Bernard Bolzano, a Genius Unnoticed in his Time

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Bernard Bolzano was a Czech, son of Italians, who wrote in German. He was born in 1781, and from his writings we know he first discovered mathematics in 1795, with him stating 15 years later that "this science has always been one of my favourite studies, the theoretical part in particular, as a branch of philosophy and as a way of practising correct thinking"[2, p.88]. One year later, in 1796, he entered the University of Prague to study mathematics, philosophy, and physics. In 1800, he began to study theology as well, and he became a catholic priest in 1804.

Through this time, he kept his interest in mathematics, and in 1804 he also released his first mathematical writings, translated as "Considerations on Some Objects of Elementary Geometry". In this book, he first introduces some of the ways that he looks at mathematical concepts and proofs. Most notably, he states that proofs should not make use of "some fortuitous, alien, intermediate concept" [2, p.32], such as the use of planes in proofs about angles, or the use of motion in proofs about space. Because of this, he sought to reprove basic geometrical concepts, limiting himself as much as possible to clearly defined terms, and to prove things in their proper order, only using that which had been proven, without reliance on intuitive concepts. Further, he stated that it was important not only to move mathematics forward, but also to trace "back all truths of mathematics to their ultimate foundations" [2, p.31]. Bolzano believed that many ideas in mathematics seem intuitive, but that all things should be explained in their entirety in order to find the most basic ideas upon which others are based. As an example of this, he noted that it is obvious to intuition that the angles at the base of an isosceles triangle are equal, but the early geometers sought to prove it regardless, in order to better understand the reasons behind their judgments. In short, Bolzano wanted to reconstruct mathematics in a more rigorous manner, a project which his later papers would continue as well. As a result, Bolzano's mathematical work is all done with thorough explanation, so that a student might be able to read and understand it. Regrettably, this 1804 work attracted little attention.

In 1810, Bolzano tried reaching out to the mathematical community again, this time with a writing titled "Contributions to a Better-Grounded Presentation of Mathematics". In this paper, Bolzano again seeks to correct the flaws he sees in mathematics. However he saw deeper flaws than he did in 1804 – in looking at Euclid, Bolzano states that as Euclid first proved “the equality of triangles; [and] only much later their similarity, which is derived through an outrageous detour through a consideration of parallel lines as well as the area of triangles etc.!” [4, p.16], then Euclid's entire method of proof must be faulty. Hence, he sought to rectify the situation by attempting to put forward a complete reorganization of mathematics and all conceptual sciences, done through the framework of logic [4, p.17]. However, the logic of the day was not sufficient, and “Bolzano's project to reform mathematics thus also demanded that logic be dragooned into the ranks of the modern sciences” [4, p.18]. To accomplish his goals, Bolzano wrote about the subject of mathematics as a whole, with sections on what mathematics is and how it is relative to the sciences, as well as seeking to give explicit meaning to such things as definitions, axioms, theorems, corollaries, and many others. Bolzano also wrote heavily on the proper use of logic to reach conclusions, and further discussion of how intuition should not be used as part of proof. An important part of this was when he noted that any true proof must make use of all of its hypothesis – either the unused parts of the hypothesis are unnecessary, so that “the theorem itself must be expressed too narrowly” or “that the proof itself contains some false conclusion” [2, p.132]. Bolzano also wrote that, while many mathematicians of the time believed that axioms should be those concepts which were obvious, he believed that axioms should be “a ground but never a consequence of other propositions”, and that finding proofs for propositions considered to be obvious was important, “since they often reveal unnoticed but vital elements of the intrinsic structure of the sciences to which they belong” [1, p.11]. However, as with his first paper, Bolzano received little response from the mathematical community.

Despite his setbacks, Bolzano continued his attempts to push for a more rigorous and grounded form of mathematics, publishing another paper in 1816. In this paper, whose title I will shorten to “The Binomial Theorem”, Bolzano noted that existing proofs that the generalized form of the binomial expansion was valid all had errors. A deep problem with existing proofs, one that he had recognized as a sign of a false proof in 1810 is that although it was known that the generalized binomial theorem did not work for values of x greater than 1 or less than -1, existing proofs did not use this hypothesis. In other words, if the existing proofs were valid, they would have proven that the binomial theorem held for all values of x, which was known to be untrue, and hence these proofs must be invalid. Furthermore, Bolzano was opposed to the use of what he referred to as infinitely small quantities, more commonly known at the time as infinitesimals – numbers which were considered to be greater than zero and less than any fixed quantity. Instead, he said, he "made use, with equal success, of the concept of those quantities, which can become smaller than any given quantity, or [...] quantities which can become as small as desired" [2, p.158], replacing numbers that are smaller than any given number with those that can become smaller than any fixed number – a seemingly small distinction, but insightful and forward-thinking. Thanks to the realization of this problem, Bolzano was able to construct a definition of convergence far more well-defined that of other mathematicians of his time. In the following selection, I will show how Bolzano sought to introduce students to these quantities and their properties, in §14 through §18 of "The Binomial Theorem" § is the symbol Bolzano uses to denote each section of his writings. All comments in square brackets are mine.

§14

> *Convention*. To designate a quantity [allowed to be either positive or negative] which can become smaller [closer to zero] than any given quantity, we choose the symbols ω, Ω or something similar.

In the next section, Bolzano uses the < symbol, but without the modern meaning. To Bolzano, "less than" meant "closer to zero". Hence, whenever Bolzano writes $a < b$, what he means in modern notation is $|a| < |b|$. His format will be kept as is.

§15

Lemma. If each of the quantities $\omega, \overset{(1)}{\omega}, \overset{(2)}{\omega}, \ldots, \overset{(m)}{\omega}$ [Bolzano uses I in place of 1 in his texts , but we change it to 1 for convenience] can become as small as desired while the (finite) number of them does not alter, then their algebraic *sum* or *difference* is also a quantity which can become as small as desired, i.e.

$$\omega \pm \overset{(1)}{\omega} \pm \overset{(2)}{\omega} \pm \ldots \pm \overset{(m)}{\omega} = \Omega.$$

Proof. For if the *sum* of these quantities is to be < *D*, where *D* designates some finite quantity, then if there is a constant number *n* of them, each of them may be taken < $\frac{D}{n}$, which is possible as a consequence of the assumption [that each ω can be made less than any given quantity]. Then certainly $\omega \pm \overset{(1)}{\omega} \pm \overset{(2)}{\omega} \pm \ldots \pm \overset{(m)}{\omega} < D$, even if the terms of this sum should all be positive, and all the more so in any other case. [End of Proof]

§16

Corollary. Therefore also $(A + \omega) \pm \left(B + \overset{(1)}{\omega}\right) \pm \left(C + \overset{(2)}{\omega}\right) \pm \ldots \pm \left(R + \overset{(r)}{\omega}\right) = A \pm B \pm C \pm \ldots \pm R + \Omega$, if the number of these terms does not change, while $\omega, \overset{(1)}{\omega}, \ldots, \overset{(r)}{\omega}$ can [each] become as small as desired.

[Proof of Corollary] For in fact, $(A + \omega) \pm (B + \overset{(1)}{w}) \pm (C + \overset{(2)}{\omega}) \pm \ldots \pm (R + \overset{(r)}{\omega})$

$= A \pm B \pm C \pm \ldots \pm R + (\omega \pm \overset{(1)}{\omega} \pm \overset{(2)}{\omega} \pm \ldots \pm \overset{(r)}{\omega})$

$= A \pm B \pm C \pm \ldots \pm R + \Omega$ (§15). [End of Proof]

§17

Lemma. Every product of a quantity which remains constant, and another which can become smaller than any given quantity, can itself become smaller than any given quantity. That is, $A.\omega = \Omega$ [Note that Bolzano uses a period to represent multiplication, and we will keep his representation].

Proof. For if A.ω is to be < D, then just take $\omega < \frac{D}{A}$. [End of Proof]

§18

Lemma. $(A+\omega)(B+\overset{(1)}{\omega}) = A.B+\Omega$.

Proof. For $(A+\omega)\left(B+\overset{(1)}{\omega}\right) = A.B+\omega.B+\overset{(1)}{\omega}.A+\overset{(1)}{\omega}.\omega = A.B+\Omega$(§§17, 15).

[End of Proof]

Sadly, this 1816 paper was not well noticed by mathematicians at the time either. Bolzano, not discouraged, pushed forward, publishing in 1817 another paper whose title will be shortened to "A Purely Analytic Proof". In this paper, he seeks to prove that given a continuous function *f*, if *f*(*a*) < 0 and *f*(*b*) > 0 for some *a* and *b*, then there exists some *c* between *a* and *b* such that *f*(*c*) = 0. Though this concept may seem obvious, Bolzano notes that proofs up to this time had relied far too heavily on geometric intuition, incorrect concepts of continuity of a function, or far more complicated assumptions than should be permitted. As such, he first seeks to provide a correct definition of continuity for him to work from, stating that "the expression *that a function fx varies according to the law of continuity for all values of x inside or outside certain limits* means only that, *if x is any such value the difference f(x + ω) – fx can be made smaller than any given quantity, provided ω can be taken as small as we please*, or

f(*x* + *ω*) = *f*(*x*) + *Ω*." [2, p.256]. In this simple statement placed in his preface, Bolzano gave the most correct definition of continuity of his time. Later in his preface he also notes that to prove this theorem, it would be sufficient to prove the more general statement that "if two continuous functions of *x*, *fx* and *ϕx*, have the property that for *x* = *α*, *fα* < *ϕα*, but for *x* = *β*, *fβ* > *ϕβ*, there must always exist some value of *x* lying between *α* and *β* for which *fx* = *ϕx*" [2, p.260], and it is this that he seeks to prove first. We will proceed to examine some of the statements Bolzano proves in order to obtain this result in selected sections from "A Purely Analytic Proof". We first note that a sequence is an ordered list, while a series is the sum of all of the terms of a sequence, but Bolzano refers to sequences as "series" and to what we call series as "the value of the series".

§1

> *Convention*. Suppose that for a *series of quantities* the special case does not occur that from a certain term onwards all the terms are each *zero,* as happens for example after the (n + 1)th term in the *binomial series* for every positive whole-numbered exponent n. Then it is obvious that the *value of this series*, that is, the quantity resulting from its summation of its terms, cannot always remain the same if the number of terms is arbitrarily increased. In particular this value must certainly change every time the number of terms is increased, even by a *single one* which is not zero. Hence the value of a series depends not only on the *rule* determining the formation of the individual terms but also on their *number*. So this value represents a *variable* quantity even though the *form* and *magnitude* of the individual terms remain unchanged. With this in mind, we denote a *function* of x, which consists of an arbitrarily extendible *series* of terms and whose value therefore depends not only on x but also on the *number of terms r*, by $\overset{(r)}{F}(x)$ or $\overset{r}{F}x$ [The rth partial sum of a power series in x]. So, for example,
>
> $$A + Bx + Cx^2 + \ldots + Rx^r = \overset{r}{F}x, \text{ while } A + Bx + Cx^2 + \ldots + Rx^r + \ldots + Sx^{r+s} = \overset{(r+s)}{F}x.$$

Although Bolzano introduces, and continues to work with, partial sums of power series, his arguments would still have held for partial sums of arbitrary series, as any series can be easily represented as a power series with $x = 1$. §2 through §6 are corollaries of §1. §2 states that each further term in a series can be either constant or variable, and introduces a geometric series $a + ae + ae^2 + ae^3 + \ldots$, where e is the common ratio. §3 states that there exist series which can exceed any given value if enough terms are summed. In §4 and §5, Bolzano notes that there exist series that do not exceed some given value, and some of these series have the characteristic that the difference between the value of the series after

r terms and after $r + s$ terms can be made as small as desired, provided r is made large enough. This is what is now known as the Cauchy Criterion for convergence of a series, named for Augustin-Louis Cauchy, who discovered it separately in 1821, 4 years later. Further, he notes that the previously introduced geometric series has this property for the common ratio $-1 < e < 1$, and that it must also hold for all series that decrease more quickly than a geometric series. We begin again with §6.

§6

Corollary 5. If the values of the sums of the first n, $n + 1$, $n + 2$, …, $n + r$ terms of a series [now called partial sums of the series] like those of §5 [so the difference between the values of the series after r and after r + s terms can be made arbitrarily small – in this corollary he uses n and r instead of r and s] are denoted (§1) by $\overset{n}{F}x,\ \overset{n+1}{F}x,\ \overset{n+2}{F}x, \ldots,\ \overset{n+r}{F}x,$ [so that they are represented by power series] respectively, then the quantities

$$\overset{1}{F}x, \overset{2}{F}x, \overset{3}{F}x, \ldots, \overset{n}{F}x, \ldots, \overset{n+r}{F}x, \ldots$$

$$[\overset{1}{F}x = A + Bx; \overset{2}{F}x = A + Bx + Cx^2; \ldots]$$

represent a *new* series (called the *series of sums* of the previous one) [Bolzano does not write this new "series" as a sum, so we would refer to it as a sequence today]. By assumption this has the special property that the difference between its nth term $\overset{n}{F}x$ and every later term $\overset{n+r}{F}x$ (no matter how far from that nth term) stays smaller than any given quantity, provided n has first been taken large enough. This difference is the increase produced in the *original* series by a continuation beyond its nth term, and by the assumption, provided n has been taken large enough, this increase should be as small as we please.

In the following section, Bolzano makes an unstated assumption about the real numbers – what is now known as the "Axiom of Completeness", which states that the real number line has no "holes" or “gaps”. This axiom must be assumed for Bolzano's proof, but he does so without seeming to realize his assumption. This axiom may be stated in multiple equivalent ways, and we will examine some of these equivalent statements when appropriate.

§7

Theorem. If a series [sequence] of quantities

$$\overset{1}{F}x, \overset{2}{F}x, \overset{3}{F}x, \ldots, \overset{n}{F}x, \ldots, \overset{n+r}{F}x, \ldots$$

has the property that the difference between its *n*th term $\overset{n}{F}x$ and every later one $\overset{n+r}{F}x$, however far this latter term may be from the former, remains smaller than any given quantity if *n* has been taken large enough, then there is always a certain *constant quantity*, and indeed only *one*, which the terms of this series [sequence] approach and to which they can come as near as we please if the series [sequence] is continued far enough.

[In modern terms, if a sequence meets the Cauchy Criterion for convergence of a sequence, the value of its *n*th term converges as *n* increases]

Proof. It is clear from §6 that a series [sequence] such as that described in the theorem is possible. But the hypothesis that there exists a quantity *X* which the terms of the series [sequence] approach as closely as we please when it is continued ever further certainly contains nothing impossible, provided it is not assumed this quantity be *unique* and *constant*. For if it is to be a quantity which may vary then it can, of course, always be taken so that it is suitably near the term $\overset{n}{F}x$ with which it is just now being compared – even exactly the same as it [The value of the terms of this sequence can clearly approach some variable term, as we can simply set the variable to always be as near as we wish to the *n*th term of the sequence]. [Claim] But also the assumption [the existence of] of a *constant* quantity with this property of proximity to the terms of our series [sequence] contains nothing impossible because on this assumption it is possible to determine this quantity as accurately as we please. [Proof of Claim] For suppose we want to determine *X* so accurately that the difference between the assumed value and the true value of *X* does not exceed a given quantity *d*, no matter how small.

This is where Bolzano makes his earlier-mentioned assumption of the completeness axiom. It is now known that one way of stating the completeness axiom is that every sequence which meets the Cauchy Criterion converges to some real number. When Bolzano assumes that if we can approximate a real number *X* as closely as we wish, then it exists, he is assuming the completeness axiom without stating it.

Then we simply look in the given series for a term $\overset{n}{F}x$ with the property that every succeeding term $\overset{n+r}{F}x$ differs from it by less than $\pm d$. By the assumption [in the statement of the theorem] there must be such an $\overset{n}{F}x$. Now I say that the value of $\overset{n}{F}x$ differs from the true value of the quantity *X* by at most $\pm d$. For if *r* is increased arbitrarily, for the same *n*, the difference $X - \overset{n+r}{F}x = \pm\omega$ can become as small as we please. But the difference $\overset{n}{F}x - \overset{n+r}{F}x$ $[Ox^{n+1} + \ldots + Rx^{n+r}]$ always remains < $\pm d$, however large *r* is taken. Therefore the differences

$$X - \overset{n}{F}x = (X - \overset{n+r}{F}x) - (\overset{n}{F}x - \overset{n+r}{F}x)$$

must also remain < $\pm(d + \omega)$ [Recall from above that Bolzano means < $|d + \omega|$]. But since for the same *n* this is a *constant* quantity, while ω can be made as small as we please by increasing *r*, then $X - \overset{n}{F}x$ must be = or < $\pm d$. For if it were greater [than *d*] and = $\pm(d + e)$, for example [where *e* is some arbitrary value], it would be impossible for the relation *d* + *e* < *d* + *ω*, i.e. *e*

$< \omega$, to hold if ω is reduced further [as e is some fixed quantity, and ω can be made less than any fixed quantity by definition] . The true value of X therefore differs from the value of the term $\overset{n}{F}x$ by at most d, and can therefore be determined as accurately as we please since d can be taken arbitrarily small. There *is* therefore a *real quantity* to which the terms of the series under discussion [the sequence which Bolzano refers to as the series of sums] approach as closely as we please if the series is continued far enough. [Proof that X is unique] But there is only *one* such quantity. For suppose that besides X there was another *constant* quantity Y which the terms of the series approach as much as we please if it is continued far enough, then the differences $X - \overset{n+r}{F}\ x = \omega$ and $Y - \overset{n+r}{F}\ x = \overset{1}{\omega}$ can be made as small as we please if r is allowed to be large enough. Therefore this must also hold for their own difference, i.e. for $X - Y = \omega - \overset{1}{\omega}$ which, if X and Y are held to be *constant* quantities, is impossible unless one assumes $X = Y$.

[End of Proof]

In §9, Bolzano states as a corollary that as the terms of this sequence approach some specific value, so must the value of the original series approach some specific value as more terms are added. In §10, Bolzano notes that this statement only holds if the change after an arbitrary number of further terms can be made as small as we please, rather than the change after an individual term, and gives the harmonic series as an example of when such a series fails to approach a specific value. We continue with §11.

§11

Preamble. In investigations of applied mathematics it is often the case that we learn that a definite property M applies to *all* values of a *variable quantity x* which are *smaller* than a certain u, without at the same time learning that this property does not apply to values which are greater than u. In such cases there can still perhaps be some $\overset{1}{u}$ that is $> u$ for which, in the

> same way as it holds for *u*, all values of *x* lower than $\overset{1}{u}$ possess property M. Indeed this property M may even belong to *all values of x without exception*. But if *this alone* is known, *that M does not belong to all x in general*, then by combining these two conditions we will now be justified in concluding: *there is a certain quantity U which is the greatest of those for which it is true that all smaller values of x possess property M*. This is proved in the following theorem.

In the following theorem, Bolzano proves the existence of what is now known as the least upper bound, or the supremum, of a set of real numbers. This is now known to be one of the ways of stating the completeness axiom. This theorem is also known for its connection to another statement of the completeness axiom, known as the Bolzano-Weierstrass Theorem. The Bolzano-Weierstrass Theorem states that any bounded sequence of real numbers has a convergent subsequence, or alternatively, that any infinite bounded set of real numbers, *A*, has a limit point. This connection can be seen as follows - if we let *U* be the supremum of values *u* for which there are infinite values in *A* less than *u*, then it can be seen that between $U - \omega$ and any $U + \omega$ there must be infinite values in *A*, meaning that there are an infinite number of points from *A* in any open interval around *U*, which makes *U* a limit point of *A*. This theorem was proven independently by Weierstrass around 50 years later, and Bolzano's name was later attached due to its connection to the existence of the supremum.

§12

> *Theorem*. If a property *M* does not apply to *all* values of a variable quantity *x* but does apply to *all* values smaller than a certain *u*, then there is always a quantity *U* which is the greatest of those of which it can be asserted that all smaller *x* possess the property *M*.
>
> *Proof*. 1. Because the property *M* holds for all *x* *smaller* than *u* but nevertheless not for *all x*,

there is certainly some quantity $V = u + D$ (where D represents something positive) of which it can be asserted that M does not apply to all x which are $< V = u + D$. If I then raise the question of whether *M in fact applies to all x which are* $< u + \frac{D}{2^m}$ where the exponent m is in turn first 0, then 1, then 2, then 3, etc., I am sure that the *first* of my questions will have to be answered *'no'*. For the question of whether M applies to all x which are $< u + \frac{D}{2^0}$ is the same as that of whether M applies to all x which are $< u + D$, which is ruled out by the assumption. What matters is whether all *succeeding* questions, which arise as m gradually gets larger, will also be ruled out. Should this be the case, it is evident that *u itself* is the greatest value for which the assertion holds that smaller x have the property M. For if there was an even greater value, for example $u + d$, i.e. if the assertion held that also all x which are $< u + d$ have the property M, then it is obvious that if I take m large enough, $u + \frac{D}{2^m}$ will at some time be = or $< u + d$. Consequently if M applies to all x which are $< u + d$ it also applies to all x which are $< u + \frac{D}{2^m}$. We would therefore have said 'no' to this question but would have had to say 'yes' [a contradiction, which implies our assumption that all $x < u + d$ have the property M is incorrect]. Thus it is proved that in this case (when we say 'no' to all the above questions) there is a certain quantity U (namely u itself) which is the greatest for which the assertion holds that all x below it poses the property M.

2. However, if one of the above questions is *answered 'yes'* and m is the particular value of the exponent for which this happens *first* (m can be 1 but, as we have seen, not 0), then I now know that the property M applies to all x which are $< u + \frac{D}{2^m}$ but not to all x which are $< u + \frac{D}{2^{m-1}}$ [As we answered 'no' for all values less than m]. But the difference between $u + \frac{D}{2^{m-1}}$ and $u + \frac{D}{2^m}$ is $= \frac{D}{2^m}$ [because $\frac{D}{2^{m-1}} = 2 \times \frac{D}{2^m}$]. If I therefore deal with this as I did before with the difference D, i.e. if I raise the question of whether M applies to all x which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}}$$

and here the exponent *n* denotes first 0, then 1, then 2, etc., then I am sure once again that at least the *first* of these questions will have to be answered 'no'. For to ask whether *M* applies to all *x* which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+0}}$$

is just the same as asking whether *M* applies to all *x* which are $< u + \frac{D}{2^{m-1}}$, which has previously been denied. But if all my *succeeding* questions are also to be answered negatively as I gradually make *n* larger and larger, then it would appear, as before, that $u + \frac{D}{2^m}$ is that greatest value, or the *U*, for which the assertion holds that all *x* below it possess the property *M*.

3. However, if one of these questions is answered positively and this happens first for the particular value *n*, then I now know *M* applies to all *x* which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}}$$

but not to all *x* which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n-1}}.$$

The difference between these two quantities is $= \frac{D}{2^{m+n}}$ and I deal with this again as before with $\frac{D}{2^m}$, etc.

4. If I continue this way as long as I please it may be seen that the result I finally obtain must be one of two things.

(a) Either I find a value of the form

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r+s}}$$

which appears to be the greatest for which the assertion holds that all *x* below it possess the property *M*. This happens in the case when the question of whether *M* applies to all *x* which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r+s}}$$

are answered with 'no' for every value of s.

(b) Or I at least find that *M* does indeed apply to all *x* which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r}}$$

but not to all *x* which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r-1}}.$$

Here I am always free to make the number of terms in these two quantities even greater through new questions.

5. Now if the *first* case occurs the truth of the theorem is already proved. In the *second* case we may remark that the quantity

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r}}$$

represents a series whose number of terms I can increase arbitrarily and which belongs to the class described in §5. This is because, depending on whether *m*, *n*, …, *r* are all =1, or some of them are greater than 1, the series decreases at the same rate, or more rapidly than, a geometric progression whose ratio is the proper fraction $\frac{1}{2}$. From this it follows that it has the property of §9, i.e. there is a certain *constant quantity* to which it can come as close as we please if the number of terms is increased sufficiently. Let this quantity be *U*; then I claim the property *M* holds for all *x* which are < *U*. For if it did not hold for some *x* which is < *U*, e.g. for $U - \delta$, then the quantity

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r}}$$

must always keep at the distance δ from *U* because for all *x* that are smaller than it, the property *M* is to hold. Since every *x* that is

$$= u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r}} - \omega$$

however small ω is [assuming ω is positive], possesses the property *M*, while on the other hand, *M* is not to apply to $x = U - \delta$, it must therefore be that

$$U - \delta > u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r}} - \omega$$

or

$$U - [u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r}}] > \delta - \omega$$

Hence the difference between *U* and the series cannot become as small as we please, since $\delta - \omega$ cannot become as small as we please because δ does not change, while ω can become smaller than any given quantity [This results in a contradiction, as *U* is, by assumption, what the value of the series approaches. Therefore, the assumption that *M* does not hold for some $x < U$ must be false]. But just as little can *M* hold for all *x* which are $< U + \varepsilon$. For the value of the series

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r-1}}$$

can be brought as close to the value of the series

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r}}$$

as we please because the difference of the two is only $\frac{D}{2^{m+n+\ldots+r}}$. Further, the value of the latter series can be brought as close as we please to the quantity *U*. Therefore the value of the first series can also come as close to U as we please. So

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r-1}}$$

can certainly become $< U + \varepsilon$. But now by assumption *M* does not hold for all *x* which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \ldots + \frac{D}{2^{m+n+\ldots+r-1}};$$

> so much less therefore for all x which are $< U + \varepsilon$. Therefore U is the greatest value for which the assertion holds that all x below it possess the property M. [End of Proof]

After this, in §13 and §14, Bolzano notes the importance of his theorem, and seeks to correct a number of errors that he believes those reading may believe or have believed, and then proceeds to prove the general case of the mean value theorem he introduced in his preface in §15, followed by a more specific result that any polynomial function must have a root between its positive and negative values.

Bernard Bolzano published another mathematical work in 1817, but in 1819, the murder of a conservative writer was used as reason to purge governmental and university positions of liberals. As a result, Bolzano lost his position at the University of Prague, was only allowed to publish his works in foreign journals, and was even put on trial by the Catholic Church, though he was declared innocent [2, p.11-12]. Punished and censored, he continued to write, completing his great work on logic, “Theory of Science”, in 1827, but not publishing it until 1837 due to this censorship. During this time he also published works on religion and metaphysics, but after completing “Theory of Science”, he began work on another great mathematical work – “Theory of Quantity”. Though this work was still unfinished by the time of Bolzano's death in 1848, a student posthumously published parts of it under the title of “The Paradoxes of the Infinite” in 1851. This work, along with the “Theory of Science”, gained him greater renown among the logicians, and to some degree among mathematicians as well. Cantor referred to his proof of the existence of an infinite set in “Paradoxes”, and may have been influenced by his mention of the fact that an infinite set can have one-to-one correspondence with its proper subsets [3]. However, the majority of his mathematical results continued to be unknown, and large sections his work on “Theory of Quantity” remained unpublished. This work included, among other things, the first construction of a function everywhere continuous and nowhere differentiable over its domain, a feat not accomplished again until Weierstrass did so in 1872, though Bolzano was only able to prove that between any two points at which his function was non-differentiable there existed a third. [2, p.351-352]. However, Bolzano's older mathematical writings were eventually rediscovered by a mathematician named Otto Stolz in 1881, and he finally came to gain the notice of the mathematical community, for they could, by then, view his approach as far more modern than that of his peers. At this late date, his works were not able to bring to mathematics the logical soundness that he had wished for. However, his goals were accomplished regardless, for other mathematicians, such as Cauchy and Weierstrass, had followed Bolzano's path toward rigor.

Works Cited

[1] Bolzano, Bernard, Rolf George, and Paul Rusnock. *On the mathematical method and correspondence with Exner*. Amsterdam: Rodopi, 2004. Web. 25 April 2014. <http://books.google.com/books?id=YYG5kshSxVIC>.

[2] Bolzano, Bernard and Steve Russ. *The mathematical works of Bernard Bolzano*. Oxford: Oxford University Press, 2004. Print.

[3] Morscher, Edgar. "Bernard Bolzano." The *Stanford Encyclopedia of Philosophy* (Spring 2013 Edition). <http://plato.stanford.edu/archives/spr2013/entries/bolzano/>.

[4] Rusnock, Paul. *Bolzano's philosophy and the emergence of modern mathematics*. Amsterdam: Rodopi, 2000. Web. 21 April 2014. <http://books.google.com/books?id=HuUjhwIonC4C>.