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Indivisibles and the Cycloid in the Early 17th Century

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At the heart of the differential and integral calculus lie the notions of the infinite and the infinitely small. Whether characterized by the continuous, the instantaneous, or the infinite in quantity, whether described vaguely in the 17th century [2, 200-202] or rigorously in the 19th , one will seldom see a concept in the calculus which fails to invoke these fundamental notions. Although the historical development of ideas of the infinite was plagued with skepticism and doubts about mathematical arguments which utilized them, owing to the seeming incomprehensibility inherent to the meaning of "infinite," there have nevertheless been a few mathematicians who momentarily withheld their suspicions and applied their intellect to the massive concept. Archimedes, for example, considered the infinite and the infinitesimal "as suggestive heuristic devices, to be used in the investigation of problems concerning areas and volumes which were preliminary to the intuitively clear and logically rigorous proofs given in the classical geometrical method of exhaustion" [2, 96]. Indeed, it was perhaps Archimedes' relatively lax views towards the infinite (along with his genius) which allowed him to exploit so well the method of exhaustion (a classical technique developed by Eudoxus which determined areas of non-polygonal regions by inscribing in them polygons with an ever-increasing number of sides, thus finding their areas by "exhausting" the space of that region [8, 84-85]) in determining curvilinear areas, volumes, surfaces, and arcs—works of his which have "frequently been referred to as genuine integration" [1, 34] ("integration" in this context refers to techniques for finding areas of curved regions). As Carl B. Boyer describes in [2], the infinite, and related concepts such as the continuum, began to be discussed and used more freely during the later Middle Ages, albeit in a much more metaphysical and scientific setting. However, mathematicians of the late 15th and 16th centuries rejected these Scholastic and Aristotelian views and instead attempted to reconcile them with the views of Archimedes. This attempt at reconciliation during the 16th century, mixed with influences from Hindu and Arabic algebra, as well as a gradual acceptance of new kinds of numbers (irrational, negative, imaginary), allowed for and helped to lead to the eventual development of various methods of "indivisibles" [2, 96-98], with which this paper will concern itself. In particular, I want to pay special attention to the application of methods of indivisibles to finding the area under one arch of a curve known as the "cycloid"—the curve traced by a point fixed on the circumference of a circle which rolls on a horizontal line (figure 1).

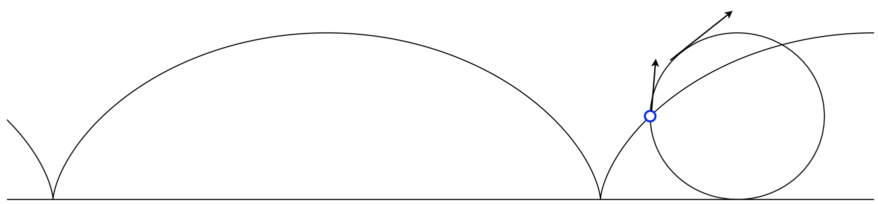

Figure 1; generation of the cycloid curve

The mathematicians whose solutions to this problem we will observe in depth are Gilles Personne de Roberval (1602-1675) and Pierre de Fermat (1601-1665), but it will first be necessary to introduce the mathematician who popularized indivisibles in the early 17th century, Bonaventura Cavalieri (1598-1647), and briefly explain his method for using them.

Geometria indivisibilibus continuorum nova quadam ratione promota (1635) and *Exercitationes geometricae sex* (1647), both by Cavalieri, rapidly became the most quoted sources on geometric integration (again, "integration" refers to techniques for finding areas under curved regions) in the 17th century [1, 122-123], and it is in these two texts in which Cavalieri develops and applies his method of indivisibles. It should be noted that Cavalieri never explains precisely what he understands by the word "indivisible," but he conceives of a surface as being composed of an indefinite number of parallel equidistant lines, and a solid as being composed of an indefinite number of parallel equidistant planes (figure 2), and it is these lines and planes which he called the "indivisibles" of the surface and of the volume, respectively [2, 117].

Figure 2; Cavalieri's conception of "indivisibles" [1, 124]

Cavalieri held a view towards the infinite which could be described as "agnostic" [2, 117], and in fact he held that his use of the infinite and the indivisible was purely auxiliary, similar to the "sophistic" quantities used by Cardano in his solving of the cubic equation; "inasmuch as it did not appear in the conclusion, its nature need not be made clear" [2, 118]. This is to say that, although he could not precisely explain their nature, the "indivisibles" of Cavalieri's arguments played the role of a useful tool or "blackbox," and his conclusions were not concerned with "indivisible" facts, but rather with *geometrical* facts. Cavalieri's method worked by establishing ratios between the individual indivisibles of one figure to those of another, and then from these ratios he developed relations between the areas (or volumes) of the figures (figure 3).

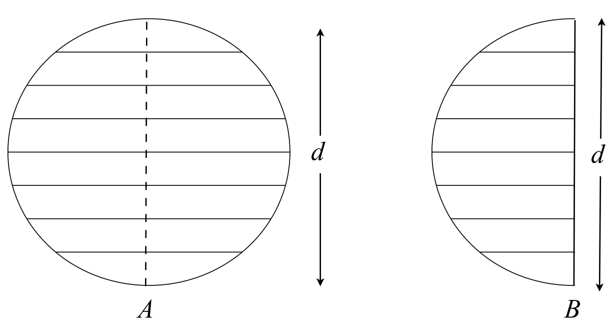

Figure 3; at any given height, the "indivisibles" of circle A are twice the length of those of semicircle B, and we conclude that area(circle(A)) = 2 area(semicircle(B))

The foundation of Cavalieri's method of indivisibles rests upon two complimentary notions: the *collective* and the *distributive*. In the first notion, the *collective* sums, $\Sigma\, l_1$ and $\Sigma\, l_2$ of line (or surface) indivisibles (note: 'Σ' is used here to denote a collection of indivisibles; it does *not* refer to a numeric sum), of two figures P_1 and P_2 are obtained separately and then used to obtain ratios of the areas (or volumes) of the figures themselves [1, 125]. The second, *distributive*, notion was a concept which Cavalieri developed primarily to defend against anticipated philosophic objections to the comparison of indefinite numbers of lines and planes [2, 126]. It is used to tell us *in what way* we may compare this collection of indivisibles. The two concepts are best summarized in what is commonly known as *Cavalieri's Theorem* (for solids), which essentially states:

> "If two solids have equal altitudes, and if sections made by planes parallel to the bases and at equal distances from them are always in a given ratio, then the volumes of the solids are also in this ratio" [2, 118].

I have mentioned Cavalieri's method because it plays a large underlying role in the following arguments from Roberval and Fermat in determining the area under one arch of the cycloid curve.

Let us first introduce the cycloid curve itself. As I have mentioned, and as can be seen in figure 1, the cycloid is that curve which is traced by a point fixed upon a circle which itself rolls along a horizontal line. The cycloid possesses many peculiar properties, but here we will only demonstrate that the area under one arch of the curve is equal to three times that of the circle which generated it.

The cycloid was first referred to Roberval by Marin Mersenne (1588-1648) in a letter from 1628 [10, 167]. In it, Mersenne suggested that the curve might look like a semi-ellipse. It would not be until a letter dated January 6th, 1637, almost 9 years later, that Roberval would provide Mersenne with a sample, but not a full dossier, of his long-awaited work on the cycloid (Roberval was extremely secretive with his methods and results, which he attributed to an examination held every three years to retain his position of the Chair of Ramus at the Collège Royal [2, 140]). This letter to Mersenne was expanded in Roberval's *Traité des indivisibles,* which he seems to have written sometime between 1634 and the end of 1636, though it was not published until 1693 in *Divers ouvrages de mathématiques et de physiques*, by the French Académie des Sciences. In it, Roberval explains how to construct the curve, how to find tangents to the curve, how to find the area under one arch of the curve, and how to find the volume of the solid produced by rotating one arch of the curve about the horizontal line of its base.

As a preliminary in *Traité des indivisibles*, Roberval outlines his conception of indivisibles, which is very similar to that of Cavalieri, except for the fact that Roberval considered a line as being made up not of an infinity of points, but of an infinity of little lines; likewise he held the surface to be composed not of an infinity of lines, but of an infinity of little surfaces, and so on in any given dimension [9, 190-191]. Notice that in Roberval's conception of indivisibles, he supplies "the essential element found in our conception of the definite integral, in that, after dividing a figure into small sections, he allowed these continually to decrease in magnitude, … the result being obtained by summing an infinite series" [2, 142]. Thus, even though he does not rigorously define this concept, Roberval avoids the logical hole of considering a line as being made up of elements which have no length (for then, how could such a line have length?), a surface as being made up of elements which have no width (likewise, how could such a surface have area?), etc., which is an inconsistency of Cavalieri's conception of indivisibles. That being said, however, their two methods are actually very similar.

Roberval's solution to finding the area under one arch of the cycloid curve.

[This text is taken from the original *Traité des indivisibles*, found in the 1693 *Divers ouvrages…*, and translated from the French by me; all comments in [square brackets] are mine.]

We pose that the diameter AB of the circle AEFGB is driven in parallel to itself, as if it were carried by some other body, until arriving at CD to complete the semi-circle or half-turn.

> [See figure 4. He imagines diameter AB moving right to a line segment CD equal and parallel to AB such that the rectangle ABCD is formed, where sides AC and BD are each of length equal to "the semi-circle or half-turn," meaning one half of the circumference of the circle AEFGB. So, if our circle AEFGB is of radius r, then AC is of length πr, and AB and CD are each of length 2r. Let circle AEFGB have center X.]

While it [AB] walks on [as AB moves along AC towards CD], the point A at the extremity of the said diameter [AB] goes through the circumference of the circle AEFGB, and makes its way as

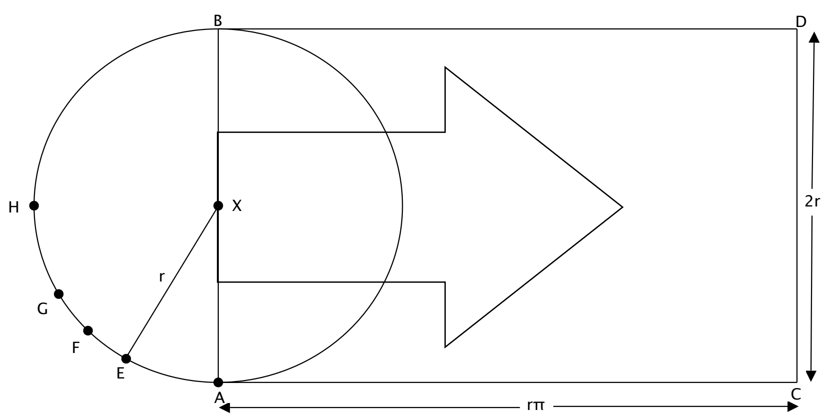

Figure 4; preliminary diagram

much as the diameter [makes its way], so that when the diameter is on CD, the point A has come to [rolled up to] B, and the line AC finds itself equal to the [semi-]circumference AGHB.

> [So, Roberval begins his construction of one half of one arch of the actual cycloid curve by supposing that, as diameter AB moves towards CD and the circle rolls towards the right, the point A also moves around the

circumference of the circle AEFGB (clockwise), a distance equal to that travelled by the diameter AB in its horizontal movement towards CD. Notice that Roberval does not yet let the point A make its horizontal movement towards CD. This relationship is given as follows: as A moves through an angle of θ along the circumference of AEFGB while the circle moves rightward, A moves a distance of (arc length) $r\theta$, and AB simultaneously moves a distance of $r\theta$ along AC towards CD (figure 5), so that the arc length traversed by A and the horizontal movement of AB towards CD are each $r\theta$ from the original position of A.]

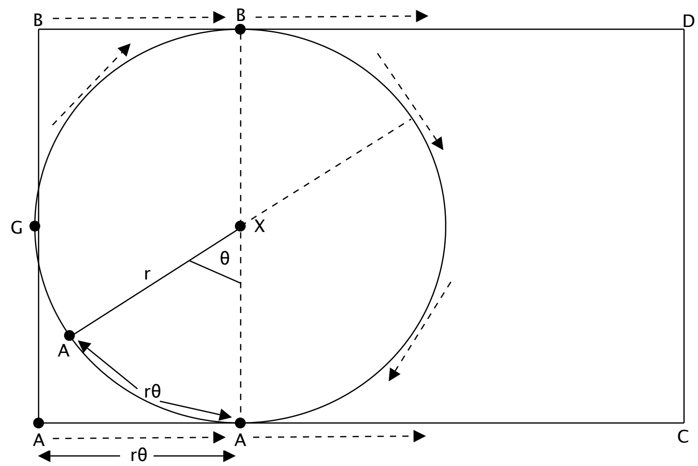

Figure 5; the relationship between the different movements of the point A

Thus this course [AC] of the diameter is divided into parts infinite and equal as well between them as to each part of the circumference AGB, which is divided also into infinite parts each equal between them and to the parts of AC traversed by the diameter, as has been said.

[Roberval characterizes "continuous" movement by saying that as AB travels towards CD, the "course of the diameter" AC is divided into tiny, "indivisible" segments which are equal to one another as well as equal to similar tiny, "indivisible" segments of the semi-circumference AGB (Figure 6). Roberval makes this observation so that later in the solution we may make comparisons between the "indivisibles" of the circumference AGB with those of the line AC.]

In afterwards I consider the path which the aforementioned point A made carried by two movements, the one from the diameter in front, the other from its own on the circumference.

[Now he begins to describe the actual cycloid curve.]

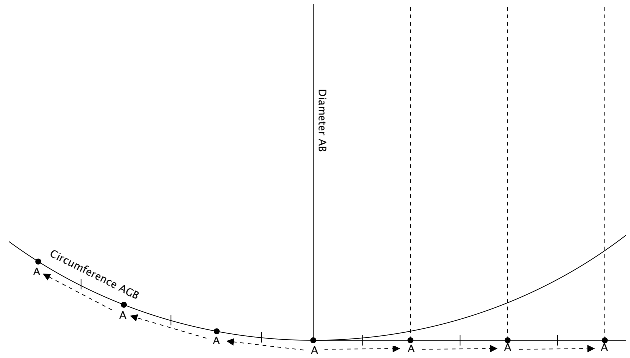

Figure 6; the horizontal movement of AB towards C, as well as the movement of A along the semi-circumference AGB (see figure 4), are characterized by infinitely small ("indivisible") segments of the line AC and of the semi-circumference AGB which are described by A at equal intervals, so that the segments are all equal

To find the said path, I see that when it [the point A] has come to E [a point on the semicircle; see figure 7] it is lifted above its first place from which it left; this height is marked drawing from the point E [perpendicular] to the diameter AB a [distance] sine E_1, and the sinus verse A_1 is the height of A when it has come to E.

[See figure 7. Here "sine E_1" is the length of the half-chord from E at angle E_1 on the circle to a point labeled "1" (Roberval's notation) on the circle's diameter AB. In modern trigonometry, we call this distance "$r\sin(E_1)$." Also, "sinus verse A_1" is given today by $r - r\cos(E_1)$.]

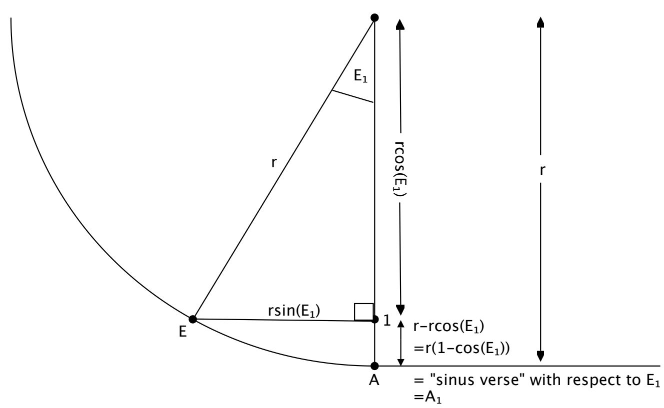

Figure 7; how to find the height of the cycloid curve

The same when it has come to F, from the point F on AB I draw the sine F_2, and A_2 will be the height of A when it has done two portions of the circumference, and drawing the sine G_3, the sinus verse A_3 will be the height of A when it has arrived at G; and doing thus from all places of the circumference that A traverses, I find all its heights and elevations over the extremity of the diameter A [AB], which are A_1, A_2, A_3, A_4, A_5, A_6, A_7 [see figure 8];

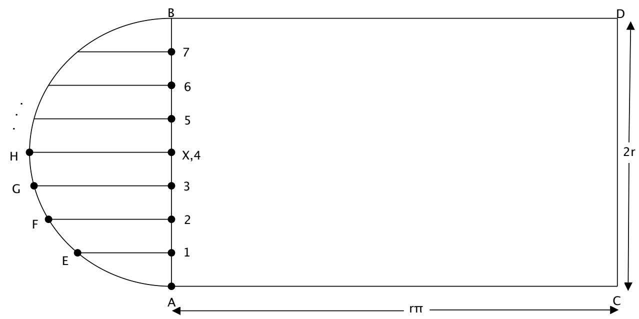

Figure 8; heights of the cycloid curve; each of the points "2," "3," ... , and each of the heights A_2, A_3, ..., is determined in the same way as the point "1" and the height A_1 (see figure 7); note: the points X and "4" are not necessarily the same

thus, in order to have places whereby pass [both] the said point A, and to-wit the line that it forms during its two movements [the cycloid], I carry all of its heights on each of the diameters M, N, O, P, Q, R, S, T, and I find that M_1, N_2, O_3, P_4, Q_5, R_6, S_7 are the same [heights] as those taken on AB.

[See figure 9. Roberval is actually describing the curve which he elsewhere called the *Companion* curve to the cycloid [1, 157]. The "diameters M, N, …, T," are the line segments parallel and equal in length (2*r*) to the diameter AB which intersect AC at the points M, N, …, T. Roberval does not express it explicitly here, but it should be noted that the points M, N, …, T are chosen so that the line segments AM, AN, …, AT are equal in length to the arcs AE, AF, … . So, on the "diameters M, N, … S," we mark respectively the heights 1, 2, …, 7 corresponding to those marked on the diameter AB. He refers to these later as "M_1, N_2, … , S_7."]

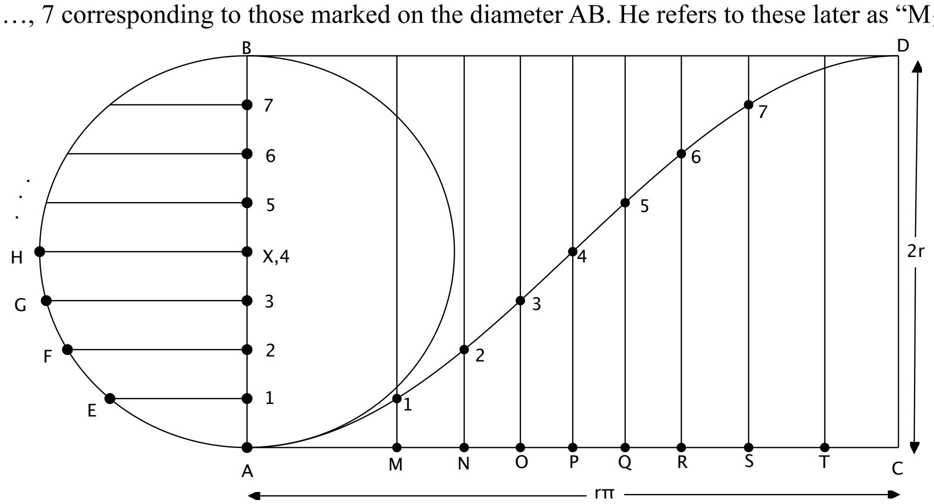

Figure 9; the "companion" curve to the cycloid

Then I take the same sines E_1, F_2, G_3, etc. and I carry them on each height found on each diameter, and I draw them towards the circle [from the companion curve], and from the [two] ends of these sines are formed two lines [curves], of which one is A 8 9 10 11 12 13 14 D [the cycloid], and the other A 1 2 3 4 5 6 7 D [the companion curve of the cycloid].

[See figure 10. So, on each "diameter M, N, O, … ," we translate rightward the corresponding half-chords from the semicircle with lengths $r\sin(E_1)$, $r\sin(F_2)$, $r\sin(G_3)$, … so that the point 1 coincides with height M1 (which Roberval calls M_1), 2 coincides with N2 (N_2), 3 coincides with O3 (O_3), and so on. Once again, Roberval's indivisibles are implicitly invoked, since constructing the complete curves A 8 9 … D and A 1 2 … D in this manner requires an infinity of "diameters," of "sines," and of heights, so that the curves A 8 9 … D and A 1 2 … D are indeed continuous, and not sets of discrete points. The curve A 8 9 … D is half of one arch of the cycloid curve.]

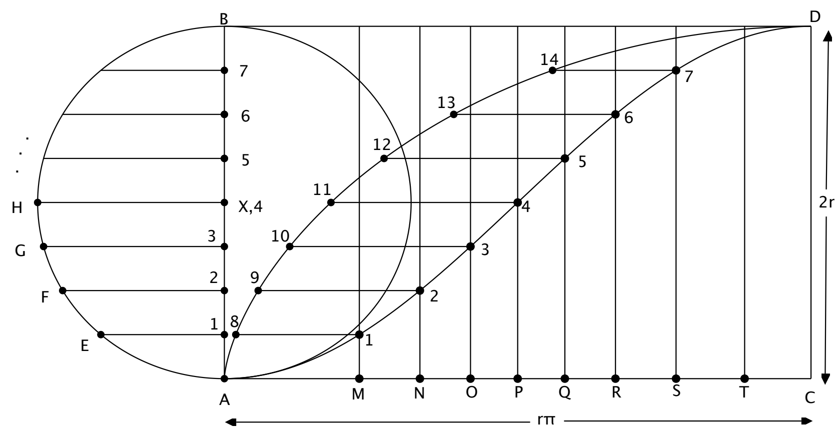

Figure 10; the cycloid curve and its companion curve; similar to Roberval's original diagram [9, 192]

I know how the line A 8 9 D [the cycloid] is made; but to know what movements have produced the other [the companion curve], I say that while AB traversed the line AC, the point A had climbed the line AB, and marked each of the points 1, 2, 3, 4, 5, 6, 7, the first space [from A to 1] while AB has come to M, the second [from 1 to 2] while AB has come to N, and thusly always equally from one space to the other until the diameter has arrived at CD; so the point A has climbed to B. And that is how the line A 1 2 3 D is formed.

[Suppose, in general, that, instead of using the notations E_1, F_2, G_3, etc., we let θ be the angle of the arc which A traverses at a given time during its movement along the circumference of the circle. As A passes through such an angle, as we have seen, the circle moves a distance of $r\theta$ towards the line CD. Observe that, as in figure 11, if we take the point A to be the origin, then, with $u = (x,y)$ representing an arbitrary point on it, the cycloid curve has been parameterized as

$$x = r(\theta\text{-}\sin\theta),\ y = r(1 - \cos\theta).$$

We also see that likewise with $v = (x,y)$ representing an arbitrary point on it, the companion curve has been parameterized as

$$x = r\theta,\ y = r(1 - \cos\theta).]$$

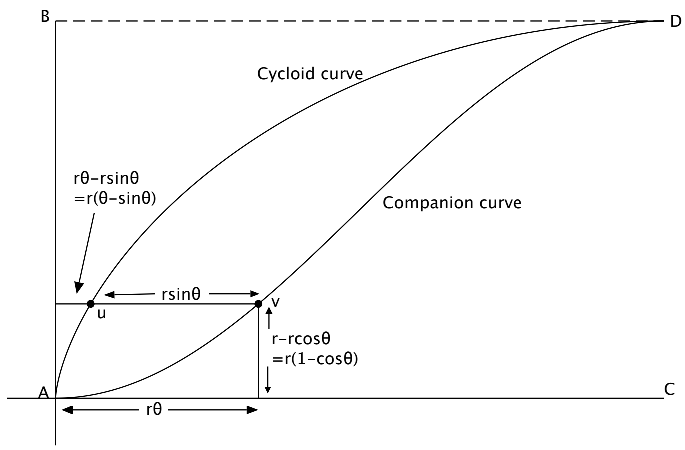

Figure 11; parameterizing the cycloid curve and its companion curve

Thus these two lines [curves] enclose a space, being separated from one another [horizontally] by each sine and rejoining together at the two ends of AD. Thus each part [line segment = "indivisible"] contained between these two lines [curves] is equal to each part [line segment = "indivisible"] of the area of circle AEB contained in the circumference of this one [semicircle]; knowing the heights A_1, A_2, etc., and the sines E_1, F_2, etc., which are the same as those of the diameters M, N, O, etc., thus the figure A 4 D 12 [see figure 10; the area enclosed by the cycloid and its companion curve] is equal to the [area of the]semi-circle AHB.

[Hence, we conclude, using Cavalieri's method of indivisibles, that the area between the cycloid and its companion curve is equal to the area of the semicircle, namely $\frac{1}{2}\pi r^2$.]

Thus the line A 1 2 3 D [the companion curve] divides the parallelogram ABCD in two [in half] equally, because the lines of one half are equal to the lines of the other half, and the line AC to the line BD;

> [See figure 12. We show congruence between the spaces APDC and APDB, the spaces inside the rectangle below and above the companion curve, respectively [1, 157]. Recall from figure 11 that if P lies on the companion curve with horizontal distance from A of $r\theta$, then the length of the vertical line segment PQ is $r(1 - \cos\theta)$. Now consider the point P' on the companion curve with horizontal distance from A of $r\pi - r\theta = r(\pi - \theta)$. Notice that the length of the vertical line segment P'Q' is
>
> $$2r - r(1 - \cos(\pi-\theta)) = 2r - r + r\cos(\pi - \theta) = r(1 - \cos\theta),$$
>
> since $\cos(\pi - \theta) = -\cos\theta$. So PQ ≅ P'Q'. This means that for every vertical line segment like PQ contained in the space APDC at a distance of $r\theta$ from AB, there is a corresponding and congruent line segment like P'Q' in the space APDB at a distance of $r\pi - r(\pi-\theta) = r\theta$ from CD, and so the two spaces have equal areas, according to Cavalieri's method of indivisibles.]

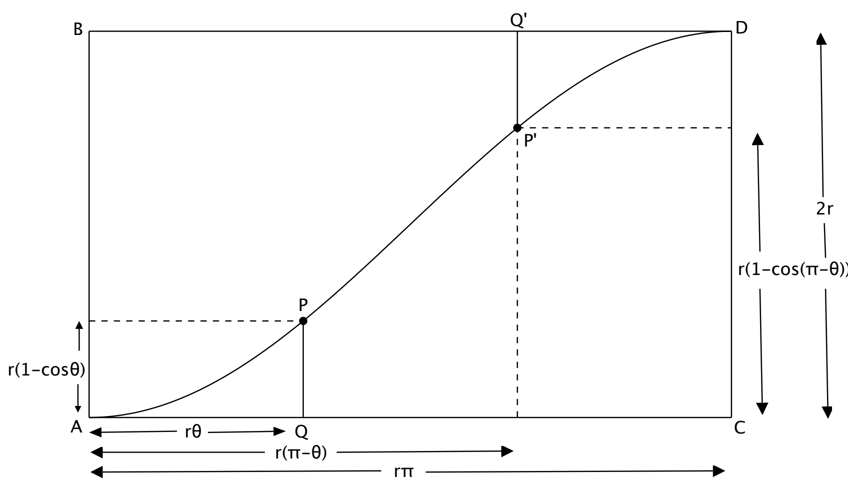

Figure 12; equating the areas above and below the companion curve

and therefore, according to Archimedes, the half is equal to the circle, to which adding the semicircle, knowing the space understood between the two curved lines, one will have a circle and a half for the space A 8 9 D C [see figure 10; the space under the half-arch of the cycloid]; and doing the same for the other half, all of the figure of the cycloid is worth three times the circle.

[The reference to Archimedes is to his determination of the area of the circle [4, 92-95], which states that the area of a circle is equal to the area of a triangle which has a base equal to the length of the circle's circumference and a height equal to the length of the circle's radius. Since the side CD is length $2r$ (the diameter of the circle), and side AC is length $r\pi$ (half of the circumference of the circle), and according to Archimedes the area of the whole circle centered at X with radius r is $\frac{1}{2}r(2\pi r) = \frac{1}{2}(2r)(\pi r) = \pi r^2$, one half of the area of the rectangle ABCD is πr^2. But as we have shown, the area under the companion curve is also one half that of the rectangle ABCD. Thus, the area under the half-arch of the cycloid is given by

$$\text{[Area under companion curve]} + \text{[Area between the companion curve and the cycloid]}$$

$$= \pi r^2 + \tfrac{1}{2}\pi r^2 = \tfrac{3}{2}\pi r^2,$$

and doubling this, we see that the area under one arch of the cycloid is given by $3\pi r^2$, meaning three times the area of its generating circle.] ❖

One can see that the above argument is at its heart a geometrical one: we establish relationships between known areas (i.e. the semicircle and the rectangle) and our principal unknown area: the area under a half-arch of the cycloid curve. However, the argument is not pure geometry in its method of establishing these relationships, which is a Cavalierian "indivisible" method, albeit tempered by Roberval's own conception of what an "indivisible" actually is. Hence we see in this argument a kind of segue between geometry and calculus: its core form is that of geometry, while its core mathematical utensil is that of the integral calculus— the notion of measuring an area using infinitely small rectangles or "surfaces."

After having solved this problem, Roberval notified Mersenne of his success in his 1637 letter, but as we have said, without providing any kind of actual solution. Mersenne, excited by Roberval's letter and curious as to whether his results were actually correct, notified his correspondents of the news in an effort to verify whether the secretive mathematician had actually discovered the area under one arch of the cycloid or not [5, 57]. One such correspondent was Toulouse lawyer Pierre de Fermat, who in his first reply in February of 1638 to Mersenne expressed doubt over the validity of Roberval's solution. Not knowing anything of Roberval's method, Fermat responded to Mersenne in another letter from July1638 which begins: "You remember that I had at another time written you that I had found the proposition of M. de Roberval doubtful and that I had apprehended that he had ambiguity in his research. Here is the confirmation." [6, 377]. Indeed, Fermat's solution to the area under one arch of the cycloid is framed in his letter to Mersenne as a "confirmation" of Roberval's mistake. This claim is, of course, not true: Roberval and Fermat both provide differing yet correct arguments. In fact, Fermat retracted his doubts in a letter sent to Mersenne roughly one week after the letter from which we obtain the following solution; we will consider the follow-up letter after having explored the present work.

Fermat's solution to finding the area under one arch of the cycloid curve.

[This text is taken from a letter appearing in *Correspondance du P. Marin Mersenne, vol. VII*, pp. 376-380. Again, the work has been translated from the French by me, and all comments in [square brackets] are mine.]

Note in the following figure [The figure to which Fermat refers is given below as figure 14; figure 13 is a preliminary diagram created by me.] the description of the curve described by a point upon a circle which rolls.

> [Note the difference between this construction of the cycloid and Roberval's construction; Fermat simply tells us how this curve came about, whereas Roberval went through the work of defining the movement of the point on the circle by relating it to other movements. The difference in rigor is most likely due to the fact that Mersenne, to whom Fermat was writing, was already aware from Roberval's letter of how to construct the cycloid curve, and so Fermat found it unnecessary to go through such labors again.]

Its base [the base of the cycloidal arch] is PR, cut equally [bisected] at the point F: its summit is A [that is, the highest point on the cycloidal arch]; the line AF, perpendicular on PR, is the diameter of the rolling circle; AIGF is half of the said circle; PLAR is the curved line [the cycloid curve]; PS is a rectangular parallelogram [Fermat defines this rectangle by its opposite vertices P and S, but we use here the convention "rectangle(PQSR)" to avoid confusion with the diagonal line PS.]

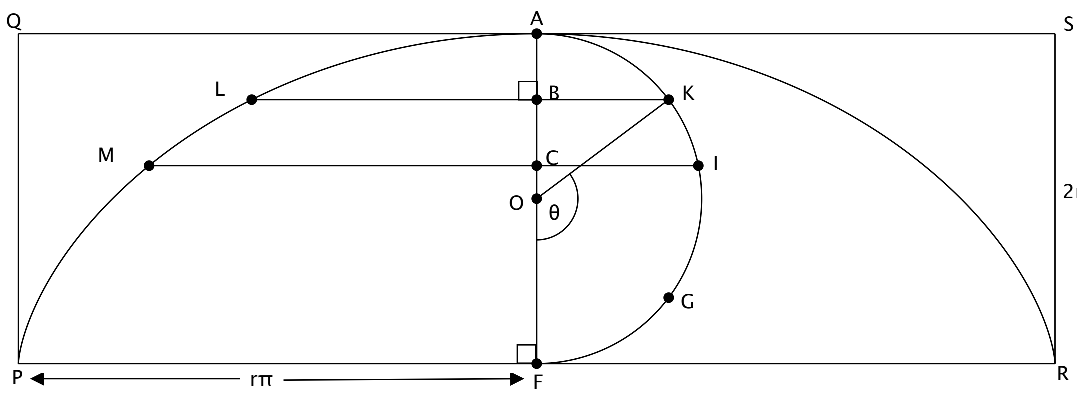

Figure 13; preliminary diagram

The principal property of the curve, which it is very easy to demonstrate, is that, if you take a point on this one [i.e., on the cycloid curve] like L, from which you draw LBK perpendicular on AF [see figure 13], the line LB is equal to the line BK and to [i.e. added to [6, note 2, p. 378]] the portion of the semi-circumference AK;

> [This is to say, for any line LBK such that L is on the cycloid, B is on the diameter AF of the circle, and K is on the semi-circumference of the circle, where LBK⊥AF, we have that LB = BK+ arc(AK). Let us demonstrate this fact. Remember from Roberval's solution we derived that the horizontal distance of a point L on the cycloid from the line QP is given by $r(\theta\text{-}\sin\theta)$, where θ is as shown in figure 13, and r is the radius of the circle AIGF. Since PF has length $r\pi$, then LB must have length
>
> $$r\pi - r(\theta-\sin\theta) = r(\pi-\theta) + r\sin(\theta) = r(\pi-\theta) + r\sin(\pi-\theta),$$
>
> since $\sin(\theta) = \sin(\pi-\theta)$ from a trigonometric identity. Notice that the length of arc(AK) is given by $r(\pi-\theta)$ and the length of BK is given by $r\sin(\pi-\theta)$, from which it follows that LB = BK + arc(AK). In particular, this gives us that PF = 0 + arc(AF) = AIGF, the semi-circumference of the circle.]

all the same the line MC is equal to the line CI and to [added to] the portion of the semi-circumference IA, and thusly of the others until the line PF finds itself equal to the entire semi-circumference [AIGF; so PR is thus equal to the circumference of the rolling circle].

If the base PR is double the circumference of the rolling circle, then in this case the line LB is equal to the line BK and to double the portion of the circle AK, and thusly the line MC is equal to the line CI and to double the portion of the circle IA. And if the line PR is triple the circumference of the rolling circle, all the same, etc.

> [It is somewhat unclear what Fermat is trying to say here ("Que sy la base PR est double de la circonféronce du cercle qui roule, en ce cas la ligne LB est esgale à la ligne BK et au double de la portion du cercle AK, …"[6, 377-378]) We ignore these remarks, as they have little bearing on his argument as a whole.]

That thus supposed, to find the proportion of the rectangle PA [i.e. rectangle (PQAF) from figure 13] to the half-figure APF [the space under one half-arch of the cycloid, we will call it space(ALPF) to avoid confusion with the triangle(APF)], let us divide AF in as many equal parts as we will like, as AB, BC, CD, DE, EF. And next from points B, C, etc., let us draw the lines BLT, CMV, DNX, EOY parallel to PF.

> [See figure 14. As with Roberval's solution, here when dividing AF into equal parts, the number of equal parts is arbitrary, "as many equal parts as we will like"; the choice of five parts is merely for convenience.]

To find the proportion that we are looking for [that of rectangle(PQAF) to space(ALPF)], it is necessary to compare every line PF, EY, DX, CV, BT, AQ to the lines PF, EO, DN, CM, BL.

> [So, Fermat's strategy in finding the area of space(ALPF) is to first compare all the lines of the larger rectangle(PQAF) (PF, EY, DX, …) to the lines of the space (ALPF) (PF, EO, DN, …) so as to find a ratio between the areas of the two spaces. In fact, when Fermat says, "all the lines," he is implying that we must compare the *sum* of the lines of the rectangle to the *sum* of the lines of the space(ALPF) [6, note 1, pp.

378]; this implies, once again, that we use a Cavalierian method of indivisibles in proceeding, where the "sum of lines" of one space refers to the collectivity of indivisibles which comprise the area of that space.]

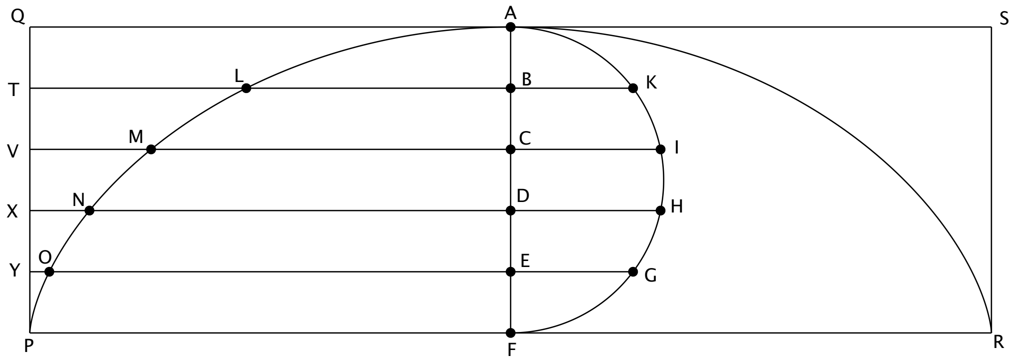

Figure 14; Fermat's original diagram

Thus the lines PF, EO, DN, CM, BL [those of space(ALPF)] are equal, as we have said, to portions of the circle FA, GA, HA, IA, KA, and to [added to] straight lines [half-chords] EG, DH, CI, BK. It is thus necessary to compare every line PF, EY, DH, CV, BT, AQ [of the rectangle(PQAF)] with the portions [arcs] of the circle FA, GA, HA, IA, KA and with [added to] the straight-lines EG, DH, CI, BK.

[From Fermat's preliminary remark earlier ("The principal property of the curve…"), each line of the area under the cycloid is equal to a corresponding half-chord plus a corresponding arc-length. For example, EO = EG + arc(GA). So, we recast the problem to finding a proportion between "the lines of the rectangle(PQAF)" and "half-chords of the circle plus arcs of the circle."]

Let us make this comparison separately. In comparing firstly every line PF, EY, DX, CV, BT, AQ, [the lines of the rectangle(PQAF)] with the straight lines BK, CI, DH, EG, etc.,[those of the semicircle AGF] it is obvious that all of these lines [that is, the "sum" of all of these lines] PF, EY, etc., to infinity, will have the same proportion to [the "sum" of all of] BK, CI, etc., than the one which is from the rectangle AP [rectangle (PQAF)] to the semi-circle FGA (because the lines AB, BC, CD, etc. are equal between them). And the aforementioned proportion [the proportion of area(rectangle(PQAF) to area(semicircle(AGF))] is of 4 to 1in the first revolution (of 8 to 1 in the second, of 12 to 1 in the third, etc., to infinity) of which alone we will speak, the consequences for the others being too easy and the demonstration too obvious.

[Here we see that Fermat's conception of the "indivisible" is decidedly more Cavalierian than Roberval's. Fermat does not state that the infinity of lines represent an infinity of "little surfaces," and in fact he does not seem to concern himself with their nature at all; he uses indivisibles only to allow himself to make comparisons between individual line segments of one space and another, and deduce ratios between the areas of the two. This conception carries with it some logical difficulties, since as we have said, the line having only a length, it would be impossible for any multitude of lines, no matter how compactly placed, to comprise a two-dimensional area. But, logical inconsistencies aside, Fermat's observation is correct; the area of the semicircle is $\frac{1}{2}\pi r^2$, and the area of the rectangle is $\pi r \cdot 2r = 2\pi r^2$, and so

$$\text{area(rectangle(PQAF))} = 4 \cdot \text{area(semicircle(AGF))}.$$

After the second revolution of the circle, we have that

$$\text{area(rectangle(PQAF))} = 2r \cdot 2\pi r = 4\pi r^2 = 8 \cdot \text{area(semicircle(AGF))},$$

and after the n^{th} revolution of the circle,

$$\text{area(rectangle(PQAF))} = 2r \cdot n\pi r = n\pi r^2 = 2n \cdot \text{area(semicircle(AGF))}.$$

However, the consideration of multiple revolutions is unnecessary in solving the problem at hand.]

Let us now compare every line [of rectangle(PQAF)] PF, EY, DX, CV, BT, AQ with portions [arcs] of the circle FA, GA, HA, IA, KA. If the portions AK, KI, IH, etc. were equal between them [that is, if the differences in length (AK, KI, IH, …) between each of the successive arcs (AK, AI, AH, …) were all equal to one another], we could say that every line PF, YE, XD, VC, BT, AQ [that is, the sum of the lines of the rectangle] would be double the portions FA, GA, HA, IA, KA [the sum of the arcs of the semicircle], because each one of the lines PF, YE, etc. is equal to the semi-circumference FA, which would consequently be the greatest of the arithmetic progression, in which the difference of the progression is equal to the smallest term [which is AK].

[Fermat assumes that the lengths AK, KI, IH, … are all equal, call this length d. Thus, the arcs AF, AG, AH, … have lengths which form an arithmetic progression, with the common difference between lengths of consecutive arcs being d. Observe that, in figure 15, we may arrange the arc lengths into a right-triangle [1, pp. 158] with base AF, where the points K, I, H, … lie on the hypotenuse, each a horizontal distance of d from one another, because for each arc length, there is a corresponding point on the diameter (see figure 14), meaning, if we "stack" each of the arc lengths on top of one another such that the vertical distance between each is equal to the common length c of segments of the diameter AB, BC, …, then the endpoints of the arc lengths actually align along a line of slope $-c/d$ (figure 15).The triangle must then have a height of $2r$, so the area of the triangle, i.e., the "sum" of all of the arcs AF, AG, AH, … is equal to $\frac{1}{2} \cdot 2r \cdot \pi r = \pi r^2$. So the area comprised of all of these arc lengths is in fact the area of the circle which they partly describe, and is actually ½area(rectangle(PQAF)), which is the "sum" of the lines PF, YE, XD, …, each having length equal to AF (see figure 14), and so we conclude as Fermat says, that

$$\sum(\text{arc lengths of semicircle AGF}) = \tfrac{1}{2}\,\Sigma(\text{lines of rectangle(PQAF)} = \tfrac{1}{2}\,\text{area(rectangle(PQAF))}.]$$

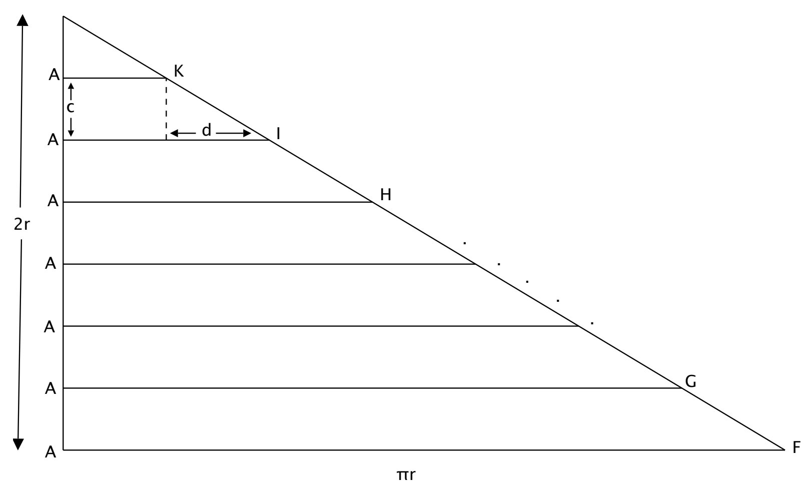

Figure 15; determining the relationship between arcs of the circle and lines of the rectangle

But we cannot without paralogism [fallacious reasoning] determine this double proportion, because the straight lines AB, BC, etc., [on the diameter AF of the semicircle] being equal between them, it manifestly follows that the portions AK, KI, IH, etc. are unequal between them. And therefore we cannot say that all lines PF, EY to infinity, are double the portions FA, GA, etc. to infinity, which nevertheless I estimate that M. Roberval will have believed, not being himself amused to consider the inequality of the portions AK, KI, etc.

[Fermat thus concludes that we cannot assume that there exists an arithmetic progression between the arc lengths AK, AI, …, since we have previously assumed that AB, BC, …, which lie on the diameter, are each equal to one another, hence that Roberval must have erred at some point in his solution, despite Fermat having no knowledge of the route Roberval had taken in deducing his solution. As has been said, this mistake is one which Fermat retracts almost a week later in another letter to Mersenne, having discovered that it is irrelevant whether the arc lengths are in arithmetic progression or not; we cover this correction shortly. But, in the letter from which the current solution is taken, Fermat assumes anyway an arithmetic progression between the arc lengths and arrives at Roberval's conclusion.]

And in fact, if this double proportion were true, the proposition of M. Roberval would be as well. Because if all of [the "sum" of] the lines PF, EY, etc. [those of the rectangle] were double the [sum of] portions FA, GA, etc. [the arcs], since we have proven that the same lines PF, EY, etc. are quadruple the lines EG, DH, etc. [those of the semicircle (AGF)] it would follow that the aforementioned lines PF, EY, etc. would be to the sum of the portions AF, GA, and the straight lines EG, DH, etc. as 4 to 3.

[We have shown that, since

$$\text{area}(\text{rectangle}(PQAF)) = 2 \cdot \sum(\text{arcs of semicircle}(AGF), \text{ and}$$

$$\text{area}(\text{rectangle}(PQAF)) = 4 \cdot \text{area}(\text{semicircle}(AGF)), \text{ then}$$

$$\text{area}(\text{semicircle}(AGF)) + \sum(\text{arcs of semicircle}(AGF) = \tfrac{3}{4}\, \text{area}(\text{rectangle}(PQAF)) = \tfrac{3}{4} \cdot 2\pi r^2 = \tfrac{3}{2} \cdot \pi r^2.]$$

Thus the sum of portions AF, GA, and the straight lines EG, DH, etc. is equal to the straight lines PF, EO, DN, etc. [see figure 14; the lines of the space under half of the cycloidal arch]. Thus the lines PF, EY, etc. would be to the lines PF, OE, etc. as 4 to 3. [that is, area(space(ALPF)) = $\tfrac{3}{2}\pi r^2$]. And therefore, the rectangle AP [rectangle(PQAF)] would be to the semi-figure PAF [space(PLAF)] as 4 to 3 and the entire rectangle PS [rectangle(PQSR)] would be to the figure PAR [the full cycloid arch] as 4 to 3. Thus the rectangle PS is quadruple the rolling circle, thus the figure PAR would be triple the circle in the first revolution [meaning the area under one arch of the cycloid is three times that of the circle which generated it.], and in the second quintuple, etc. as it would be easy to extend the demonstration.

> [Since the arches formed by multiple revolutions of the generating circle do not overlap, then the area under the cycloid curve after *n* revolutions would be 3*n* times that of its generating circle. It is unclear as to why Fermat claims that after two revolutions the area under the cycloid curve is only quintuple that of the generating circle.]

But I believe the proposition is false. And perhaps I have made and guessed the path which M. Roberval has held. ❖

Returning to the problem roughly a week later [7, 397-399], Fermat tells Mersenne, "I take the feather to this stroke to justify M. Roberval against the all-too-hasty censure that I had made of his proposition about the Cycloid…" [7, 397], and he proceeds to argue that it does not matter whether the arcs of the semicircle are in arithmetic progression or not.

René Descartes was also among those contacted by Mersenne for verification of Roberval's result, and, accordingly, he too provided a solution in July 1638 [3, 407-412]. I do not present it here, but Descartes, in similar fashion to Fermat and Roberval, used an argument consisting of geometry and indivisibles. He too compared the "lines" or "indivisibles" of one space with those of another; where he differed from Fermat and Roberval was in the particular comparisons which he made.

The above examples illustrate the popular mathematical attitudes towards the infinite in the early 17th century, and give a good indication as to the atmosphere surrounding one of the fundamental concepts of the calculus in an era which barely preceded its official development in the works of Newton and Leibniz. They are also a testament to the richness of certain mathematical problems and concepts which allow for a variety of solutions and contain an abundance of interesting properties awaiting discovery. Of course, since we earlier parameterized the cycloid curve by $y = r(1\text{-}\cos\theta)$, $x = r(\theta\text{-}\sin\theta)$, today the determination of the area under one of its arches becomes the simple task of evaluating the definite integral

$$\int_0^{2\pi} r(1-\cos\theta)d(r(\theta-\sin\theta)) = r^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$

which yields our desired result of $3\pi r^2$. As intriguing as the inner-workings of the definite integral are, however, its evaluation above somewhat pales in comparison to the clever interweaving of geometry and the method of "indivisibles" which Roberval and Fermat each exhibited in solving the same problem. For Roberval to have conceived of a "companion" curve to the cycloid to assist him in his calculations, or for Fermat to have realized that a line under the cycloid curve is equal in length to the sum of a corresponding half-chord and arc from its generating circle, lends a certain uniqueness to either solution, as well as insight into the minds of either great mathematician; on the other hand, it also makes us appreciate the efficiency with which the integral calculus allows us to solve problems which at one point required significantly more exhausting arguments. This striking efficiency marks a turning point in the conception of the infinite, which certainly lies corollary to Cavalieri and his method of indivisibles, as well as to the mathematicians who used and refined this method in the solving of numerous problems throughout the 17th century.

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