• HISTORICALLY SPEAKING,—

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

The quadrature of the parabola: an ancient theorem in modern form

Contributed by Carl Boyer, Brooklyn College, Brooklyn, New York

The quadrature of the parabola by Archimedes is one of the best-known of the classics of the history of mathematics. His familiar squaring of the parabola depends on the following property: Let P_1, P_2, P_3, P_4, P_5 be points on the parabola $y^2=2px$ such that P_2M_2, P_3M_3, P_4M_4 are parallel to the axis, where M_2 is the midpoint of P_1P_3, M_4 is the midpoint of P_3P_5, and M_3 is the midpoint of P_1P_5. Then $\triangle P_1P_2P_3 + \triangle P_3P_4P_5 = \frac{1}{4}\triangle P_1P_3P_5$. Numerous demonstrations of this property are available; but the following proof seems to be more expeditious and should serve as an appropriate classroom exercise in analytic geometry.

Making use of determinants, we have

$$\triangle P_1P_2P_3 = 1/2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{4p}\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{64p}\begin{vmatrix} y_1^2 & y_1 & 1 \\ (3y_1+y_5)^2 & 4(3y_1+y_5) & 16 \\ y_3^2 & y_3 & 1 \end{vmatrix},$$

using the equation of the parabola $y^2=2px$ and the point-of-division formula

$$y_2=\frac{3y_1+y_5}{4}.$$

Remembering the midpoint formula

$$y_3=\frac{y_1+y_5}{2},$$

one subtracts from the elements of the second row twelve times the corresponding elements of the last row (in order to eliminate the cross-product term in the first column) and six times the corresponding elements of the first row, obtaining

$$\triangle P_1P_2P_3=\frac{1}{64p}\begin{vmatrix} y_1^2 & y_1 & 1 \\ -2y_5^2 & -2y_5 & -2 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{32p}\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_3^2 & y_3 & 1 \\ y_5^2 & y_5 & 1 \end{vmatrix}$$

$$=\frac{1}{16}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}$$

$$=\frac{1}{8}\triangle P_1P_3P_5.$$

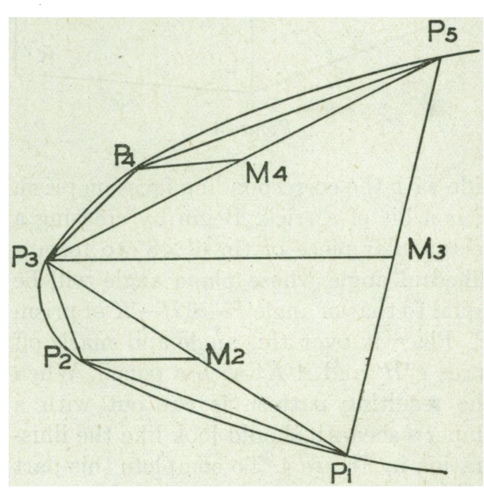

Figure 1

Noting that the area of $\triangle P_3P_4P_5$ is obtained by interchanging the subscripts 1 and 5 where they occur above, one has $\triangle P_3P_4P_5 = \triangle P_1P_2P_3$; and hence $\triangle P_1P_2P_3 + \triangle P_3P_4P_5 = \frac{1}{4}\triangle P_1P_3P_5$.

Continuing the subdivision process in the usual Archimedean manner, and applying the above property recursively, one obtains the classic result—that the parabolic segment $P_1P_3P_5$ is given by $\triangle P_1P_3P_5\,(1+\frac{1}{4}+1/16+\cdots\frac{1}{4}n+\cdots) = 4/3\,\triangle P_1P_3P_5$.

Editor's Note: On the chance that some of our readers are not as familiar with Archimedes' (230 B.C.) work as Professor Boyer assumes, we point out that this procedure uses the "method of exhaustion" credited to Eudoxus (350 B.C.) and so named by Gregory St. Vincent in 1647. Archimedes used only implicitly the limit of the sum of an infinite geometric progression. Both area and series problems are related to the calculus. The discovery in 1906 of Archimedes' lost work on *The Method* revealed how amazingly close to modern calculus he had been in the procedures he used to discover these results which he then recast into a classical and acceptable "method of exhaustion" form before "publishing" them to his contemporaries. Further details on these procedures and techniques would be found in Sir T. L. Heath's *Works of Archimedes*, recently reprinted by Dover Publications, or Professor Boyer's own *Concepts of the Calculus* (New York: Hafner Publishing Co.).—PHILLIP S. JONES.

Geometric progressions in America and Egypt

Contributed by Norman Anning, 909 Mt. View Terrace, Alhambra, California

Professor Anning sends us the page from a student's copy book which is shown in Figures 2 and 3. The book was titled:

Timothy Street's

Book

Middle Road, Trafalgar, Feb. 17th, 1841

Professor Anning adds the following comments:

Timothy Street was a distant relative.

I believe "Trafalgar" was about 30 miles northwest of Toronto, C.W. (Canada West).

Please overlook the pounds, shillings, and pence (Canadians still computed in £.s.d., but then, as now, were happy to accept Yankee $$$) and observe the clumsy way in which the student finds the "some" of a geometrical progression

$$1+4+4^2+\cdots+4^{11}.$$

He uses a formula, old as Ahmes, I believe, to sum eleven terms and then adds on the twelfth term.

Over the page is the old problem about the "nales" in the shoes of a "hors"; "there were four shoes"! The student blunders in computing the value of 3^{11} but still arrives at the *correct* price of the horse! No comment.

For a check:

$$1+3+3^2+\cdots+3^{31}=\frac{3^{32}-1}{3-1}$$

$$2\,\underline{|\,1853020188851840}$$
$$926510094425920$$

Note that $3^{32}=81^8$ and can be lifted from page 255 of the 1941 edition of *Barlow's Tables*.

Editor's Note: Professor Anning no doubt refers to Problem 79 of the Ahmes (or Rhind) Papyrus which dates back to about 1550 B.C. This problem arrives at the same result by two different procedures. One column translated reads

houses	7
cats	49
mice	343
spelt	2401
hekat	16807
total	19607

There have been several interpretations of this, but a favorite one is that each of seven houses had seven cats, each of which caught seven mice which would have eaten grain. The story behind the last two quantities is less clear (a hekat is a measure of grain). A similar problem appears in