The Reality of the Complex: The Discovery and Development of

Imaginary Numbers

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Introduction

Do complex numbers *real*ly exist? Is this a trick question? How can we possibly see the square root of a negative number and, by the definition of a square root, claim that such a number can be affirmed? Other than for the sake of representation, this question was not easily answered until around three hundred years ago. Whether or not these numbers make sense in our reality, they are nevertheless quantities that arise in abundance from study within the discipline of mathematics.

For thousands of years, early mathematics flourished with little worry about the limitations of only positive and real quantities [10]. The initial discovery of imaginary numbers in the mid-1500s was everything but an Archimedean "Eureka" moment. Mathematicians circled around similar ideas regarding complex numbers for decades. Early mathematicians had no concept of negative numbers, let alone imaginary ones. However, mathematicians that studied solutions to different cubic equations noticed something profound - quantities that weren't "possible." The fact that the concept was so unfamiliar made them believe that they had discovered different things, though. Essentially, mathematicians worked amidst a kind of "language barrier." Therefore, conflict became prevalent throughout these years. With no standard by which to evaluate their thinking, each mathematician was more or less building from the ground up. As a result, there was little progress in complex theory for many years. However, the delayed consensus that was achieved - the certain existence of a complex system - strengthened mathematical studies in areas like roots of functions, the Cartesian coordinate system, and intermediate algebraic equations.

Over the span of four centuries, the complex system was studied with increasing intensity as it became more widely accepted as mathematical truth, whether at first for representation's sake or later by choice. Proof evolved as mathematicians began to further test and question mathematics. The surfacing of more abstract proof gave mathematicians new confidence to continually develop the mathematics of the complex system. In fact, the system is actually studied as an independent subject today: complex analysis. Thus, it is evident that the development of complex numbers, facilitated by

the expansion of rigorous proof, provided new and broadening perspectives with which to approach the many branches of mathematics.

Early Beginnings: Tartaglia, Cardano, and Solving Cubics

Tartaglia's Secret

The earliest traces of imaginary numbers find themselves in Italy, nestled inside a cubic equation. During the early 1500s, the dividing line between university mathematics and informal mathematics shrunk, and there was a rapid development in algebra [8]. In the early 1530s, mathematical genius Nicolo Tartaglia arrived at the scene [4]. Tartaglia was an interesting character, often referred to as "the stammerer" because of a speech impediment acquired from a massacre of his hometown during his childhood [4]. Needless to say, Tartaglia was not highly regarded, and his plethora of mathematical thoughts were overlooked.

One of his main accomplishments was the introduction of a "secret" method for solving a certain form of cubic equations [4]. Let us examine Tartaglia's technique. Though strictly demonstrative in style, it was a breakthrough of his time in the field of algebra. The method began to use abstract reasoning rather than numerical examples, eliminating the common question, "Does this work for any number?" Tartaglia used a form of Diophantine technique, creating multiple equations to represent his scenario. He wrote his solution in expository form in what many call Tartaglia's Poem [8]. Begin by examining the depressed cubic equation given by

$$x^3 + cx = d \tag{1}$$

In order to proceed, Targaglia defines two numbers such that their difference is d and their product is equal to the cube of one-third of c. Call these two numbers u and v. Thus, we have

$$u - v = d$$
$$uv = \left(\frac{c}{3}\right)^3$$

The task remains: find the two numbers given their product and difference. This involves solving quadratics, a process for which mathematicians had previously developed a method to solve. Tartaglia follows this known method, and seeks to find the quantity $u + v$, in order to use this expression coupled with $u - v$ solve for each variable in a kind of system of equations. The method he used for finding $u + v$ is squaring their difference, adding four times their product, and taking the square root of this quantity [6]. Tartaglia then creatively claims the solution to be

$$x = \sqrt[3]{u} - \sqrt[3]{v} \tag{2}$$

Consider a numerical application of Tartaglia's method. Begin with the equation of the type found in Equation 1, specifically

$$x^3 + 3x = 4$$

To solve for x, we must essentially find u and v such that their difference is 4 and their product is 1. Thus, by using Tartaglia's method, we begin by finding $u + v$ from the description above.

$$u + v = \sqrt{(u - v)^2 + 4uv}$$
$$= \sqrt{16 + 4}$$
$$= \sqrt{20} \Rightarrow 2\sqrt{5}$$

To find u, we know that $u - v = 4$ and $u + v = 2\sqrt{5}$. We can then solve for v in both equations. Then by substitution, set these equations equal to one another and find u. Repeat this process in a similar way to find v.

$$u = \sqrt{5} + 2$$
$$v = \sqrt{5} - 2$$

Finally, we can find the solution to this cubic equation by using Equation 2.

$$x = \sqrt[3]{\sqrt{5} + 2} - \sqrt[3]{\sqrt{5} - 2} \tag{3}$$

Tartaglia's process and these formulas were shared confidentially with Italian mathematician Girolamo Cardano in 1539. Cardano, however, betrayed Tartaglia's

trust and released the method in his grand work *Ars Magna* [9]. Thus, the memory of Tartaglia's method dissolved after *Ars Magna* was published, and this technique would forever be known from this point on as Cardano's formula for the depressed cubic.

Cardano - Formula Thief or Mathematical Genius?

Cardano is known to be one of the more scandalous mathematicians in history. He was often called a "scoundrel," and with good reason [4]. He used Tartaglia's method in his grand work *Ars Magna* to develop *his* formula for the solution to the depressed cubic in Equation 1. When Cardano published the book, he almost insultingly gave credit to Ferro, an Italian mathematician who had also developed a rough form of solution to the depressed cubic, but Cardano did not give credit to Tartaglia, from whom he actually received the ideas [5]. In his work, we can find that his technique is given to the reader in a slightly different form. Cardano explains how to find x from the perspective of an actual cube. A three-dimensional representation of his idea is shown below.
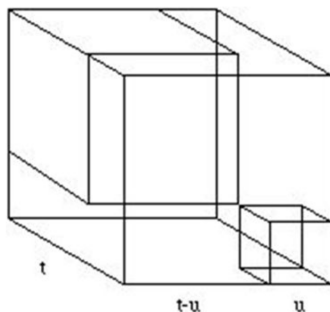
Figure 1. Gerolamo Cardano's "Cube" [12]

Cardano's method is in the more generalized form of a "rule," rather than using a system of equations and strictly algebraic steps in a methodical form, as Tartaglia's version does. In developing his rule, Cardano still used Tartaglia's solution for x from Equation 2. However, he creatively defined $x = t - u$, in relation to his cube. After solving his cube geometrically, he devised a general solution. Consider the excerpt from *Ars Magna* below.

"Cube one-third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will

duplicate this, and to one of the two you add one-half the number you have

already squared and from the other you subtract one-half the same."

Ars Magna, p. 98

Via Cardano's instructions in his rule, we can see that the formula is

$$x = \sqrt[3]{\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} + \frac{d}{2}} - \sqrt[3]{\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} - \frac{d}{2}}$$

Cardano's "recipe" represents the roots still in the radical form, thus disguising many of the roots' true integer values [6]. For example, he would have expressed the solution given in Equation 3 as is, in its radical form. When the solution is displayed this way, however, one might not naturally notice that this simplifies to just $x = 1$. We will further explore the significance of this later. For now, examine Cardano's recipe.

Cardano's method was certainly not foolproof. In certain examples of Equation 1, where c and d are negative, the method would incorporate square roots of negative numbers [6]. Cardano didn't validate these, but he did apply the "rules of ordinary arithmetic" to the imaginary quantities and showed that, if they existed, they could satisfy the cubic [10]. Since they were not recognized as a true solution, there was a search for more than 300 years to find such roots, but mathematicians finally realized it was essentially impossible because they were only using algorithms with real numbers.

However, Cardano did not just publish Tartaglia's version of solving the cubic of the form of Equation 1. He also attempted solving a cubic of a new form, depicted by

$$ax^3 + bx^2 + cx + d = 0$$

His method yet again led him to solutions for x that contained negative square roots [3]. As a result, he assumed there was no solution, and did not further study equations of this type. Because of his singular solution path, he not only came to an impasse, but essentially "missed" the real solutions that satisfied the cubic. Thus, his commentary on a topic he didn't fully comprehend seemingly backfired. The missing step for his solutions would later be provided by Rafael Bombelli.

Cardano's failure to represent complete mathematical thoughts and essentially rationalize this as acceptable did not harm his reputation, but rather illustrated the structure of mathematical proof during the mid-16th century. Mathematicians of this era were solely comfortable with examining mathematics in terms of what they knew. Cardano's *Ars Magna* is thus structured so that demonstrations are followed by general rules; the pattern of demonstration, rule, demonstration, rule, . . . carries throughout his work. This does carry early marks of deductive reasoning, but it is purely geometrical and the terms are not defined clearly, so the demonstrations are not a convincing base for the general rule that follows. Cardano actually labels each of his mathematical explanations with "Demonstration" [5]. He does this with a purpose: the mathematical demonstrations were provided so that "reasoning may reinforce belief" [5]. Elements of early proof found in *Ars Magna* serve as evidence that mathematics was essentially still taken much on faith (rather than by concrete proof) before the development of the complex system. The introduction of imaginary numbers to come, supported with concrete proof, would be incredibly applicable for developing cubics and more.

Bombelli's *L'Algebra* and the Beginnings of Complex Theory

Bombelli, The Man

Rafael Bombelli was certainly a character - he was known as the "last great sixteenth century Bolognese mathematician" [4]. Bombelli was quite an accurate representation of what we would term the 'common man,' because he did not study mathematics specifically as a scholar. Rather, he spent most of his life working as an engineer and architect [4]. He heard of Cardano's solution to the cubic, and was appalled by Cardano's dismissal of what he thought to be impossible solutions. Bombelli, unlike Cardano, could not rationalize the failure to completely represent the solution(s) to a cubic. His work, *L'Algebra*, was the turning point in the development of complex numbers, as his work became the representation of true rigor regarding the imaginary.

Bombelli's Language for the Unknown

Bombelli saw the quantities that appeared from Cardano's method as both "real" and potential for new discovery in mathematics, rather than just as the mark of a partially ineffective technique. Since he was essentially building "from the ground up," he decided to coin names for these quantities [6]. Bombelli's terms are the earliest representation of what we know as imaginary quantities. He identified $\sqrt{-1}$ to be *pdm*, an abbreviation for the Italian phrase *piu di meno*, which translates to "plus of minus". Similarly, he termed its opposite $(-\sqrt{-1})$ *meno di meno*, essentially "minus of minus" and abbreviated it as *mdm* [1]. Mathematics of the time was still mainly displayed through word form rather than through symbolic notation, especially in the case where no symbol had yet been developed for such imaginary quantities as those described by Bombelli. Thus, a mathematical expression involving Bombelli's language for the unknown would be quite different from its symbolic notation today. Consider the following expression.

$$7 + \sqrt{-3}$$

Bombelli would have represented this expression as "*seven plus three times plus of minus.*" Not only did Bombelli coin terms for imaginary quantities, but there is also evidence that he recognized some of their properties and related operations. To begin, he realized some form of the identity $i^2 = -1$. In his language, "plus of minus times minus of minus makes minus," where "minus" is essentially the negative identity, -1. Similarly, he claimed in his "plus of minus" terms that $(i)(-i) = 1$ [3].

Call Me Irreducible

Bombelli "attacked the irreducible case of the cubic, which...leads to the cube root of a complex number" [6]. He developed his own signature method for solving the cubic equation - one that boldly accepted the existence of square roots of negative numbers. Thus, Bombelli's procedure may have very well marked the beginning of complex numbers - he just did not call them numbers. Rather, he deemed them "linked radicals of a new type" [3].

Consider Bombelli's progress on the irreducible case of the cubic - that is, a cubic which has three real roots. He first showed how Cardano's formula didn't allow one to find these roots - hence, it had been termed irreducible by Tartaglia and Cardano. However, he proved how a combination of imaginary roots could lead to a real number, which provided the "missing step" that validated Cardano's method when considered in a creatively different way [4]. Below, we can see Bombelli's demonstration of real numbers being engendered from complex ones. Consider the cubic equation given by

$$x^3 = 15x + 4$$

Solving this via the Cardan method yields the solution

$$x = \sqrt[3]{\sqrt{-121} + 2} - \sqrt[3]{\sqrt{-121} - 2}$$

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \tag{4}$$

However, the equation described contains three real roots, namely 4, $-2 + \sqrt{3}$, and $-2 - \sqrt{3}$. Bombelli acknowledged that where there should have been three roots derived, there was only one root derived, and it was none of those three [4]. A greatly intrigued Bombelli began his own work on algebra in 1560 [4]. He actually went on to show that the imaginary expression given by Cardano's technique could be manipulated to produce a real quantity [4].

Let us look at an example, where he set out to essentially prove that a real value could be extracted. Bombelli claimed that the two expressions in Equation 4 only differed in a sign. So to begin, he creatively interpreted these expressions in terms of a and b, where a and b are integers. It is clear that the two expressions, as he said, only differ by a sign. In common notation, we have

$$\sqrt[3]{2 + \sqrt{-121}} = a + b\sqrt{-1}$$
$$\sqrt[3]{2 - \sqrt{-121}} = a - b\sqrt{-1}$$

Now, examine the first expression above. [1] Notice that

$$2 + \sqrt{-121} = (a + b\sqrt{-1})^3$$
$$= a^3 + 3a^2b\sqrt{-1} + 3ab^2(\sqrt{-1})^2 + b^3(\sqrt{-1})^3$$
$$= a(a^2 - 3b^2) + b(3a^2 - b^2)\sqrt{-1}$$

In order for this equality to be true, the quantity $a(a^2 - 3b^2)$ must equal 2 and the quantity $b(3a^2 - b^2)$ must equal 11 [4]. Since we are considering integer values for a and b, we know a must either be 1 or 2, and likewise b must be either 1 or 11, since both 2 and 11 are prime. The only two combinations of these solutions that satisfy both of these equalities are

$$a=2, b=1$$

When these quantities are substituted into Bombelli's "a and b" expressions above, we have

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1}$$
$$\sqrt[3]{2 - \sqrt{-121}} = 2 - \sqrt{-1}$$

At this point, return to Equation 4. The solution becomes

$$x = (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4$$

His idea is absolutely ingenious and astounding, because it inadvertently showed that real numbers are also complex. Essentially, Bombelli more empirically proved that any real number can be expressed in complex form. Bombelli's theory on this matter was a mathematical breakthrough, which gave new depth to developing set theory regarding the definition of the organization of the set of real numbers and the close relationship between the set of reals and the set of imaginaries.

Bombelli's work was so foundational for algebra that he is known to mark the beginning of the "new algebra" [8]. This is remarkable - because Bombelli initially recognized imaginary quantities, we see the emergence of an entirely new perspective to

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[1]This proof was reproduced from *The History of Mathematics*, p.325-6.

algebra as mathematicians knew it. This perspective was substantiated and accepted due to his bold demonstration that resembled a more concrete idea of proof. He actually took something and showed why there were misconceptions to it, instead of forming rules from calculations. His introduction of complex numbers thus provided the opportunity for mathematicians to expand upon cubic functions and study algebra in a new light.

Developing New Mathematical Language for Imaginary Quantities

Leibniz' Use of "Imaginary"

In the late 1600s, Descartes developed his rule of signs, and mentioned the term "imaginary" for complex numbers, but did not discuss this idea to any beneficial degree [11]. At the turn of the 18th century, we begin to see a more modern take on complex numbers. They became more widely studied and accepted as mathematical truth. During this century, mathematicians studied them enough "to acquire some confidence in them" [9]. One man that pioneered this re-introduction was Gottfried Wilhelm Leibniz, a German mathematician and philosopher. Leibniz noticed some interesting properties of complex numbers, specifically that a linear combination of complex conjugates would result in a positive rational number. For instance, the expression $\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}$ is equivalent to $\sqrt{6}$. Let us examine how this is so. Consider a slightly different imaginary expression given by

$$\sqrt{2 + \sqrt{-5}} + \sqrt{2 - \sqrt{-5}}$$

Squaring this quantity, one obtains

$$(2 + \sqrt{-5}) + 2\sqrt{2 + \sqrt{-5}}\sqrt{2 - \sqrt{-5}} + (2 - \sqrt{-5})$$

$$= 4 + 2\sqrt{(2 + \sqrt{-5})(2 - \sqrt{-5})}$$

$$= 4 + 2\sqrt{4 + 5}$$

$$= 4 + 2(3) = 10$$

Thus, we can say

$$\sqrt{2 + \sqrt{-5}} + \sqrt{2 - \sqrt{-5}} = \sqrt{10}$$

Because $\sqrt{10}$ is a real number, Leibniz's assertion holds for this case. Essentially, this creates a real quantity because of the nature of the conjugate - the validity of this does not hinge on the numerical values. In fact, Leibniz' claim can actually be proved, using the algebraic technique above, for any two positive integers. Call them a and b, where a is the integer adding the imaginary expression, and b is the real integer coupled with the imaginary quantity, $\sqrt{-1}$.

Proof. Consider the expression given by

$$\sqrt{a + \sqrt{-b}} + \sqrt{a - \sqrt{-b}}$$

Squaring this expression in a and b yields

$$\left(\sqrt{a + \sqrt{-b}} + \sqrt{a - \sqrt{-b}}\right)^2 = a + \sqrt{-b} + 2\sqrt{a + \sqrt{-b}}\sqrt{a - \sqrt{-b}} + a - \sqrt{-b}$$

$$= 2a + 2\sqrt{(a + \sqrt{-b})(a - \sqrt{-b})}$$

$$= 2a + 2\sqrt{a^2 + b}$$

Thus, we have

$$\sqrt{a + \sqrt{-b}} + \sqrt{a - \sqrt{-b}} = \sqrt{2a + 2\sqrt{a^2 + b}}$$

∎

Leibniz also studied cubics and was inspired by Bombelli's *L'Algebra*, so much as to give his own commentary [11]. Leibniz sought to prove that Cardano's formula was actually universally valid, and need not be redefined, as Bombelli had done. In examining the solution to cubics, he said they'll "either have three real roots... or two *imaginary* roots and one real root" [11]. Leibniz applied his reasoning to the "basic laws of algebra" in order to give this claim more credibility and rigor [11].

However, his most substantial contribution to the development of the complex was his use of the term "imaginary." Indeed, this was one of the first appearances of modern terminology for these quantities. Leibniz didn't just coin the term out of necessity, as was the case with Bombelli's terms. Leibniz actually had a strong reason as to why

he called the quantities imaginary. He said, "The Divine Spirit found a sublime outlet in the wonder of analysis..., that amphibian between being and not-being, which we call the imaginary root of negative unity" [9]. Leibniz used these numbers, but was still very unsure of their nature - this goes back to early mathematics being founded in geometry and what can be measured. Therefore, because there was little support for the geometrical validity of complex numbers, he said they were not really numbers at all. It is from this perspective that the term "imaginary" surfaced and expanded in use.

Leibniz's mathematics regarding complex numbers not only brought about a more modern representation of imaginary quantities and expressions, but his discoveries about the complex provided an even deeper understanding of algebraic properties such as quadratics and binomials, as well as a touch on the number theory properties of conjugates.

Euler Creates More Modern Language

Perhaps one of the greatest mathematical minds in history was Leonhard Euler. We find Euler's ideas embedded in almost every area of mathematics studied today. Though Euler was not necessarily famous for his work with complex numbers, he did provide one of the most basic but foundational terms regarding the complex system. Euler coined the symbol i to represent the square root of the negative identity; that is, he provided $i = \sqrt{-1}$ [4]. Euler offered his personal perspective on i, saying that if the quantities were used in solving a problem, then this provided key evidence that the problem could not actually be solved [9]. Essentially, Euler coined the term i for more concise representation, but he was one of many mathematicians who still saw complex numbers as nonexistent.

Questioning the Scope of Complex Numbers

As the study of complex numbers continued throughout the 18th century, many mathematicians came to believe that there might be different "orders" or "types" of complex numbers [9]. This was a plausible thought, since there are many different categorizations and types of real numbers - that is, rational and irrational, natural numbers

and integers, and even categories like perfect squares and primes. However, in 1747 d'Alembert published a work in which he proved that every different "form" of complex number could essentially be encompassed into the expression $a + bi$ [9]. This was the first representation of a complex number as a sum, broken down into two distinct components. Indeed, the paradox of the set of complex numbers is that it is infinite, just as the reals, yet each of its elements can only be expressed in this single form.

The Complex Plane Lands in Mathematical Thought

Around the time of Leibniz, mathematician John Wallis took a major leap toward exploring complex numbers from the field of geometry. Wallis seemed to believe that complex quantities were actual numbers. In 1685, he developed a geometric representation of complex numbers based on the well-known geometric solution to a quadratic. Wallis used Descartes' method of deriving the solutions to a quadratic using the Pythagorean theorem and properties of isosceles triangles [6]. In one example, an isosceles triangle with height 12 units and side lengths 15 units is created, along with the inclusion of an extending side from the vertex to point P of length 20 units.
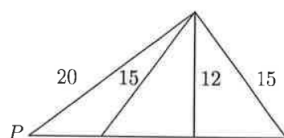
Figure 2. Wallis' Geometric Solution to a Quadratic, gathered from Cooke's *History of Mathematics* [6].

Wallis claimed that the distance from point P to each of the two vertices on the base of the isosceles triangle gives the solutions to the quadratic. Solving for these distances via the Pythagorean theorem yields the roots of 7 and 25 [6]. To address this idea in the case of imaginary numbers, Wallis then essentially reversed the criteria used for the quadratic above to create an "imaginary" or impossible triangle [6]. Instead of the isosceles triangle having side lengths 15 units and height 12 units, perhaps the side lengths were 12 units, and the height 15, such that the triangle formed with such criteria was disconnected and therefore not a true triangle. The figure below illustrates this scenario.
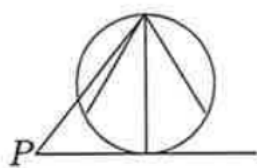
Figure 3. Wallis' Demonstration of Comlpex Solutions, gathered from Cooke's *History of Mathematics* [6].

He pointed out that even though such a triangle was theoretically impossible, it was still acceptable to connect point P to the base vertices of such a triangle [6]. Thus, the solutions would be imaginary ones, because they were not straight line segments, but they were still solutions. He claimed they were "no more absurd than negative numbers" [9]. This, however, is humorous considering negative numbers had actually been considered absurd for many centuries prior. Wallis, as a result of his discoveries, claimed that a complex number could essentially be represented as a segment in a plane, with the real part on one axis, and the real number component of the imaginary part on the other axis [9]. At first glance, this seems akin to the complex plane used today. However, Wallis failed to actually connect this with the xy-coordinate plane. As Kline says, Wallis had originally intended the imaginary axis be equivalent to our "y-axis," and the coefficient for the imaginary part is what was graphed as the coordinate, but this was not clearly conveyed. He missed denoting the y-axis to be "the axis of imaginaries" and so his idea was not widely accepted [2].

A century and a half later, mathematician William Rowan Hamilton built on the geometry of an ever-developing complex theory. Another advocate for the reality of the complex, he addressed the incompleteness of Wallis' work with the complex system as a coordinate plane. He claimed that in the common complex number representation a+bi, the real quantity could not actually be added to the imaginary one - this is not a genuine sum [9]. Thus, it must be represented solely as an ordered pair (a, bi) on a kind of "ab$i$"-coordinate plane, a parallel to the known xy-coordinate system [9]. In this representation, the i is absolved into the axis, since the value itself cannot be graphed as a distance in the plane.

More Modern Times: Arrival of the Fundamental Theorem of Algebra

Some time after the age of John Wallis, mathematicians began to separate complex numbers from geometry and algebra and place the idea into its own "category," so to speak. Complex numbers were certainly not the only mathematics being studied in the 17th and 18th centuries. Algebra was significantly developing during this age also, and it may very well have been enhanced by the study of imaginaries. Take into consideration what we know as the fundamental theorem of algebra. Although mathematicians like d'Alembert (1746), Euler (1749), and Lagrange (1772) have been credited with representing some form of the idea that every polynomial of degree n has n roots, they were not concise, and they either did not attempt or were unable to correctly prove it [4]. Mathematician Carl Friedrich Gauss was the first mathematician to introduce by name and correctly prove the fundamental theorem of algebra [4].

Theorem 1. [2] *Every polynomial equation of the form*

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots + N = 0$$

with $n \in N$ has precisely n roots in the complex field (which may or may not be in $\mathbb{R}$).

Contrary to what his mathematics shows, Gauss was thought to be a better astronomer than mathematician [4]. Nevertheless, he dominated almost all fields in mathematics around the turn of the 18th century [8]. In 1799, Gauss examines the fundamental theorem as the topic for his dissertation (with slightly different notation) [2]. In the dissertation, he provides his own correct proof for Theorem 1. Before examining this theorem, consider the following. He mentions imaginary numbers in the context of a polynomial having either real or impossible roots, but claims that they cannot be made axiomatic to this theorem - they need be proved [7]. Gauss was certainly one to challenge unsupported ideas. He held accountable mathematicians that idealized the complex without extensive study. Examine the quote from Gauss below.

> "Certain authors who seem to have perceived the weakness of this method
> assume virtually as an axiom that an equation has indeed roots, if not possible

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[2]The general contents of this theorem were gathered from [4] and [7].

ones, then impossible roots. What they want to be understood under possible and impossible quantities, does not seem to be set forth sufficiently clearly at all. If possible quantities are to denote the same as real quantities, impossible ones the same as imaginaries: then that axiom can on no account be admitted but needs a proof necessarily". [3]

Gauss essentially claims that complex numbers cannot be taken as a self-evident alternative to "possible" roots, that is real ones. Hence, proofs assuming imaginary numbers to be one of two root possibilities must first include a proof for why such numbers are the default possibility when there are no real roots [7]. This was one of the reasons he refrained from working with complex numbers in his first three proofs. Doing so would have required more examination than was needed to establish a more basic proof. Nevertheless, we can see that Gauss' knowledge of such quantities broadened his understanding of the theorem. For our purposes, let us examine Gauss' first proof of the fundamental theorem of algebra. [4]

Proof. Every algebraic equation can be represented by the form $x^n + Ax^{n-1} + Bx^{n-2} + ... + M = 0$, where $m \in \mathbb{Z}^+$. Let the first part of this equation be represented by X and suppose that $X = 0$ is true for several different values of x, namely $x = \alpha$, $x = \beta$, $x = \gamma,\ldots$, then X will be divisible by the product of the factors $(x - \alpha)$, $(x - \beta)$, $(x - \gamma)$, etc. Conversely, the product of these factors will satisfy $X=0$ when x is set equal to each of the quantities α, β, γ, etc. Finally, when X is equal to the product of n such factors (which may be different, or some may be repeated), then there are no other factors than these that can divide X. Thus, an equation of degree n cannot have more than n roots.

Consider now, because of what was stated before, it is clear that the nth-degree equation can have fewer than n roots, namely if the factors are identical, then the distinct number of factors will be fewer than n. However, for the sake of agreement, we will say in this case these repeated roots are distinct, and thus the equation has n roots, only that some of these turn out to be equal. ∎

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[3]This quote was gathered from Gauss' dissertation [7].

[4]This proof was reproduced from Gauss' dissertation [7].

Notice Gauss does not consider examples, namely because this proof will require more support when one encounters equations with complex roots, and he intended specifically not to deal with imaginaries at the time. Consider, though, the overall structure. Not only is Gauss' proof abstract, but it is also direct. Gauss doesn't prove by contradiction, as many mathematicians had prior to his time. On another note, the latter part of Gauss' dissertation contains an examination of both d'Alembert's and Euler's proofs. Gauss boldly dissects these, giving each of his "objections," pointing out the errors in his colleagues' reasoning [7]. This added strength to his proof - not only did he explain his reasoning well, but explained how other proofs were faulty.

It is thus clear from this example of direct proof that the development of the complex system couples with more precise proof style. In more modern cases where the complex system was considered almost "normal" on the mathematical playing field, complex numbers and proof actually affected each other in a mutual way. The validity of complex numbers was enhanced by proof which recognized their existence, but complex numbers also called for more concrete and concise proof, especially in the case of Gauss with the fundamental theorem of algebra. In addition, since Gauss worked with the theorem from the perspective of accepting complex numbers, the ever-developing algebra could be examined from a more well-rounded view. Essentially, the complex system provided clarity for the roots of polynomial equations.

Conclusion

In conclusion, it must be said that the development of complex numbers was, to say the least, a complex process. The newness of the concept coupled with its inability to be represented geometrically hindered early mathematicians from forming complete and concrete conclusions about these quantities. Complex numbers were, therefore, more of a development than a stark discovery by a glorious mathematician of antiquity. After all, it took some 300 years for mathematicians to recognize the $\sqrt{-1}$ as anything but absurd. But the history of the complex shows just how extensive and profound the idea is within the larger mathematics discipline.

As scholars began to question the complex system - as any good mathematician would - they increasingly relied on proof to solidify their claims. The emergence of proof affected the validity of imaginary quantities. Not only so, but complex numbers impacted and were applied to studies of numerous mathematical topics, through their ability to broaden perspectives and solidify claims, such as the solution to the cubic and the fundamental theorem of algebra. Complex numbers are still widely used to aid mathematical inquiry today. From quadratics and cubics to number theory to geometry and the coordinate plane and beyond, complex numbers have been woven into mathematics in a way that solidifies its connectedness, depth, and beauty.

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