Huygens and *The Value of all Chances in Games of Fortune*

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The science of probability is something of a late-bloomer among the other branches of mathematics. It was not until the 17th century that the calculus of probability began to develop with any seriousness. Historians say the cause of this late development may have been due to the ancient belief that the outcome of random events were determined by the gods [5, pp. 8-10]. For instance, the ancient Greeks, who developed many branches of mathematics and logic, never touched probability. Random events, such as the tossing of an astragalus (a kind of die), were methods of divination and were not studied mathematically. Also, given the certainty of mathematics and logic, predicting the outcome of uncertain events may not have been immediately obvious. These certain assertions of uncertain events were frowned upon by the Greeks. Huygens references this idea himself, "Although games depending entirely on fortune, the success is always uncertain, yet it may be exactly determined at the same time" [4, p. 8]

The 17th century mathematicians who developed probability were pioneers of a new field. And unlike other areas of mathematics, the spark that ignited this study was gambling. Some of these mathematicians were gamblers themselves, such as Blaise Pascal (1623 – 1662). These men developed answers to questions about the odds of games, the expected return on a bet, the way a pot should be divided, and other questions; and in the meantime, they developed a new branch of mathematics.

One of these men was Christianus Huygens. Huygens was a Dutch mathematician, astronomer, physicist, and inventor who was born on April 14, 1662 [2, p. 110]. At the age of 16, he moved to University of Breda and studied mathematics under Francis van Schooten and J. Pell. By the age of 25, he had written two mathematical treatises—*Theorems on the Quadrature of Hyperbolae, Ellipses, and Circles* and *New Inventions Concerning the Magnitude of the Circle*. In 1655, he travelled to France to study for his doctorate in Law, and during the trip, conversed with mathematicians in Paris about the mathematics of games of chance. Shortly after, he wrote his most influential treatise on probability—

The Value of all Chances in Games of Fortune. During his time, he also studied spherical lenses, studied the rings of Saturn, discovered a moon of Saturn, and invented a pendulum clock. During his time in France, he also tutored Leibniz (1646 – 1716) in analytic geometry. Later in his life, he moved back to the Netherlands and designed a tubeless telescope. He died in July of 1695.

The *Libellus de Ratiociniis in Ludo Aleae*, or "The Value of all Chances in Games of Fortune," is Huygens' seminal treatise on probability. It was published in 1657 in Latin and was widely circulated. It is the first published text on probability and remained the standard textbook on probability for the next 50 years [3, p. 68]. According to Campe in *The Game of Probability*, " . . . Huygens's work on games of chance came to form the mathematical basis for what would become probability theory in the modern sense of the word."

Huygens relied heavily on other mathematicians in France, especially Pascal, Fermat, and Carcavi [3, p. 68]. In his letter to van Schooten, which serves as a preface to the treatise, he states that, "for some time, the best mathematicians of France have occupied themselves with this kind of calculus so that no one should attribute to me the honour of the first invention." Although Huygens never met Pascal and Fermat, he was aware of their famous correspondence on gambling [1, p. 74]. He learned of the methods and arguments they used for calculating probabilities and expectation on his trip to Paris 1655 [3, p. 67]. He also directly used several of Fermat's problems in his treatise.

Not only was Huygens's work a standard textbook for a half-century, but it also influenced the work of the great probabilists of the 18th century such as James Bernoulli (1654 – 1705), de Moivre (1667 – 1754), and Montmort (1678 – 1719). Significantly, Huygens's treatise is incorporated into James Bernoulli's famous work *Ars Conjectandi*, or the "Art of Conjecturing," another seminal work on probability [2, p. 115].

Huygens organized his work in a systematic way that is akin to other mathematical systems—axioms, definitions, theorems, corollaries, etc. The treatise can be broken into four sections—an introduction, a postulate, 14 propositions, and five exercises. In the introduction, he poses several problems that he will answer throughout the 14 propositions including what one should bet that he will throw a 6 on a die on the first throw and how to divide up a pot in the middle of the game. Throughout, we will use the 1714 London translation [4] titled, "The Value of all Chances in Games of Fortune; Cards, Dice, Wagers, Lotteries, &c. Mathematically Demonstrated", and all comments in [square brackets] are mine.

> [Introduction]
> Although in Games depending entirely upon Fortune [randomness], the Success is always uncertain; yet it may be exactly determin'd at the same time, how much more likely one is to win than lose. As, if any one shou'd lay that he wou'd throw the Number *Six* with a single Die the first throw, it is indeed uncertain whether he win or lose; but how much more probability there is that he shou'd lose than win, is presently determin'd, and easily calculated. So likewise, if I agree with another to play the first Three Games for a certain Stake, and I have won one of my Three, it is yet uncertain which of us shall first get his third Game; but the Value of my Expectation and his likewise may be exactly discover'd; and consequently it may be determin'd, if we shou'd both agree to give over play, and leave the remaining Games unfinish'd, how much more of the Stake [pot] comes to my Share than his; or, if another desired to purchase my Place and Chance, how much I might just sell it for. And from hence an infinite Number of Questions may arise between two, three, four, or more Gamesters: The satisfying of which being a thing neither vulgar [common] nor useless, I shall here demonstrate in few Words, the Method of doing it; and then likewise explain particularly the Chances that belong more properly to Dice.

Huygens, in his 'Postulat', lays out an axiom from which he builds the remaining 14 propositions. He has a certain context or scenario in mind when he states this postulate, namely, a game. The only facts about the game that are given are that it involves two or more players who each provide an entry fee to play the game. All of the entry fees are gathered in a pot, and the winner takes the pot. Each player has a certain chance of winning the game. After winning, the winner gives some of the winnings back to the other players and keeps the rest for himself or herself. Huygens is interested in what is a fair entrance fee, which he calls a 'fair lay.'

Postulat

As a Foundation to the following *Proposition*, I shall take Leave to lay down this Self-evident Truth:

[Postulate]: *That my Chance or Expectation to win any thing is worth just such a Sum, as wou'd procure me in the same Chance and Expectation at a fair Lay.*

[Example]: As for Example, if any one shou'd put 3 Shillings in one Hand, without letting me know which, and 7 in the other, and give me the Choice of either of them; I say, it is the same thing as if he shou'd give me 5 Shillings, because with 5 Shillings I can, at a fair Lay, procure the same even Chance or Expectation to win 3 or 7 Shillings.

Generally, the term 'expectation,' which is better translated 'value of the game,' should be understood as 'the stake per participant in the equivalent lottery' [3, p. 69]. In other words, it is the fair amount that one should pay or bet ['fair lay'] that is equivalent to the expected payoff of the lottery or game. In the following three propositions, Huygens explains how this kind of expectation can be calculated. In modern probability theory, mathematical expectation is a definition, and it requires no proof. But, given his postulate, Huygens must derive his proposition about expectation and how to calculate it.

Proposition I: If I expect a or b, and have an equal Chance of gaining either of them, my Expectation is worth $\frac{\mathrm{a+b}}{2}$.

In this proposition, two players have an equal probability of winning the game and the winning prize is either *a* or *b*. Like many of his proofs, the argument for Huygens' first proposition is broken into three sections: a proof, a demonstration in symbols, and a demonstration in numbers.

[Proof]

To trace this Rule from its first Foundation, as well as demonstrate it, having put x for the value of my Expectation, I must with *x* be able to procure the same Expectation at a fair Lay. Suppose then that I play with another upon this Condition, that each shall stake *x*, and he that wins give the Loser a [here, the winner gains b]. 'Tis plain, the Play is fair, and that I have upon this Agreement an even Chance to gain a, if I lose the Game; or $2x - a$, if I win it: for I then have the whole stake $2x$, out of which I am to pay my Adversary *a*. And if $2x - a$ be supposed equal to b, then I have an even Chance [equal probability] to gain either a or b. Therefore, putting $2x - a = b$ [so, $2x = a + b$], we have $x = \frac{a+b}{2}$ for the Value of my Expectation. Q.E.I.

Huygens ends this proof with "Q.E.I," an abbreviation for *quod erat inveniendum*, the Latin phrase for 'that which was to be found out.' He generally ends his proofs with this phrase while the demonstrations end with the usual Q.E.D or 'that which is to be demonstrated.'

Huygens is flexible with his use of the word 'expectation' throughout this treatise. When he says "the Value of my Expectation" in the proof, he means our modern definition of expectation, that is, the probability of an event multiplied by its value. He generally means this when he uses the word 'expectation' alone as well. However, when he says "equal Expectation" in the demonstration in symbols, he means an equal probability.

> [Demonstration in symbols]
> The Demonstration of which is very easy: For having $\frac{a+b}{2}$, I can play with another who shall likewise stake $\frac{a+b}{2}$ upon the Condition that the Winner shall pay the Loser *a*. By which means I must necessarily have an equal Expectation [equal probability] to gain *a* if I am loser, or *b* if I am winner. For then I win $\frac{a+b}{2}$ [This is an error: $\frac{a+b}{2} + \frac{a+b}{2} = a + b$], the Whole Stake, out of which I am to pay the Loser *a*. Q.E.D.
>
> [Demonstration in numbers]
> In Numbers. If I have an equal Chance to *3* or *7*, then my expectation is, by this *proposition*, worth *5*. And it is certain I can with *5*, again procure the same Expectation: For if the Two of us stake *5* a piece upon this Condition, That he that wins pay the other *3*, 'Tis plain the Lay is just and that I have an even Chance to come off with *3*, if I lose, or *7* if I win—for then I gain *10*, and pay my Adversary *3* out of it. Q. E. D.

This numerical example circles back to the example in the postulate. Five shillings is the expectation for an equal probability of winning three shillings or seven shillings since $5 = \frac{3+7}{2}$.

The next two propositions follow naturally from the first and are proved in the exact same manner. The second proposition builds upon this first proposition and states, "If I expect *a*, *b*, or *c*, and each of them be equally likely to fall to my share, my expectation is worth $\frac{a+b+c}{3}$." After showing the result for three terms, he states the result for four terms: $\frac{a+b+c+d}{4}$. He ends the proposition with 'and so on," implying that the rule can be generalized to $\frac{a_1 + a_2 + \ldots + a_n}{n}$ for n terms.

> Prop. III: If the number of Chances [opportunities] I have to gain *a*, be *p*, and the number of Chances [opportunities] I have to gain *b*, be *q*, supposing the Chances [the probability of winning *a* or *b*] be equal; my Expectation will then be worth $\frac{ap+bq}{p+q}$.

Again, Huygens is flexible with his wording. The word "chances" can mean opportunities when he says "the number of Chances," or it can mean probability when he says "the Chances be equal."

It is important to note here that the *p* and *q* are both natural numbers, not probabilities between 0 and 1. As a gambler, he is stating the chances as odds. Odds are probabilities stated as natural numbers, and the probability can be calculated from the odds by creating a fraction with the natural number as the numerator and the sum of the two natural numbers as the denominator. For instance, if the probability of winning *a* or *b* is $\frac{3}{5}$ or $\frac{2}{5}$ respectively, then the proposition would read: "If the number of Chances I have to win *a* be 3, and the number of chances I have to gain *b* be 2, supposing the Chances equal; my Expectation will be worth $\frac{3a+2b}{3+2}$." Note that his expectation is $\frac{3}{5}a + \frac{2}{5}b$ exactly as we would calculate it today.

> [Proof]
> To investigate this Rule, I again put *x* for the value of my Expectation which must consequently procure me the same Expectation in fair Gaming. I take therefore such a Number of Gamesters as may, including my self, be equal to *p + q*, every one of which stakes *x*; so that the whole stake *px + qx* [(*p + q*)*x*], and all play with an equal Expectation [probability] of winning. With so many Gamesters as are expressed by the Number *q*, I agree singly [individually], that whoever of them wins shall give me *b*; and if I win, he shall have *b* of me: and with the rest, expressed by *p – 1*, I singly [individually] make this Agreement, That whoever of them wins, shall give me *a*; and if I win, he shall receive *a* of me. It is evident, our playing upon this Condition is fair, no Body having any injury done him;

Without loss of generality, Huygens places himself in the group of *p*, and thus, there are *p -1* players other than himself in the group. It is not clear from the problem what the other players win if Huygens loses the game. All we know is that 1) if he wins, Huygens will give *a* to the players in the group of *p – 1*, and *b* to the players in the group of *q*; and 2) if he loses, Huygens will receive *a* from a winner of the group of *p – 1*, and *b* from a winner of the group of *q*. No other rules for how the winnings are distributed are given.

> and that my Expectation of *b* is *q* [*q* ways to gain *b*]; my Expectation of *a* is *p – 1*; and my Expectation of *px +qx – bq – ap + a* (i. e. of winning) is *1* [This is a mistake. The Expectation is *a*, not *1*] : for then I gain the whole Stake *px + qx*, out of which I must pay *b*, to every one of the Gamesters *q*, and *a* to every one of the Gamesters *p – 1*, which together makes *bq + ap – a*. If therefore *px + qx – bq – ap + a* be equal to *a*, I shou'd have *p* Expectations of *a* [*p* ways to gain *a*] (for I had *p -1* Expectations of *a*, and 1 Expectation of *px + qx – bq – ap + a*, which is supposed equal to *a*) and *q* Expectations of *b* [*q* ways to gain *b*]; and consequently am again come to my first Expectation. Therefore *px + qx – bq – ap + a = a*, [so *(p + q)x = ap + bq*] and consequently $x = \frac{ap+bq}{p+q}$, is the value of my Expectation. Q.E.I.
>
> [Demonstration in numbers]
> In Numbers. If I have 3 Expectations of 13 and 2 Expectations of 8, the value of my Expectations wou'd by this Rule be 11 [$\frac{3(13)+2(8)}{3+2} = \frac{55}{5} = 11$] . And it is easy to show, that having 11, I cou'd again come to the same Expectations. For playing against Four more, and every one of us staking 11; with Two of them I agree singly, that he that wins shall give me 8; or to give him 8, if I win: And with the other Two in like manner, that which soever wins, shall give me 13; or give him so much if I win. The Play is manifestly fair, and I have just 2 Expectations of 8, if either of the Two that promis'd me 8 should win; and 3 Expectations of 13, if either of the Two that are to pay me 13 shou'd win, or if I win my self, for then I gain the whole Stake, which is 55; from which, deducting 13, a piece for the last Two I bargain'd with, and 8 a piece for the other Two, there remain 13 for my self [(2)11 + (3)11 – (3)13 – (2)8 + 13 = 13]. Q.E.D.

Compared to our modern definition of expectation [$a\frac{1}{p} + b\frac{1}{q} = \frac{ap+bq}{p+q}$], Huygens proof seems long and torturous. This was, of course, the beginning of probability theory, and methods for calculation were simplified over time. Although he does not state it, this rule too could be generalized to *n* terms: $\frac{a_1p_1 + a_2p_2 + \ldots + a_np_n}{p_1 + p_2 + \ldots + p_n}$, where the a_i's are the winnings and p_i's are the odds.

Propositions 4-9 all deal with a similar problem, namely, how to divide up the pot while a game is in progress. This is the problem that he proposed in the introduction to the book. In these propositions, the winner gains the entire pot without giving any to the loser as in the previous problems. Proposition 4 gives a general rule about dividing a stake and solves a simple example where one player needs one game to win and the other player needs two games to win.

> Prop. IV. *To come to the Question first propos'd, How to make a fair Distribution of the Stake* [pot] *among the several Gamesters, whose Chances* [probabilities of winning] *are unequal? The best way will be to begin with the easy Cases of that Kind.*

The first half of the proof explains the scenario and states that it is sufficient to know only how many games are remaining, not how many transpired before the present moment. For brevity's sake, we will examine only the second half of the proof of the proposition where he calculates this expectation.

> Moreover to find how to share the Stake, we must have regard to what would happen, if both play'd on: For it is manifest, that if I win the next Game, my Number is completed, and the Stake, which call a, is mine. But if the other gets the next Game, then both our Chances will be even, because we want but one Game apiece, and each of them worth $\frac{1}{2}a$. But it is plain, I have an equal Chance to win or lose the next Game; and consequently and equal Chance to gain a, or $\frac{1}{2}a$; which by Prop. 1 is worth half the sum of them both, i. e. $\frac{3}{4}a$ [which is $\frac{1}{2}(a + \frac{1}{2}a)$]. Q.E. I.
> My Playfellow's Share, which of course must be the remaining $\frac{1}{4}a$, might be first found after the same manner. From whence it appears, That he who would play in my room [position], ought to give me $\frac{3}{4}a$ for my Chance; and consequently that whoever undertakes to win one Game, before the other win two, may lay 3 to 1 Odds.

Similarly, proposition 5 asks how the pot should be divided if player one needs one game to win and player 2 needs three games. Relying on the previous result, he concludes that player one should get $\frac{7}{8}$ of the pot. In propositions 6 and 7, Huygens takes these problems a step further and asks how the stakes should be divided if player one needs more than one game to win the pot.

Huygens includes a Corollary after proposition 7 in which he compares the results from propositions 4 and 7. He concludes that it is better for player one to need two games before player two wins four games ($\frac{13}{16}$ of the pot in proposition 7), than to need one game before player two gets two games ($\frac{3}{4}$ of the pot in proposition 4). "*From whence it appears, that he that is to get two Games, before another shall get four, has a better Chance than he is to get one, before another gets two Games. For in this last Case, namely of 1 to 2 his Share by Prop. 4 is $\frac{3}{4}a$, which is less than $\frac{13}{16}a$.*" This result is not immediately obvious since the proportion of games needed to win is the same in each case: 1:2 is equivalent to 2:4. The counterintuitive nature of the result makes it a fine addition to the treatise. Propositions 8 and 9 extend the problem of dividing stakes to three players.

The final five propositions, propositions 10-14, deal with questions concerning expectation for dice throwing. The topic of proposition 10 was mentioned in his introduction: "if any one shou'd lay that he wou'd throw the Number *Six* with a single Die the first throw, it is indeed uncertain whether he will win or lose; but how much more probability there is that he shou'd lose than win, is presently determin'd, and easily calculated."

> Proposition X: To find how many Throws one may undertake to throw the Number 6 with a single Die.

This proposition is confusingly worded. At first reading, this proposition sounds as if it is finding the expected number of throws to roll a 6. However, the proposition seeks to find the probability of rolling a six in a given number of throws. In modern probability theory, this exact problem is modeled by the Geometric probability distribution function. This proposition is one of the pioneering statements to develop this distribution.

> [Proof]
> If any one wou'd venture to throw *Six* the first Throw, 'tis plain there is but 1 Chance [opportunity] by which he might win the Stake; and 5 Chances [opportunities] by which he might lose it: For there are 5 Throws against him and only 1 for him. Let the stake be called *a*. Since therefore he has one Expectation of [way to gain] *a*, and 5 Expectations of [ways to gain] 0, his Chance [expectation] is, by *Prop*. 2 worth $\frac{1}{6}a$; and consequently there remains to his Adversary [an expectation of] $\frac{5}{6}a$. So that he that wou'd undertake to throw Six the first Throws, must lay only 1 to 5 [a probability of $\frac{1}{1+5} = \frac{1}{6}$].
> He that wou'd venture to throw *Six* once in two Throws, may calculate his Chance after the following manner: If he throws *Six* the first Throw, he gains *a*; if the contrary happens, he has still another Throw remaining, which by the foregoing Case, is worth $\frac{1}{6}a$. But he has only 1 Chance to throw *Six* the first Throw and 5 Chances to for the contrary: Therefore before he throws, he has one Chance for *a* and five Chances for $\frac{1}{6}a$, the Value of which, by *Prop*. 3 is $\frac{11}{36}a$. And consequently there remains $\frac{25}{36}a$, to the other that lays with him. So that their several Chances, or the Values of their several Expectations, bear the Proportions of 11 to 25, *i.e.* less than 1 to 2.

Recall from proposition 3 that the expected value of winning *a* or *b* with odds *p* and *q* respectively is $\frac{ap+bq}{p+q}$. In this case, *p* is 1; *q* is 5; *a* is *a*; and *b* is $\frac{1}{6}a$. So, $\frac{ap+bq}{p+q} = \frac{1a+(5)\frac{1}{6}a}{1+5} = \frac{11}{36}a$.

> Hence, after the same Method, the Chance of him who wou'd venture to throw *Six* one in three Throws, may be investigated and found worth $\frac{91}{216}a$ [that is, $\frac{11a+(25)\frac{1}{6}a}{11+25}$] , so that he may lay 91 against 125 [216 = 91 + 125]; which is a little less than 3 to 4.
> He who undertakes to throw it once in four Times, has a Chance worth $\frac{671}{1296}a$, and may lay 671 to 625, i.e. something more than 1 to 1.
> He who undertakes to throw it once in five Times, has a Chance worth $\frac{4651}{7776}a$, and may lay 4651 against 3125, i.e. something less than 3 to 2.
> He who undertakes the same in 6 Throws has a Chance worth $\frac{31031}{46656}a$, and may lay 31031 to 15625, i.e. a little less than 2 to 1.
> And thus the Problem be resolv'd in what Number of Throws we please. Q.E.I.

Proposition 11 shows how to calculate the probability of rolling a 12 with two dice in any given number of throws. In proposition 12, he demonstrates how to calculate the probability of rolling two sixes in a given number of throws, which is akin to the negative binomial distribution in modern probability, again pioneering an important distribution in probability theory. Proposition 13 calculates the expected value of each player in a game where two dice are rolled until a 7 or 10 appear—one player wins if 7 appears first, and the other if 10 appears. Finally, proposition 14 calculates the expected value in the same game with 7 and 6, rather than 10.

Huygens ends the treatise with five problems for the reader to solve, and he provides answers to four of these five problems. Two of these problems involve gamblers drawing white or black balls blindfolded, and the first gambler to draw a white ball wins the game. Two of the problems involve gamblers playing various games with dice. And the other problem deals with gamblers playing a game with drawing from a deck of 40 cards, a problem which Huygens got from Fermat.

With this treatise, Huygens made strides toward developing a new branch of mathematics. He helped to define and systematize many of the concepts such as expectation, and some of his calculations were later developed into probability distributions. His propositions about games of chance paved the way for many others after him to follow and develop the field of probability and statistics.

Works Cited

1. Campe, Rudiger. *The Game of Probability: Literature and Calculation From Pascal to Kleist*. Stanford University Press, 2012.

2. David, F. N. *Games, Gods, and Gambling: The Origins and History of Probability and Statistical Ideas from the Earliest Times to the Newtonian Era*. New York: Hafner: Publishing Company, 1962.

3. Hald, Anders. *A History of Probability and Statistics and Their Applications Before 1750*. John Wiley and Sons, 2003. Pg. 64- 77.

4. Huygens, Christian. *The Value of Chances in Games of Fortune*. English translation 1714. https://math.dartmouth.edu/~doyle/docs/huygens/huygens.pdf

5. Mlodinow, Leonard. *The Drunkard's Walk: How Randomness Rules our Lives.* Vintage Books, 2008.