Thinking Inside the Box: Geometric Interpretations of Quadratic Problems in BM 13901

by Woody Burchett

Georgetown College

Dr. Homer S. White, Adviser

wburche0@georgetowncollege.edu

101 Westview Drive
Versailles, KY 40383

From the beginning of human civilization, mathematics has been an integral part of society, so much so that even the very first civilized society, Mesopotamia, developed an extensive system of mathematics. What began simply as a need for counting things and conducting trade soon became much more complex as surveyors and builders found a need for geometry. These surveyors advanced Babylonian mathematics to the point of square roots and Pythagorean triples, and eventually to what modern mathematicians call Algebra (A History of Mathematics, Katz 13).

What makes the Babylonians stand out is not only their chronology, but also their method of recording mathematics. They wrote their work with styluses on clay tablets, and this is extremely fortunate for both historians and mathematicians, because these clay tablets turned out to be nearly indestructible, and as a result, many have survived to this day (A History of Mathematics, Katz 10). The example that I will be examining in this paper comes from such a tablet, problem 9 from BM 13901.

To properly analyze a primary source of Babylonian mathematics, it is obviously important to know the extent of Babylonian mathematical knowledge. For starters, the Babylonians used a base 60, or sexagesimal, number system that included place values. They could also derive square roots and solve linear systems. The Babylonians could also solve quadratic, and even cubic equations, and for their time, were extremely advanced algebraically (Allen 5-17).

It is also worthwhile to note that, as far as we can tell, the Babylonians didn't have any practical applications for these types of quadratic problems. The Indians, however, did have some practical applications for these problems, and a particular example occurs in the *Sulbasutras*, when the need arises for brick altars in the shape of falcon. In the texts, the only details given about the falcon's trapezoidal tail are the area, length of the side that touches the body, and the slope of each side of the trapezoid, given indirectly by the method of cutting the square bricks. As a result, the rest of the figure must be deduced through mathematics. The elements of the trapezoid (a rectangle and two right isosceles triangles) can be rearranged to form a rectangle if the two right triangles are combined to make a square. The equation to find the area is then given by $x^2 + ax = A$, where a is the length of the side that touches the body, x is the height of the rectangle and original trapezoid, and A is the total area. When the quadratic is solved for x, you have all the information needed to build the falcon's tail (Knudson 64-65). Math historian Jens Hoyrup even remarks that the general design of the quadratic problems in the *Sulbasutras* are very similar to those found in Ancient Mesopotamia, although he isn't certain whether or not the two cultures ever shared mathematical knowledge (Knudson 67).

This particular paper, however, is not about Babylonian mathematics as a whole, but rather a comparison of interpretations of a specific Babylonian problem. The comparison will involve both my own analysis and interpretation of the problem and Jens Hoyrup's.

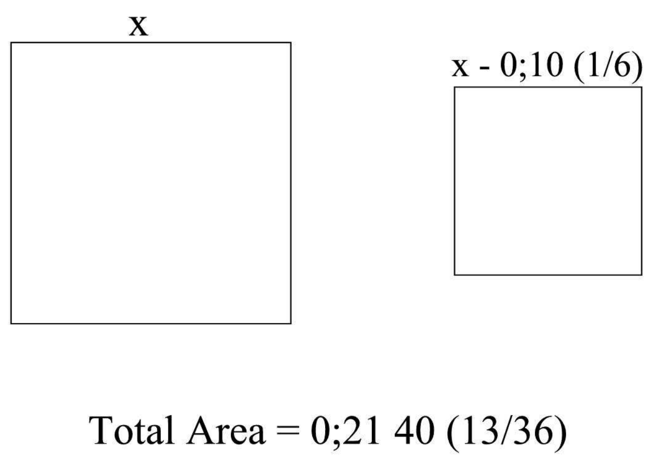
For my interpretation of problem 9 from the Babylonian math text BM 13901, I used the translation from the Katz sourcebook, which was done by Eleanor Robson. As I mentioned before, Babylonian mathematics was done in sexagesimal or base 60, so in order to make the arithmetic more clear, I added the corresponding base 10, decimal number in parentheses next to every sexagesimal number. The following is the Robson translation:

> *"I summed the areas of my two square-sides so that it was 0;21 40 (13/36). A square-side exceeds the (other) square-side by 0;10 (1/6). You break off half of 0;21 40 (13/36) and you write down 0;10 50 (13/72). You break off half of 0;10 (1/6) and then you combine 0;05 (1/12) and 0;05 (1/12). You take away 0;00 25 (1/144) from the middle of 0;10 50 (13/72) and then 0;10 25 (25/144) squares 0;25 (5/12). You write down 0;25 (5/12) twice. You add 0;05 (1/12) that you combined to the first 0;25 (5/12) so that the square-side is 0;30 (1/2). You take away 0;05 (1/12) from the middle of the second 0;25 (5/12) so that the second square-side is 0;20 (1/3)"*(Katz Sourcebook 105).

For my interpretation of this translation, I attempted to make each step correspond to a geometric or arithmetic operation. Let's begin with the first step.

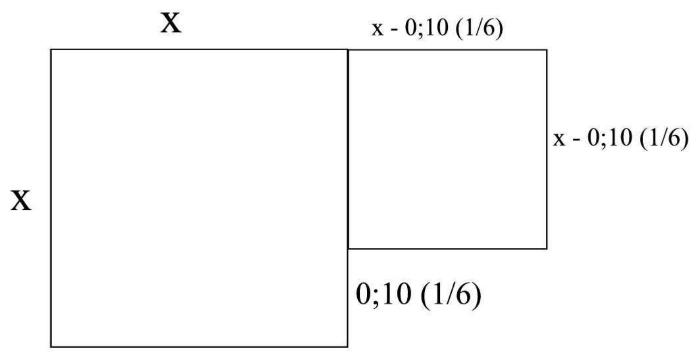
> *"I summed the areas of my two square-sides so that it was 0;21 40 (13/36). A square-side exceeds the (other) square-side by 0;10 (1/6)."*

This step is simply states the dimensions of the figures that are referenced in this problem. The square side of the larger square is denoted with the variable *x* and the square side of the smaller square with *x*-0;10 (1/6). This step also gives the total area of the two squares combined: 0;21 40 (13/36). These squares are shown in the figure below:



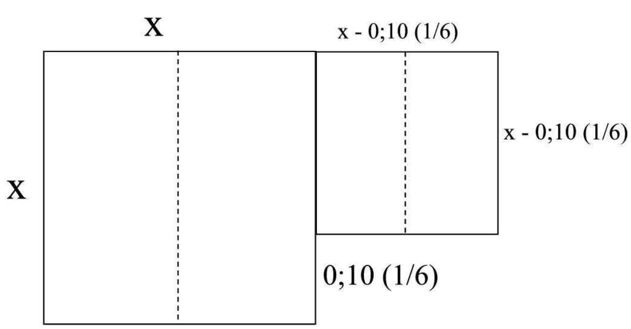
“You break off half of 0;21 40 (13/36) and you write down 0;10 50 (13/72).”

This step instructs us to break the sum of the areas of the squares in half. This may have been done by positioning the two figures next to each other as shown in the figure below:

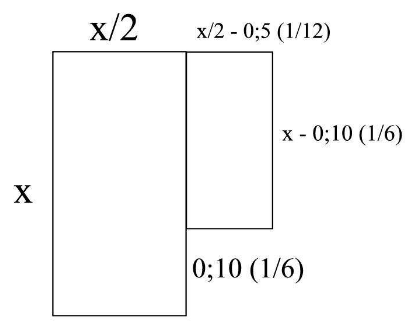


Total Area = 0;21 40 (13/36)

To break the figure exactly in half, take half off of each of the squares, as shown below. This makes the total area of the figure 0;10 50 (13/72).



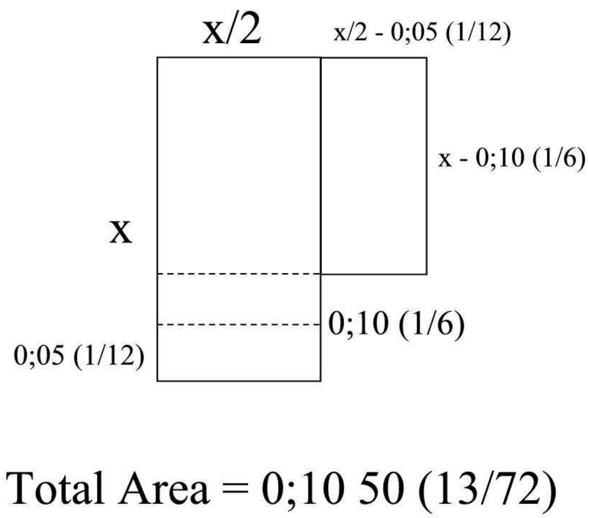
Total Area = 0;21 40 (13/36)



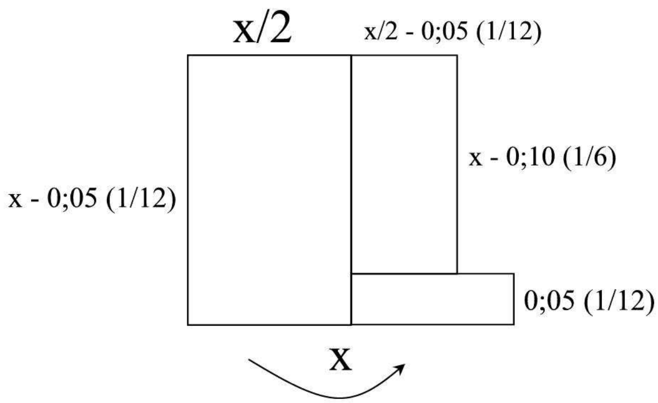
Total Area = 0;10 50 (13/72)

> *"You break off half of 0;10 (1/6) and then you combine 0;05 (1/12) and 0;05 (1/12)."*

This is the first step in the problem that is difficult to interpret. I suggest that the "half of 0;10 (1/6)" refers to taking half of the rectangle formed by the excess length of the larger square. This rectangle has a base of *x*/2 and a height of 0;10 (1/6). This operation can be seen in the figure below:

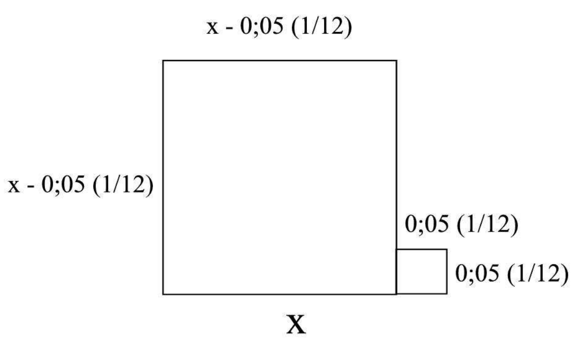


The “combining of the 0;05 (1/12) and 0;05 (1/12)” may refer to the newly formed rectangle taking one of its sides, with a height of 0;05 (1/12), and combining it with its other half, also with a height of 0;05 (1/12). This operation can be seen in the figure below.



Total Area = 0;10 50 (13/72)

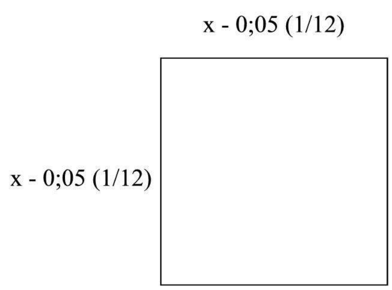
As you can observe in the new figure, two new squares have been formed. One square has the square side *x*-0;05 (1/12), while the other square has the square side 0;05 (1/12). This can be more clearly seen in the figure below:



Total Area = 0;10 50 (13/72)

"You take away 0;00 25 (1/144) from the middle of 0;10 50 (13/72)"

The total area of the smaller square with the sides 0;05 (1/12) is now clearly 0;00 25 (1/144). This 0;00 25 (1/144) is what the translation is referring to, and the "taking away" simply means to remove the smaller square from the larger one, as shown in the figure below. It is also important to note that this is the one instance where my interpretation doesn't follow the text very closely. The text refers to taking away "from the middle", while my diagrams clearly show a taking away from the side. This now makes the total area of the figure 0;10 25 (25/144).



Total Area = 0;10 25 (25/144)

"and then 0;10 25 (25/144) squares 0;25 (5/12)."

This is simply referring to the fact that the square root of 0;10 25 (25/144) is 0;25 (5/12). In modern terms, this gives us x-0;05 (1/12) = 0;25 (5/12). This fact will be used in the next step of the translation.

"You write down 0;25 (5/12) twice."

This tells us to write down the value we proved to be equal to *x*-0;05 (1/12). The reason it asks us to write it down twice is because we are about to find both the length of the side of the original large square, and the length of the side of the original small square.

"You add 0;05 (1/12) that you combined to the first 0;25 (5/12) so that the square-side is 0;30 (1/2)."

This operation will give us the length of the side of the original large square. Due to the fact that 0;25 (5/12) = *x*-0;05 (1/12), we simply add 0;05 (1/12) to each side to obtain the value of *x*. The result, as stated in the translation is *x* = 0;30 (1/2).

"You take away 0;05 (1/12) from the middle of the second 0;25 (5/12) so that the second square-side is 0;20 (1/3)."

Very similar to the previous step, this step gives us the length of the side of the original small square. Due to the fact that 0;25 (5/12) = *x*-0;05 (1/12), we simply subtract 0;05 (1/12) from each side to obtain the value of *x*-0;10 (1/6). The result, as stated in the translation is *x*-0;10 (1/6) = 0;20 (1/3).

That concludes my interpretation of the translation of problem 9 from the Babylonian math text BM 13901. As you can see, I tried to follow the translation word for word, and using some neat geometry tricks and simple algebra I was able to come to a solution pretty easily.

Next comes the interpretation of the same problem by Jens Hoyrup, who has both translated and interpreted a wide range of Babylonian mathematical tablets. The translation that Hoyrup uses is slightly different from the translation I used, so we should start with examining the differences.

> *"The surfaces of my two confrontations I have accumulated: 0;21,4[0]. Confrontation over confrontation 0;10 goes beyond. The moiety of 0;21,40 you break: 0;10,50 you inscribe. The moiety of 0;10 you break: 0;5 and 0;5 you make hold. 0;0,25 inside 0;10,50 you tear out: 0;10,25 makes 0;25 equilateral. 0;25 until twice you inscribe. 0;5 which you have make hold to the first 0;25 you append: 0;30 the confrontation. 0;5 inside the second 0;25 you tear out: 0;20 the second confrontation"* (Hoyrup 168).

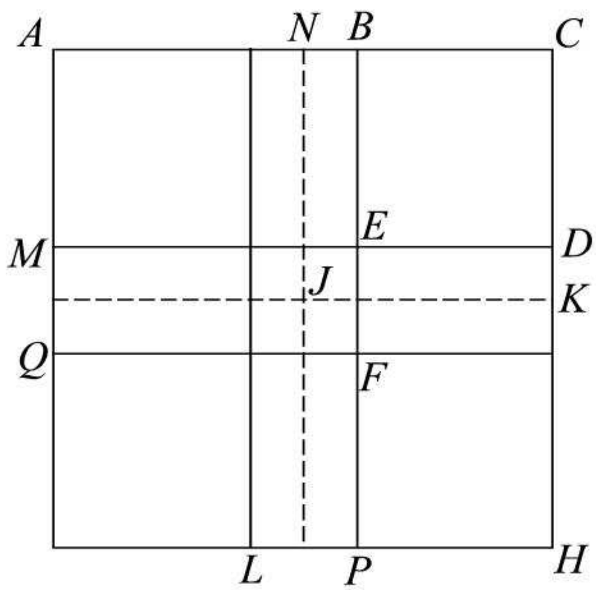
Hoyrup's translation appears to be more literal than Robson's: indeed it is scarcely readable without first consulting the detailed index of mathematical procedures and the terms used to describe those procedures that Hoyrup provides in the beginning of his article. It will be extremely helpful if we give the definitions for all the words used in this translation before we examine it.

- When the translation uses the word "accumulate," Hoyrup says that it is referring to, "a genuinely numerical addition, and can thus be used to assess the measuring numbers of, for example, lengths and areas."

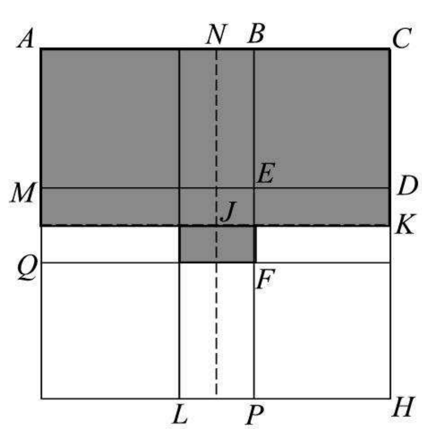
- Hoyrup describes the subtractive procedure of “tearing out” to be the opposite of “to append.” Hoyrup describes the adding procedure of “appending” as operating on concrete entities and only joining entities of the same kind and dimension.
- The term, “to hold,” indicates multiplication and is usually used when a length and width are being used to create a rectangle. The process of “making hold” also refers to a length and width coming together to make a rectangle.
- “To confront” refers to the geometrical procedure of “squaring,” ad naturally the noun “confrontation” simply refers to a geometric square.
- A “moiety” is a “necessary half,” meaning that it is a half derived for a purpose, and not coincidental. It is always half of some entity, and does not refer to the number, ½.
- “Breaking” is the process of deriving a “moiety.” It is used for no other purpose in Babylonian mathematics other than to find a necessary half.(Hoyrup 158-162).

Now that we know all the terms and their meanings, it is possible to examine Hoyrup’s interpretation. Hoyrup begins by suggesting that the problem is referring to a standard diagram. His goal is to see if the text hints at a specific diagram that was likely used to solve the problem. The first thing Hoyrup does is derive the “moiety” or “necessary half” from the total area. His method of doing so is radically different than my own.

Hoyrup proposes a method he believes is in line with what the mathematicians were actually doing, based on the fact that other problems in BM 13901 can be interpreted in a similar way. This method employs a diagram that, according to Hoyrup, lies in the background of many problems in Babylonian mathematics. The diagram can be seen below:



Hoyrup explains that within the square AH, are the two original squares: ABFQ and BCDE. It's also easy to see the sum of each square side: AB and BC. Also important is the average square. It can be seen as the squares AJ and JC. The deviation, *d*, is represented by NB. Hoyrup observes that this is most likely the diagram used to solve this problem and states that the problem can be followed quite easily with it, but doesn't spell out exactly how. The easiest way I can see a solution is by creating the two average squares AJ and JC and having the two squares *d* touching at the bottom, like in the diagram below:



As you can see, this creates a symmetrical figure that will be very easy to take half of, and still has the correct area for the sum of the two squares. After the step of "breaking" the figure into a "moiety," the figure that remains is identical to the figure I created in my solution. The area of the square can then be computed, and the square-sides can be easily obtained after that (Hoyrup 180-183).

I would first like to discredit the differences in interpretation due to the inconsistencies of the translations. If I had only examined Hoyrup's translation, I almost certainly would have interpreted the passage in the same way. The only real difference between Hoyrup's translation and my own is how he arrives at the figure made up of the average square (*z* in Hoyrup's interpretation and *x*-0;05 in my interpretation), because after this figure is obtained, the areas of each square become easy to compute.

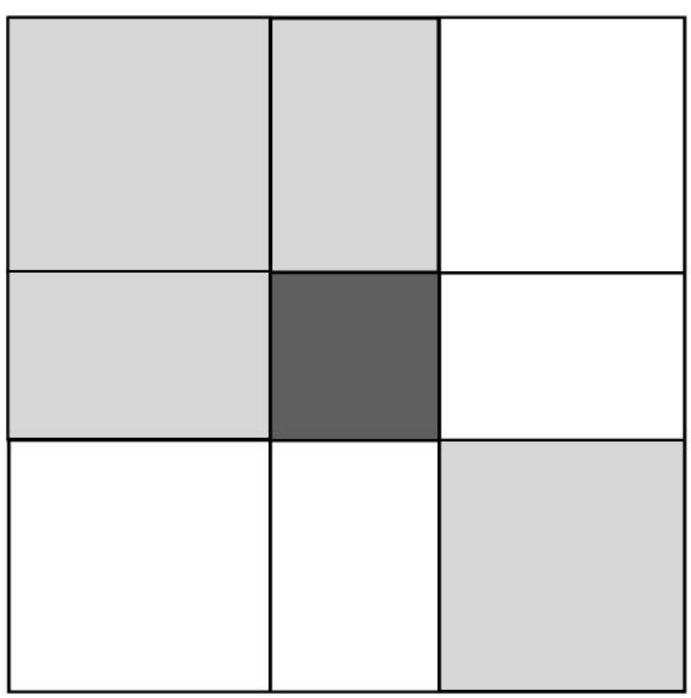
When it comes to the question of which translation is superior, I would have to give the edge to the Hoyrup interpretation, due to the fact that it utilizes a diagram that appears to be common throughout Babylonian mathematics. Hoyrup even makes use of this diagram again on the BM 13901 in problem 19 where it is used, once again, to obtain the sides of a square. The following is the translation used by Hoyrup in his analysis (with added base 10 numbers in parentheses):

> *"My confrontations I have made hold: T[he] surface I have [accum]ulated. So much as the confrontation over the confrontation goes beyo[nd] together with itself: I have made hold, t[o inside the surface] I have appended: 0;23,20 (7/18). My confrontations I have [accumulated, 0;50 (5/6).] 0;23,20 (7/18) until 2 you repeat, 0;46,40 (7/9) you inscri[be] [0;50 (5/6) and 0;50 (5/6) you make hold,*

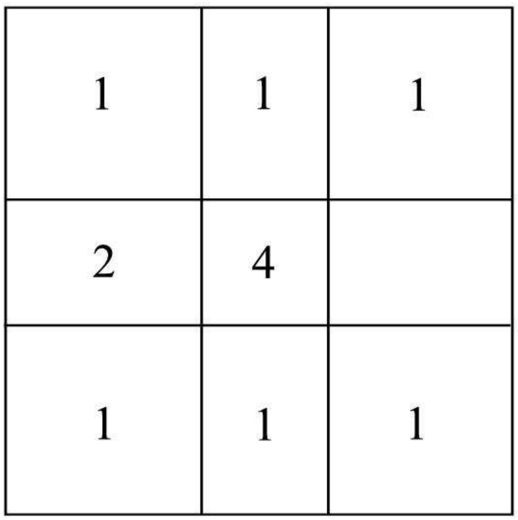
0;41,40 (25/36) inside 0;46,40 (7/9) you tear out: 0;5 (1/12). The igi of 12 is 0;5 (1/12) to 0;5 (1/12) you raise, 0;0,25 (1/144) makes 0;5 (1/12) equilateral. The moiety of 0;50 (5/6) you break, 0;25 (5/12) to 0;5 (1/12) you append: 0;30 (1/2) the first confrontation. 0;5 (1/12) inside 0;25 (5/12) you tear out: 0;20 (1/3) the second confrontation.] (Hoyrup 172).

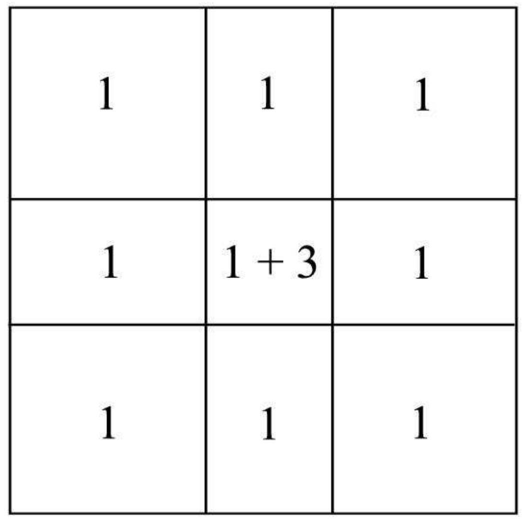
First it is important to note that Hoyrup translates the term "igi" to mean the reciprocal of a number. In this case it simply refers to the reciprocal of 12 being 0;5 (in base 60).

In algebra terms, we are given that the sides $x+y$ =0;50 (5/6). We are also given that the surface has an area of 0;23,20 (7/18). This surface consists of x^2, y^2, and $(x–y)^2$. With this information, we are to find both x and y. The following diagram shows this surface, with the dark shaded region being counted twice (once as part of the larger x^2 square and again as the $(x\text{-}y)^2$):



To use the diagram effectively, though, one must first double the figure, which is done in the translation. The resulting equation is $2x^2+2y^2+2(x-y)^2=0;46,40$ (7/9), and the picture of the figure is given by the two diagrams below (the second merely being a simplified version of the first), with numbers to denote how many times each area is counted:





It becomes pretty clear what the next step is when you look at the second figure: take away area the big square $(x+y)^2$, which is 0;41,40 (25/36) from the total area of the figure, which is 0;46,40 (7/9), leaving only the area of a $3(x-y)^2$, which is 0;5 (1/12). After dividing the area by three, leaving you with 0;1,40 (1/36), you are able to find the length of each side of the center square simply by taking the square root of 0;1,40 (1/36), which is 0;10 (1/6). This is extremely useful, because we now know that half of the middle side is 0;5 (1/12), $(x-y)/2$ in algebra terms, and we can add and subtract this from the average side of 0;25 (5/12), $(x+y)/2$ in algebra terms, to obtain the side of the larger square, calculated by adding 0;5 (1/12) to 0;25 (5/12), and to obtain the side of the smaller square, calculated by subtracting 0;5 (1/12) from 0;25 (5/12). These two operations complete the problem, giving us the larger square side of 0;30 (1/2), and the smaller square side of 0;20 (1/3).

It is worthwhile to note that this problem presents extremely strong evidence that the Babylonians used geometric manipulations to solve algebraic problems rather than our modern methods. This is extremely clear due to the fact that rather than simply finding "$x+y$" and "$x-y$", the text states that they instead found "$(x+y)/2$" and "$(x-y)/2$". This is unnecessary in modern algebra, but the modern notation would render the diagram we employed useless, as it would become necessary to add $x+y$ and $x-y$. This would leave you with a $2x$, which will exceed the side of the diagram by 0;10 (1/6), making it extremely difficult to understand what you do next when using only the diagram to solve the problem.

Clearly the standard diagram that Hoyrup suggests for problem 9 has been seen before in Babylonian algebra, and due to this fact, his interpretation is, in all likelihood, superior to my own. Regardless, it's very useful to examine these geometric methods for solving problems simply because it shows you the ingenuity of these Ancient Mathematicians, and inspires us to approach different problems in new, and often easier ways.

Bibliography

Allen, G Donald. "Babylonian Mathmatics." 1 Jan. 1997. 19 Nov. 2007 <http://www.math.tamu.edu/~dallen/masters/egypt_babylon/babylon.pdf>.

Hoyrup, Jens. "The Old Babylonian Square Texts BM 13901 and YBC 4714: Retranslation and Analysis." 2001

Katz, Victor. A History of Mathematics: Brief Edition. New York: Pearson Education Inc., 2004.

Katz, Victor., ed. The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook. Princeton: Princeton University Press, 2007.

Knudson, Toke Lindegaard. "On the Application of Areas in the Sulbasutras." Ancient Indian Leaps in Mathematics. Ed. BS Yadav and Mon Monhan. Basel: Birkhäuser, forthcoming.