Constructivism: A Realistic Approach to Math?

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Occasionally in mathematical history, results surface that raise eyebrows and cause mathematicians to take a step back and say, "Wait, that shouldn't be so. We should work to avoid results like that!" Favorite examples of such results include Cantor's transfinite ordinals, the Banach-Tarski paradox, and the Axiom of Choice; all three of these results caused groups of mathematicians to balk. These mathematicians are signaling that the discipline has strayed too far from the realm of ideas accessible to the human mind, as major results are so counter-intuitive or inconceivable as to be deemed paradoxical. In some ways, this reaction makes sense; what is math (at least, in practice) if not a product of human minds? The constructivists, also known as intuitionists, are the quintessential conservative group; they took extreme pains to pare math down to exactly what is accessible to the human mind. In the end, though, their tactics failed to sway any more than a handful of other mathematicians, and classical math proceeded forth undeterred, counter-intuitive results and all. This result indicates to me that constructivism was, in fact, not a plausible approach to math—if for no other reason than its complete failure to convince working mathematicians.

I will focus on the three foremost intuitionistic authors: Errett Bishop, Arend Heyting, and L. E. J. Brouwer. Of these three, Brouwer was the first to publish about intuitionism. Although there were some intuitionistic tendencies in math from the Greeks to Kant to Poincaré, Brouwer's 1907 doctoral thesis established intuitionism as we know it. Heyting's 1956 *Intuitionism* takes the form of a dialogue, pitting intuitionism (in the persona of Mr. Int) against classical mathematics (in the persona of Mr. Class) and several other competing foundational groups. Later, Bishop's 1967 *Foundations of Constructive Analysis* is the first serious explication of constructivist mathematics beyond the level of arithmetic. Bishop actually follows through on the foundational goal and shows what math can and can't be done constructively.

The Mechanics of Intuitionism Algorithmic computability and finite meaning characterize constructive mathematics on the level of theorems, proofs, and computations. Bishop sets out to reduce math to constructions that can be performed in finitely many steps, claiming to be motivated by the desire to make math meaningful for finite intelligences. He says, "We are not interested in properties of the positive integers that have no descriptive meaning for finite man" (Bishop (1967), 2).

All three of our intuitionist authors start by building math up from the finite integers. Bishop says that all of constructive math can be interpreted as being about statements about the finite integers.[1] This is both a re-iteration of his mental-finitist position, and an expression of the foundational status of the integers in constructivist math. Keep in mind, though, that the integers are always taken to be the basis of Bishop's constructions, but they are not the wellspring of his philosophy. Heyting extols the benefits of the natural number concept:

> "1. It is easily understood by any person who has a minimum of education,
> 2. It is universally applicable in the process of counting,
> 3. It underlies the construction of analysis." (Heyting (1956), 15)

Brouwer builds his mathematics up from the naturals. The first few lines of his thesis are:

> " 'One, two, three,...', we know by heart the sequence of these sounds (spoken ordinal numbers) as an endless row, that is to say, continuing for ever according to a law, known as being fixed." (Brouwer (1907), p. 3, [[p. 15]])

Hesseling takes care to note that Brouwer's natural numbers are an unfinished sequence, not an infinite set (Hesseling (2003), 36). Brouwer takes the counting numbers to be intuitive, and he takes ordinality to be the true content of arithmetic on the naturals. He defines negative numbers as counting "to the left," and arithmetic operations on positives and negatives "in two directions," (Brouwer (1907), p. 5, [[p. 16]]). Then, he defines rational

[1] "Everything stretches itself to number, and every mathematical statement ultimately expresses the fact that if we perform certain computations within the set of positive integers, we shall get certain results." (ibid., 3)

numbers as pairs of ordinals (ibid., p. 5–6, [[p. 16]]). He describes how to order the rationals, i.e.

$$\frac{a}{b} \gtreqless \frac{c}{d} \text{ if } (a \times d) \gtreqless (b \times c),$$

and how to situate them among the ordinals. He builds the irrationals by taking cuts of the rationals (similar to Dedekind cuts, with the notable exception that Brouwer does not start with an infinite set as his starting point). In fact, at every stage of Brouwer's development of the irrationals, he has constructed no more than countably many numbers. From there, he goes on to develop the continuum.

As we can see from Brouwer's construction, intuitionists do not reject whole types of finite numbers, as Kronecker is said to have rejected the irrational numbers.[2] Brouwer, Bishop, and Heyting have no problem with irrational (even transcendental) numbers themselves; for instance, Bishop finds e acceptable since we can approximate it to an arbitrary degree of precision using an algorithm, each step of which is finite.

$$e = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{k!}$$

As n increments, we can compute the new partial sum $\sum_{k=0}^{n} \frac{1}{k!}$ in finite time.

As for infinite numbers, Brouwer only takes issue with certain transfinites, so he is not a finitist. He says,

> "In chapter I we have seen that there exist no other sets than finite and denumerably infinite sets and continua; this has been shown on the basis of the intuitively clear fact that in mathematics we can create only finite sequences, further by means of the clearly conceived 'and so on' the order type ω, but only consisting of *equal elements* [footnote: The expression 'and so on' means the indefinite repetition of *one and the same* object or operation, even if that object or that operation is defined in a rather complex way.] ; (consequently we can, for instance, never imagine *arbitrary* infinite dual fractions as finished, nor as individualized, since the denumerably infinite sequence of digits cannot be considered

[2] Kronecker purportedly said, "God created the integers; everything else is the work of man," as well as, "What good your beautiful proof on [the transcendence of] π? Why investigate such problems, given that irrational numbers do not even exist?"

> as a denumerable sequence of *equal* objects), and finally the intuitive continuum (by means of which we have further constructed the ordinary continuum, i.e. the *measurable continuum*), but no other sets." (Ibid., 1907 p. 142–143, [[p. 80]])

Brouwer denies the existence of Cantor's second number-class, not because it is a completed infinite, but because of the non-constructive nature of the "and so on..." Cantor uses to arrive at the second number class. Heyting borrows one of Brouwer's later discussions of cardinalities, which deals with species rather than sets. Brouwer and Heyting define "species" as "a property which mathematical entities can be supposed to possess," (Heyting (1956), 37). If a one-to-one correspondence can be formed between two species, they are "equivalent." The cardinalities are as follows:

> "Definition 1. A species that is equivalent to the species N of all natural numbers is *denumerably infinite*.
>
> Definition 2. A species that contains a denumerably infinite subspecies is called *infinite*.
>
> ...
>
> Definition 3. A species that is equivalent to a detachable subspecies of N is called *numerable*." (ibid., 39–40).

As we can see, Heyting is not a finitist, but he does not go into the theory of the non-denumerably infinite. Bishop, too, is not a finitist.

The main difference between constructivists and classical mathematicians is that constructivists eschew methods and definitions which rely on the unknown, unproven, or nonexistent, while classical mathematicians rarely take issue with these. For instance, constructivists do not accept the Bolzano-Weierstrass theorem,[3] although limit points of infinite sequences are acceptable. In a similar vein, Bishop says, "Certain parts of measure theory are hard to develop constructively because limits that are classically proved to exist simply do not exist constructively" (Bishop (1967), 214).

Heyting gives an example of two definitions, one of which is acceptable constructively, and the other of which is not:

[3]Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$. (Rudin (1964), 40)

> "Let us compare two definitions of natural numbers, say k and l.
>
> "I. k is the greatest prime such that $k-1$ is also a prime, or $k=1$ if such a number does not exist.
>
> "II. l is the greatest prime such that $l-2$ is also a prime, or $l=1$ if such a number does not exist.
>
> "Classical mathematics neglects altogether the obvious difference in character between these definitions. k can actually be calculated ($k=3$), whereas we possess no method for calculating l, as it is not known whether the sequence of pairs of twin primes p, $p+2$ is finite or not. Therefore intuitionists reject II as a definition of an integer; they consider an integer to be well defined only if a method for calculating it is given. Now this line of thought leads to the rejection of the principle of excluded middle, for if the sequence of twin primes were either finite or not finite, II would define an integer." (Heyting (1956), 2)

Heyting gives another example to illustrate that "a proof of the impossibility of the impossibility of a property is not in every case a proof of the property itself," (ibid., 17). He details the construction of a number ρ, whose decimal expansion we write down at the same time as we write down the decimal expansion of π:

$$\begin{aligned} \pi &= 3.14159265... \\ \rho &= 0.33333333... \end{aligned}$$

The string of 3s in ρ's decimal expansion terminates after we finish writing the sequence of digits 0123456789 in π. So, if the 9 in this sequence occurs in the kth place of the decimal expansion of π, we have

$$\rho = 0.\underbrace{33\ldots3}_{k\text{ times}} = \frac{10^k - 1}{3 \cdot 10^k}$$

If no sequence 0123456789 occurs, we define $\rho = 1/3$.

This is as valid a construction as any, since we can determine ρ to an arbitrary degree of precision. The interesting thing about this example is that, when Heyting was writing his book, it was unknown whether the sequence 0123456789 indeed appeared in π.[4] Certain

[4] Such a sequence occurs first at the 17,387,594,880th place, according to Jonathan Borwein (Bor-

properties of ρ, then, depend on arbitrarily much information. Heyting's example concerns the rationality of ρ. For him, "rational" means "we can find two integers of which ρ is the ratio." So, although no turn of events would lead to ρ being *ir*rational, we cannot assert ρ's rationality until we compute such integers (Heyting (1956), 17–18).

One of the hallmarks of the intuitionist viewpoint is rejection of certain logical principles: most notably, the law of excluded middle $(\mathfrak{a} \vee (\neg\mathfrak{a}))$. We see this above with Heyting's examples.

Brouwer rejects logic as a foundation for math in his so-called "First Act of Intuitionism":

> "To begin with, the First Act of Intuitionism completely separates mathematics from mathematical language, in particular from the phenomenon of language which are described by theoretical logic, and recognizes that intuitionist mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time, i.e. of the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the empty form of the common substratum of all two-ities. It is this common substratum, this empty form, which is the basic intuition of mathematics." (Brouwer (1952), 1952 p. 140–141, [[509–510]])

He devotes the third chapter of his dissertation to explicating this notion of the independence of math from logic. For him, mathematics is at heart mental constructions and reasonings that take place within a person's mind. Logic, on the other hand, is a linguistic structure which does not represent true math (Brouwer (1907), p. 132, [[p. 75]]). As a matter of principle, Brouwer denies that there is reason to believe in a general law such as $\mathfrak{a} \vee (\neg\mathfrak{a})$, especially when the infinite is concerned.

Bishop claims to derive his interpretation of logic from Brouwer, but he never makes the distinction Brouwer makes between math that occurs in the mind and its communication via mathematical language. Bishop says, "The constructive interpretations of the mathematical connectives and quantifiers have been established by Brouwer," then goes on to describe

wein (1997)).

constructivist interpretations of $\wedge$, $\rightarrow$, etc. (Bishop (1967), 7). He describes constructive proofs essentially as classical mathematical proofs satisfying constructivist constraints. He never leaves the realm of mathematical language, or uses the disparateness of math and mathematical language as an explanation for his view on logic. Additionally, we see from his phrase "mathematical connectives" that he is not concerned with the distinction between math and logic as Brouwer is.

Heyting comes closer to Brouwer's stance on logic, saying,

> "Logic is not the ground on which I stand. How could it be? It would in turn need a foundation, which would involve principles much more intricate and less direct than those of mathematics itself. A mathematical construction ought to be so immediate to the mind and its result so clear that it needs no foundation whatsoever." (Heyting (1956), 6)

He clearly sets math as the foundation of logic rather than vice versa, just as Brouwer does in his thesis. However, Heyting, like Bishop, fails to draw the distinction between mental construction and mathematical language. His constructions, too, seem like classical math proofs. For instance, he says that if A means an integer is divisible by 8, and B means it is divisible by 4, then $8a$ (which can also be written $4 \times 2a$) is a construction P for A and B, which shows $A \rightarrow B$ (ibid., 6).

For Bishop and Heyting, "mathematical truth" is synonymous with having performed a construction. Heyting says, "a mathematical theorem expresses a purely empirical fact, namely the success of a certain construction" (ibid., 8). Ostensibly, some logical deductions will preserve constructions, while others will not necessarily preserve constructions. Bishop and Heyting reject the deductions in the latter category.

The constructive conception of negation is subtle, and illustrates the issue at hand. The intuitionists hold that if we simply negate the definition of "mathematical truth" as "having performed a construction," we do not get a definition of falsity; we just get the informal notion that no construction has yet been produced. Formal mathematical negation is achieved by: "I have effected in my mind a construction B, which deduces a contradiction from the supposition that the construction A were brought to an end" (ibid., 19).

For the classical mathematician, knowing (for instance) $\mathfrak{a} = 0 \vee \mathfrak{b} = 0$ does not necessitate knowing in particular $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$; it is enough to know that it is not the case that $\mathfrak{a} \neq 0 \wedge \mathfrak{b} \neq 0$. For the constructivist, on the other hand, proving $\mathfrak{a} = 0 \vee \mathfrak{b} = 0$ is *not* the same as proving $\neg(\mathfrak{a} \neq 0 \wedge \mathfrak{b} \neq 0)$ (ibid., 24). For constructive validity, we need to be able to prove one of $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$.

Ontology, Epistemology, and the Heart of the Constructivist Position Brouwer's First Act of Intuitionism (see page 6) is the source from which all his other intuitionist beliefs follow. In the First Act, he separates math from mathematical language, and establishes the true heart of math as a mental exercise. Math is constructed by the mind by performing changes on its own thought in time, then abstracting away from the particulars of these constructions. The only activity with "mathematical significance", for Brouwer, is "the pure construction of intuitive mathematical systems which, if they are applied, are turned outward in life by taking a mathematical view of the world," (Brouwer (1907), 1907 p. 173, [[p. 94]]).[5]

The other threads of intuitionistic thought follow directly from this First Act, namely: solipsistic realism, rejection of logic, denial of truth independent of human mental activity, and the failure of verbal constructions to refer to anything mathematical. Intuitionistic ontology almost certainly follows from the First Act, because Brouwer's explanation of what mathematics *is* must precede any considerations about whether mathematical structures can exist.

"Solipsistic realism" is a good name for the intuitionists' ontological position (even though they did not use this moniker) because they deny the existence of an external mathematical reality and instead believe reality is a property of mental products alone. Hesseling

[5] "For that matter, even the stages mentioned above, from the third on, are deprived of mathematical significance. Mathematics has its place only in the first; in practice it cannot remain aloof from the second, but this stage remains an unconscious non-mathematical act. It may afterwards by guided and supported by *applied mathematics*, but it can never obtain priority relative to intuitive mathematics." (ibid., 1907 p. 175, [[p. 95]])"

quotes Brouwer as saying,

> "(...) the only thing that is real to me is my own self at the moment, surrounded by a wealth of images in which the self believes and which make the self live. The question whether these images are 'factual' is devoid of meaning: for my self only the images exist and are, as such, real. A second reality, independent of my self and corresponding to these images, is out of the question." (Hesseling (2003), 27)

This shows that Brouwer is not an anti-realist, since he takes mathematical object to exist intra-mentally. The other intuitionists, too, believe in the reality of mathematical objects, but only post-construction.

Bishop takes an anti-platonist stance:

> "A set is not an entity which has an ideal existence. A set exists only when it has been defined. To define a set we prescribe, at least implicitly, what we (the constructing intelligence) must do in order to construct an element of the set, and what we must do to show that two elements are equal. A similar remark applies to the definition of a function" (Bishop (1967), 2).

Following from the First Act, Brouwer separates math and languages, which entails the separation of math and logic. He says outright that we cannot linguistically perform a construction and then presume the existence of a true intuitive mathematical construction corresponding to the linguistic product (Brouwer (1952), 1952 p. 141, [[p. 510]]). He says in his thesis,

> "Now the following question arises: suppose we have proved by some method, without thinking of mathematical interpretations, that the logical system, built up out of certain linguistic axioms, is consistent, i.e. that two contradictory theorems can occur at no stage of development of the system; suppose further that afterwards we find a mathematical interpretation of the axioms (which of cause will require the construction of a mathematical system whose elements satisfy certain given mathematical relations); does it follow from the consistency of the logical system that such a mathematical system *exists*? Such a conclusion has never been proved by axiomaticians, not even for the case where the given conditions involve that it is a mathematically constructible system that is required. Thus, for instance, it has nowhere been proved that a finite number, subjected to a provably consistent system of contradictions, must always exist." (Brouwer (1907), 1907 p. 141, [[p. 79]])

Also as a consequence of the First Act, Brouwer denies mathematical truth independent of mental construction. Mathematical-linguistic feats do not necessarily refer at all, as we see from the above quotation; so, if we prove a nice property in a linguistic system, we have not proved anything about a real mathematical system. Hence, truth can only be a property of real mathematical objects within the mind.

Bishop and Heyting both deny external mathematical truth. Bishop says, "nothing is true unless and until it has been proved," (Bishop (1967), 26). There is no fact of the matter about the truth of a theorem; the theorem is only true after it has been proved constructively, and in no sense was it always true all along. Heyting explains his stance on external mathematical truth and existence:

> "Class. One may object that the extent of our knowledge about the existence or non-existence of a last pair of twin primes is purely contingent and entirely irrelevant in questions of mathematical truth. Either an infinity of such pairs exist, in which case $l = 1$; or their number is finite, in which case l equals the greatest prime such that $l-2$ is also a prime. In every conceivable case l is defined; what does it matter whether or not we can actually calculate the number?
>
> "Int. Your argument is metaphysical in nature. If 'to exist' does not mean 'to be constructed', it must have some metaphysical meaning. It cannot be the task of mathematics to investigate this meaning or to decide whether it is tenable or not. We have no objection against a mathematician privately admitting any metaphysical theory he likes, but Brouwer's program entails that we study mathematics as something simpler, more immediate than metaphysics. In the study of mental mathematical constructions 'to exist' must be synonymous with 'to be constructed.' " (Heyting (1956), 2)

He is slightly more skeptical than Bishop; he does not deny external mathematical truth, but he simply decides math is not in the business of investigating truth of this sort.

While Bishop's and Heyting's statements on ontology and epistemology corroborate Brouwer's, their accounts follow slightly different paths of entailment. (In fact, the word "entailment" might be a bit strong, as Bishop's and Heyting's philosophical accounts are fairly cursory and leave something to be desired in the way of explaining their central tenets and motivation).

The basic statements of constructivist epistemology and ontology seem straightforward enough, but there is some subtle trouble lurking within. Consider the following statements, which relate truth and existence to proof:

i. A mathematical statement has a truth value if and only if it has been explicitly proved or refuted.

ii. A mathematical statement has a truth value if and only if it can be proved or refuted (although we might not yet have devised an explicit proof or counter-example).

iii. A mathematical object exists if and only if it has been constructed.

iv. A mathematical object exists if and only if it is constructible (although we might not yet have explicitly constructed it).

Brouwer probably accepts (i) as agreeing with his definition of mathematical truth, with the caveat that these proofs and refutations must correspond to mental processes. Bishop and Heyting, whose accounts of constructive math are considerably more rooted in language, explicitly accept (i) as their definition of mathematical truth. Heyting says, "A mathematical assertion affirms the fact that a certain mathematical construction has been effected," (ibid., 3). Furthermore, he says there is no absolute mathematical truth stemming from "some world of mathematical things existing independently of our knowledge" (ibid., 3). If we take knowledge of a mathematical concept to mean we have mentally effected a proof, then we can take Heyting's claim to mean explicitly that there is no mathematical truth without constructive proof. Since mathematical negation also relies on construction for Bishop and Heyting, we can say there is no mathematical falsity without assertion and proof.

All three of Brouwer, Heyting, and Bishop adopt (iii). As discussed above, these intuitionist authors deny the external existence of mathematical structures. Heyting says, "It [Brouwer's program] consisted in the investigation of mental mathematical construction as such, without reference to questions regarding the nature of the constructed objects, such as whether these objects exist independently of our knowledge of them" (ibid., 1).

Adopting (i) forces rejection of (ii), and adopting (iii) forces rejection of (iv). Rejecting (ii) and (iv) does impact the intuitionist program, but Brouwer is the only of the three authors to soften the blow.

If they reject (ii), they must admit that mathematical truth is not closed under logical entailment—even if they restrict themselves to constructively valid logical inferences. Belief in the closure of mathematical truth would not depend on the belief in a set of all mathematically true statements (which the intuitionists would obviously reject), so this is not why belief in such closure is anathema to intuitionism.

Classical mathematicians rely on the closure of mathematical truth by relying on the truth-preserving properties of the logical connectives. That is, if we have two true statements $\mathfrak{a}$ and $\mathfrak{b}$, under classical mathematics, we automatically get that $\mathfrak{a} \wedge \mathfrak{b}$ is true. By implicitly denying (ii), the intuitionists are denying the truth of $\mathfrak{a} \wedge \mathfrak{b}$ for arbitrary true statements $\mathfrak{a}$ and $\mathfrak{b}$. If the intuitionists tacitly believe that $\mathfrak{a} \wedge \mathfrak{b}$ will be true for all true statements $\mathfrak{a}$ and $\mathfrak{b}$ (once they get around to proving $\mathfrak{a} \wedge \mathfrak{b}$), they are essentially fixing a truth value for $\mathfrak{a} \wedge \mathfrak{b}$, which they claim they will never do by (i).

Bishop and Heyting believe in the truth-preserving capacity of the logical connectives when there is a general rule for transforming constructions for the premises into constructions for the conclusions. For instance, if we have a constructive proof of $\mathfrak{a}$ and a constructive proof of $\mathfrak{b}$, we can combine these for a constructive proof for $\mathfrak{a} \wedge \mathfrak{b}$. Implication is tricker. Heyting says,

> "The *implication* $\mathfrak{p} \rightarrow \mathfrak{q}$ can be asserted, if and only if we posses a construction $\mathfrak{r}$, which, joined to any construction proving $\mathfrak{p}$ (supposing that the latter be effected), would automatically effect a construction proving $\mathfrak{q}$. In other words, a proof of $\mathfrak{p}$, together with $\mathfrak{r}$, would form a proof of $\mathfrak{q}$." (ibid., 98)

Here, Heyting seems to say that $\mathfrak{r}$ is particular to the statements of $\mathfrak{p}$ and $\mathfrak{q}$ and is not a general rule for proving any implication. Notably, Heyting does not seem to take $\mathfrak{p} \rightarrow \mathfrak{q}$

as equivalent to $\neg\mathfrak{p} \vee \mathfrak{q}$. He does not discuss the case where we can prove "$\mathfrak{p}$ leads to a contradiction" (i.e. $\neg\mathfrak{p}$), but it is hard to imagine that he interprets the act of transforming a construction of $\mathfrak{p}$ into a construction of $\mathfrak{q}$ via $\mathfrak{r}$ to be the same as proving $\neg\mathfrak{p} \vee \mathfrak{q}$. He does not even believe $\mathfrak{p}$ or $\neg\mathfrak{p}$ cover all the cases about the first statement in the implication, since he denies the law of excluded middle, so it is unlikely that he would think $(\mathfrak{p} \wedge \mathfrak{q}) \vee (\neg p)$ is the same as $\mathfrak{p} \rightarrow \mathfrak{q}$.

Brouwer, in contrast, admits outright that we have no reason to believe that logical deductions preserve truth. As explained earlier, he does not see logic as having anything to do with the content of mathematics; instead, it is merely a mode of expressing what we already know to be true via mental construction. So, logical manipulations alone cannot increase our stock of true statements. Brouwer does not believe that if $\mathfrak{a}$ is true and $\mathfrak{b}$ is true, $\mathfrak{a} \wedge \mathfrak{b}$ should be true—the fact that this has always held in the past is a pattern and not a law.

Even though the intuitionists reject external mathematical reality, it would be problematic for them to reject (iv) if they rely on the properties of objects that have never been constructed. Bishop and Heyting do seem to rely on the closure of the natural numbers under addition; this amounts to the belief that if we take any natural numbers a and b, we can effect the construction $a + b$, and the result will be a natural number. How can they justifiably believe in the natural-numberness of an as-of-yet unconstructed number, without deferring to (iv)? Both Bishop and Heyting try to cover their bases here. Bishop speaks of "hypothetical constructions," the constructions we assume we can perform based on constructions we have already performed (Bishop (1967), 3). Heyting says something similar, relying on a general property of a whole class of similar constructions:

> "Let "N" be an abbreviation for "natural number". The first two properties (1 is an N and if x is an N, then the successor of x is an N) can immediately be seen to be true by carrying out the generating construction." (Heyting (1956), 14)

Brouwer, on the other hand, explicitly rejects (iv). As explained above, he denies up and down that it is possible to believe in the existence of a mathematical entity corresponding to a chain of syllogisms or other linguistic performance.

Historical Motivation Brouwer's first purely intuitionistic paper, his doctoral thesis of 1907, occurred just after the discovery of the set-theoretical paradoxes and just before the bombardment of foundational worries of the early 20th century. The major set-theoretical paradoxes (e.g. Russell's, Burali-Forti's, Richard's, Nelson's) were discovered between 1895 and 1910. Nonetheless, Brouwer was not inspired to pursue intuitionism in response to either of these events. Instead, intuitionism grew out of Brouwer's other views on the products and capacities of the human mind as first expressed in his 1904 *Leven, Kunst en Mysteik* (Life, Art, and Mysticism). His goal in his dissertation was to develop a philosophy of math that treats construction, logic, and language in a way that is faithful to the concept of math as a construction of human intelligence (Hesseling (2003)). Hesseling says,

> "Brouwer's objective was not to describe various views on the foundations of mathematics or to provide foundations for mathematics as it was then practised, but to work out his own ideas in the philosophy of mathematics. This becomes clear form a letter Brouwer sent to Kortweg in 1906. In the letter, Brouwer writes that he still adheres to his convictions from two years ago, but that he is glad to find that he can now support them better with mathematical arguments. Thus, Brouwer started with his own ideas and looked for mathematics that fitted in, instead of working the other way round." (ibid., 35)

So, Hesseling does not see Brouwer's work as a response to a new result in math, nor to the paradoxes, nor to the budding tradition of foundational criticism.

On the surface, Bishop is trying to take a "straightforward" approach to math, to make it comprehensible to finite intelligences. He says of the other participants in the intuitionist tradition,

> "Brouwer's disciples joined forces with the logicians in attempts to formalize constructive mathematics. Others seek constructive truth in the framework of recursive function theory. Still others look for a short cut to reality, a point of vantage which will suddenly reveal classical mathematics in a constructive light. None of these substitutes for a straightforward realistic approach has worked. It

> is no exaggeration to say that a straightforward realistic approach to mathematics has yet to be tried. It is time to make the attempt." (Bishop (1967), 10)

Bishop, who is of course following in Brouwer's footsteps, sets himself apart from the rest of the post-Brouwer intuitionist tradition and from the foundational debate of a few decades earlier. Bishop's motivation echoes Brouwer's; he says that Brouwer "wanted to strengthen mathematics by associating to every theorem and every proof a pragmatically meaningful interpretation," (ibid., 6). Bishop reflects this sentiment in his desire to take a "realistic"[6] approach to math.

Methods and Goals Bishop and Heyting communicate their position in a similar way, providing a general introduction to intuitionistic philosophy but stopping short of a philosophical discussion which fully motivates their work. Instead, the majority of *Foundations of Constructive Analysis* and *Intuitionism: an Introduction* are devoted to actually working through the details of constructive mathematics in analysis, algebra, and topology.

In this sense, they are trying to communicate their ideas on a purely mathematical ground (albeit on a constructively valid mathematical ground). They work through the system of constructive math, concretely demonstrating what it means to have performed a valid construction, and demonstrating which results from classical math can be achieved constructively. They also point out what they take to be ambiguities in classical math.

In another sense, though, they cannot communicate their positions purely mathematically, because they cannot use classical mathematical methods to disprove the results they reject. That is, they cannot disprove the Bolzano-Weierstrass theorem; they can only mention their epistemological and ontological reasons for rejecting it, and attempt to prove a similar theorem constructively.

The constructivist's goals for mathematics as a field are distinctly different than the mainstream mathematician's. In terms of quantity of results, the disadvantages of construc-

[6]He means something akin to "reasonable" or "practical," rather than "realism" as the term is used in modern philosophy.

tivism are obvious: it yields fewer and weaker results than classical mathematics, since many results of classical mathematics have no constructive interpretation. According to Bishop, "Hilbert, who insisted on constructivity in metamathematics but believed the price of a constructive mathematics was too great, was willing to settle for consistency" (Ibid., 10). Hesseling relates an amusing anecdote,

> "Also in 1924, Brouwer lectured for the Göttingen Society on the consequences of the intuitionistic point of view in mathematics. After the lecture, Hilbert is reported to have stood up and proclaimed that the goal is to obtain more, not less theorems." (Hesseling (2003), 74)

Bishop says, more flatteringly,

> "The extent to which good constructive substitutes exist for the theorems of classical mathematics can be regarded as a demonstration that classical mathematics has a substantial underpinning of constructive truth." (Bishop (1967), 9)

For Bishop, multitudinousness of theorems is to be desired, but not at the expense of constructivist principles. The many extra theorems classical math is able to prove above and beyond those that are constructively valid add no value despite their numerousness.

Not all mathematicians attracted by the intuitionists' philosophical rigor chose to give up on the extra advances of classical math. Hilbert, notably, decided the "price of a constructive mathematics was too great" (ibid., 10). Bishop also notes that Weyl, whom Gödel describes as a "half-intuitionist" (Godel (1947)), is an intuitionist at heart but "in practice suppressed his constructivist convictions," (Bishop (1967), 10). The classical paradigm prevails over intuitionism today, even though contemporary mathematicians value results with algorithmic meaning.

Heyting admits that constructivism fails to be more useful than mainstream mathematics—or useful at all for the physical sciences (Heyting (1956), 10). The physical sciences are only indirectly dependent on the foundations of math, so it is decidedly unlikely that bolstering foundatoins at the expense of results would be a boon to physicicsts. He couches this as a defecit of intuitionism, but the loose connection between his mathematical program and the physical sciences is not really a liability. Physical scientists do not concern themselves with mathematical epistemology, and mathematicians have no special responsibility to honor the preferences of physicists above their own epistemological concerns. And, perhaps more relevantly, applications to the physical sciences are only occasionally an inspiration for mathematical development; the invention of the calculus and the analytical methods developed in the vibrating string and heat diffusion problems are notable exceptions. Physical applications of mathematical results are generally a happy byproduct rather than an impetus, so a mathematical program's dissociation from the natural sciences is not inherently damaging.

Heyting suggests instead that mathematics be considered "a study of certain functions of the human mind" similar to the humanities (ibid., 10). Considering the usefulness of classical mathematics, it is disappointing to ultimately chalk up mathematics to advanced navel-gazing. But it is not exactly fair to rank constructive and classical math on the same footing in terms of usefulness and breadth, as the two stances are nigh incommensurable. Someone who believes true meaning and existence are only to be found in computation will not consider classical results genuinely stronger. Someone who does not believe meaning hinges on computability has no reason to prefer constructive math at the expense of numerous results.

Heyting claims that constructivism avoids paradoxes—but he admits that constructivism is more restrictive than would really be necessary to avoid them (ibid., 11). And at any rate, paradox-avoidance is not the main motivation for the constructivists. Consistency in math is a plausible goal, for a foundational system so often described as being "secure." Bishop claims that consistency is ensured in constructive math, up to human error, making it a non-issue for the constructivist mathematician (Bishop (1967), 353). But he never justifies any claim about the impossibility of deriving a contradiction through constructivist rules of inference. Without such a justification, constructivism is in no better position than classical math, paradoxes or no paradoxes. Brouwer is more verbose than Bishop on this topic, but similarly casts the issue off as irrelevant. Hesseling says,

> "If we have a set of logical axioms, Brouwer argues, and we can point out a mathematical system of which the logical axioms can be considered to express properties, we know that no contradiction can occur, because a constructed mathematical system cannot contain a contradiction." (Hesseling (2003), 42)

Brouwer denies the necessity and validity of consistency proofs. He holds that such proofs only prove something about a formal linguistic system for communicating math, and that they don't prove anything about the mathematical content the system sets out to represent.

There is not a trade-off, per se, between constructivism and classical math, since the choice to adopt constructivism is mainly a matter of ontological belief and preference. The primary methodological advantage of constructivism is that it offers us a rubric for determining mathematical validity: namely, whether or not a construction has been performed. None of the constructivists, though, communicate their position in such a way that convinced mainstream mathematicians such a rubric is useful or necessary.

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