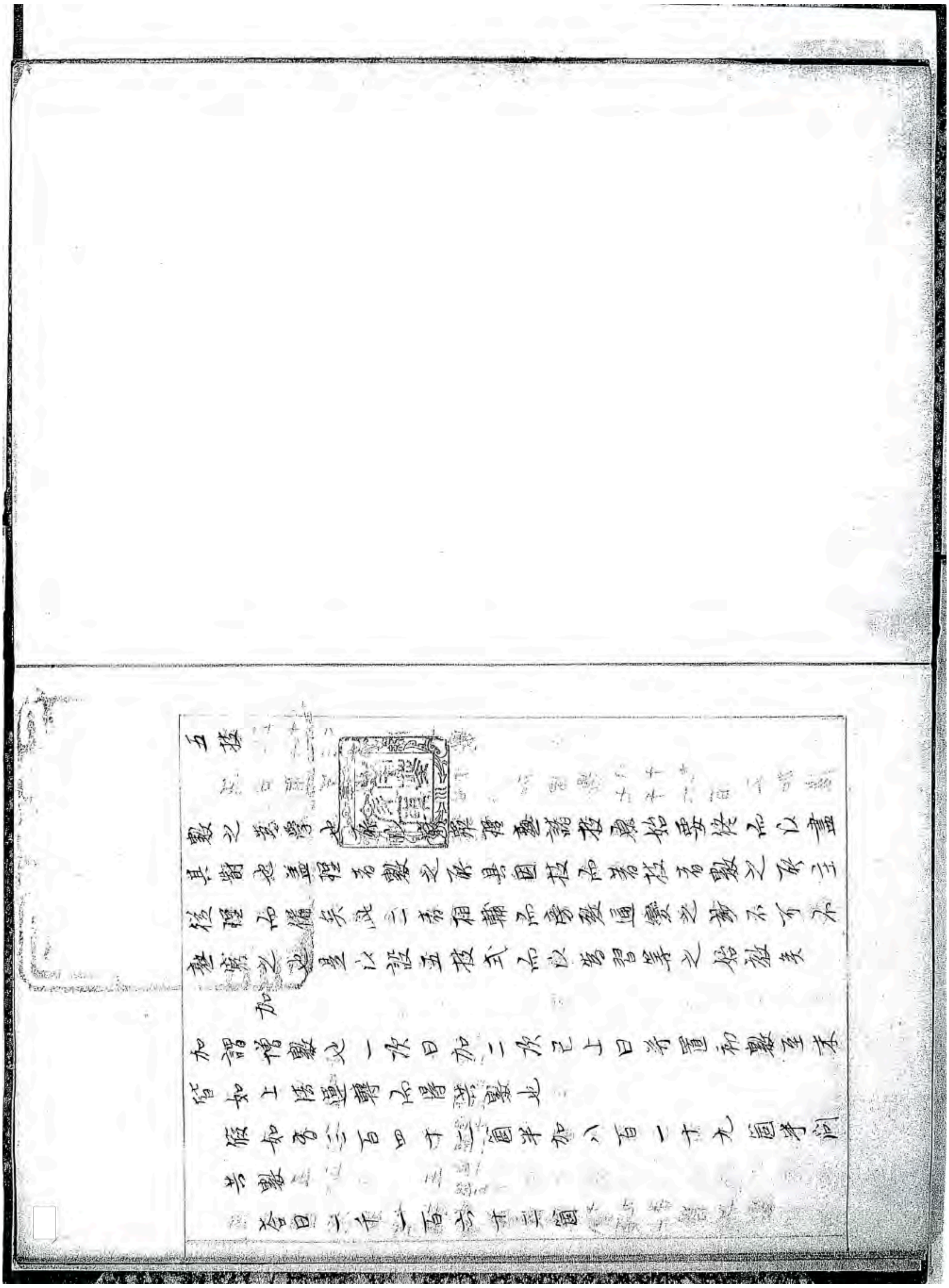
The following exercises have been taken from the *Taisei Sankei* and can be calculated either using a *soroban* or by writing down the numbers that would be represented on the *soroban* on paper if one is not available. Answers are given at the end of this article.

Rosalie Joan Hosking, Tsukane Ogawa, and Mitsuo Morimoto, "Elementary *Soroban* Arithmetic Techniques in Edo Period Japan," *MAA Convergence* (June 2018):
www.maa.org/press/periodicals/convergence/elementary-soroban-arithmetic-techniques-in-edo-period-japan

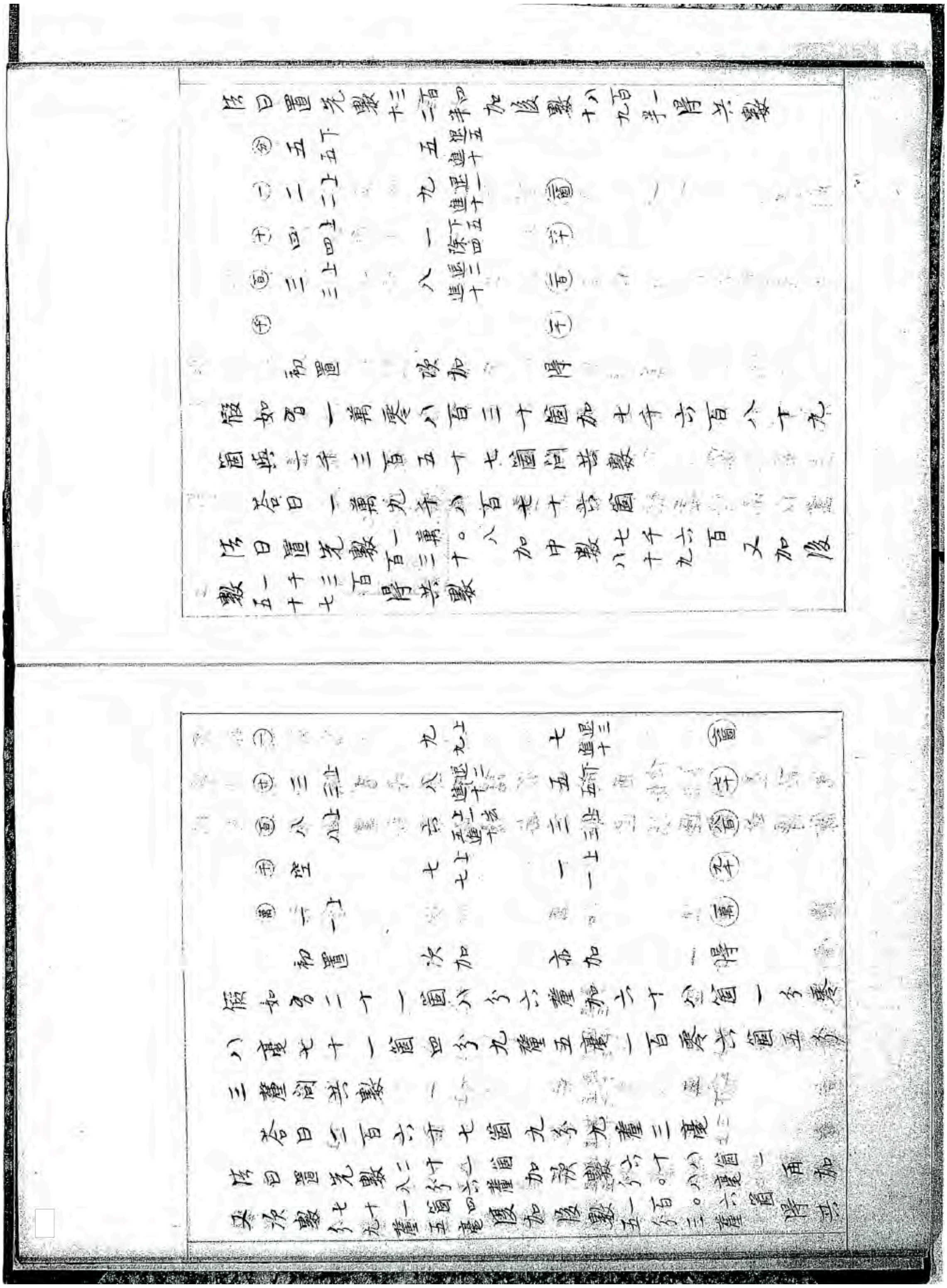
1. There is 10830 *ko* plus 7689 *ko* and 1357 *ko*. Find the total.
2. For example, there is 1083 *ko*. Subtract 514, and 27, and also 146. Find the remainder.
3. There is 2276 *ko*. Multiply this by 4. Find the product.
4. There is 795 *ko*. Multiply this by 16. Find the product.
5. There is 9104 *ko*. Divide this by 4. Find the quotient.
6. There is 3348 *ko*. Divide this by 465. Find the quotient.
7. There is 6174 *ko*. Divide this by 63. Find the quotient.

IV Original Source Material

The following images from the University of Tokyo collection contain some original pages from the *Taisei Sankei* used for this work.



Rosalie Joan Hosking, Tsukane Ogawa, and Mitsuo Morimoto, "Elementary *Soroban* Arithmetic Techniques in Edo Period Japan," *MAA Convergence* (June 2018): www.maa.org/press/periodicals/convergence/elementary-soroban-arithmetic-techniques-in-edo-period-japan



Acknowledgement

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Morimoto, Mitsuo, 2013. "Three Authors of the Taisei Sankei" in *Journal for History of Mathematics* Vol. 26, No. 1, 11–20.

Problem Solutions

1: 19876 *ko*, 2: 396 *ko*, 3: 9104 *ko*, 4: 12720 *ko*, 5: 2276 *ko*, 6: 7.2 *ko*, 7: 98 *ko*.