Name ____________________

Approximating Square Roots with Linear Algebra

In about 430 BCE, Greek geometers proved that $\sqrt{2}$ is irrational - that it cannot be exactly expressed as a ratio of whole numbers. About 500 years later, in 100 CE, Theon of Smyrna outlined an iterative method to *approximate* $\sqrt{2}$ by a rational number. (The ancient city of Smyrna is now Izmir, Turkey.) Today we'll explore Theon's method and its connection to linear algebra, and we'll adapt it to find roots of other numbers.

1. Before we start, what is the value of $\sqrt{2}$ (to at least 6 decimal places)?

Here is Theon's iterative method. We think of x and y as two sides of a triangle, and we start with $x_0 = 1$, $y_0 = 1$. Our estimate of $\sqrt{2}$, which is the ratio $\frac{y_n}{x_n}$, thus starts as 1. To get a better estimate we let $x_1 = x_0 + y_0$ and $y_1 = 2x_0 + y_0$. Thus we get that $x_1 = 2$ and $y_1 = 3$, so our new estimate is $\frac{3}{2} = 1.5$. The recursive formula is

$$x_{n+1} = x_n + y_n \qquad y_{n+1} = 2x_n + y_n.$$

2. Fill in the rest of this table to see how well Theon's method works.

n	x_n	y_n	y_n/x_n
0	1	1	1
1	2	3	1.5
2			
3			
4			

3. Does our method appear to be doing a good job of approximating $\sqrt{2}$?

4. Theon's recursive method can be thought of in terms of matrix multiplication.

$$\begin{aligned} x_{n+1} &= x_n + y_n \\ y_{n+1} &= 2x_n + y_n \end{aligned}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

Fill in the entries of the matrix A. Have this answer checked - it's important!

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5. So, another way to think about this problem is as follows. Since

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = A \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

then

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = AA \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = A^2 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Conjecture the formula for $\begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$. Check this idea by computing A^3 and then multiplying by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The result should agree with the table.

6. In general

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

So if we can find powers of the matrix A, then we can determine x and y quite easily. One way to find powers of a matrix involves eigenvalues. Find the eigenvalues and associated eigenvectors for $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$. Call the larger eigenvalue λ_1 and the smaller one λ_2, and the associated eigenvectors $\vec{w}_1$ and $\vec{w}_2$.

7. If $\vec{w} = 3\vec{w}_1 + 2\vec{w}_2$, what is $A\vec{w}$? Give your answer just in terms of $\vec{w}_1$ and $\vec{w}_2$.

8. Open the visualization tool provided by your instructor. Describe what happens to an arbitrary starting vector as you multiply by A repeatedly. What happens in the long run?

   (Bonus: does this happen for **all** starting vectors?)

9. The big question is to put together why Theon's method works to give an approximation of $\sqrt{2}$. The eigenvectors and the behavior seen above can demonstrate this. Any starting vector can be written as a linear combination of the eigenvectors $\vec{w}_1$ and $\vec{w}_2$. What happens to a multiple of $\vec{w}_1$ when you multiply by the matrix A repeatedly?

   What happens to a multiple of $\vec{w}_2$ when you multiply by the matrix A repeatedly?

10. Put together these ideas to give an explanation of why an arbitrary starting vector (with positive components) will tend to a result $\begin{bmatrix} x_n \\ y_n \end{bmatrix}$ whose ratio $\dfrac{y_n}{x_n}$ approaches $\sqrt{2}$.

11. Okay, so what if we wanted to approximate $\sqrt{3}$ instead of $\sqrt{2}$? Let's guess that the matrix $B = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$ will do the trick. Start with the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and see if this seems to work to give an approximation of $\sqrt{3}$. Show some of your calculations here.

12. Use eigenvalues and eigenvectors to provide a convincing argument that this method will converge to $\sqrt{3}$.