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Large Cardinals and Projective Determinacy

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Chapter 3

Traditionalism: 1894 to 1925

3.3 The Traditionalist View of Set Theory

The relationship of the traditionalists to the theory of sets that provided the grist for their mathematical discoveries was a tense and, at times, dissonant one. The dissonance is, in some regards, jarring: on the one hand, set theory was the core of their mathematical research program, and their work was decisive in bringing set theory into the mathematical mainstream; on the other, they felt deep mistrust toward many of set theory's central edifices—particularly Cantor's theory of cardinals and ordinals up to and including $\mathfrak{c}$ and ω_1, and later, Ernst Friedrich Ferdinand Zermelo's (1871-1953) Axiom of Choice—and largely disavowed them, setting boundaries to set theory's use that persist even in contemporary mainstream mathematical practice.

The tension is, in some regards, puzzling: in some cases, the traditonalists carried their criticisms of Cantor's even to elements of the theory that were, unbenknownst to them, indispensable for their own work. And the positions they staked were, to a certain degree, philosophically unrigorous, or, at least unsophisticated. But their cautious regard for set theory was not irrational—rather, the traditionalists were guided by the mathematical values they absorbed from their forbears. The traditionalists knew, consciously or unconsciously, that the measures of mathematical legitimacy for any new theory were applicability to established mathematical problems and the possibility of carving out a domain in which the theory could be applied without fear. In this regard, the traditionalists were reacting to two opposing forces: the proliferation of set theoretic antinomies, which lent the theory a profound *internal* instability; and set theory's untapped potential to reshape analysis. Their position was a compromise, a way of laying out a safe fragment of set theory which could be gainfully applied in a long-established and important research tradition without risk of encountering the thorny—and, importantly, unmathematical—issues that came from accepting the theory as a whole.

3.3.1 The early period: cautious acceptance

Even though the views of Baire, Borel, and Lebesgue shifted while their school of set theory was in full flourish between 1894 and 1905, the essential characteristics of their mature views were already present in their earliest publications. Three characteristics are particularly salient in this regard: a generally suspicious attitude toward set theory, a special regard for *effectiveness*—a term that for the French analysts took on a special meaning—and a profound concern for the applicability of their research. Each of these three characteristics was a manifestation of an essential belief in a realm of *real* or *legitimate* mathematics, toward which mathematical endeavors ought to aim.

Set-theoretic unease

Borel begins *Lessons* with the following goal: "to give the notion of a set the precision necessary in order to use it in rigorous research" [Bor98, p. 1]. The central question, in his view, is, when can one consider a set as *given* [Bor98, p. 2]. Cantor's answer—when one can determine *intrinsically* and on the basis of the law of excluded middle whether a given element belongs to the set or not—simply will not do [Bor98, p. 2,fn. 1]. Rather, "[w]e shall say that a set is given when, by any means, we know how to determine all the elements one after the other, without excepting one and without repeating any of them several times" [Bor98, p. 3]. Borel offers reasons for his caution:

> "[O]ne of the ideas that we should be most fortunate to give to the reader who wishes to think for himself on the theory of functions is that, in all questions to which the infinite appertains, one must be extremely wary of alleged clarity: nothing is more dangerous than to rest content with empty words in such matters" [Bor98, p. 3].

His trepidation was well-warranted. Since the publication of Borel's thesis, Cesare Burali-Forti's (1861-1931) modestly titled *A question on Transfinite Numbers* had caused a stir in the mathematical community. The crux of the article was the following observation: if one takes Ω to be the set of all ordinals, then Ω itself is a well-ordering, to which some ordinal α corresponds. Since $\alpha \in \Omega$, Ω must be an initial segment of one of its own initial segments, something Cantor had independently proven impossible in the same year [Can52, p. 144]. The failing was, in Burali-Forti's opinion, substantial: "It seems that the order types thus fall short of one of their most important objectives" [BF67, p. 111].

Burali-Forti's discovery was a shattering blow to a theory that had raised hackles from the outset; it was made only the more terrible by the rapid discovery of further paradoxes by Bertrand Arthur William Russell (1872-1970), Jules Antoine Richard (1862-1956), and Julius König (1849-1913). The "intrasubjective immanent reality" that Cantor had thought secured his theory seemed to grow feebler with each passing day [Can05, §8.1].

Thus, it is no surprise that Baire too, fearful of building on sand, dispenses with any set theory he can do without:

> "I will point out once and for all on this subject that we shall never have to worry about the difficulties included in the abstract notion of *transfinite number*... In actuality, for example, the set P^α, α being a determined number of the second number class, represents something perfectly determined independent of all abstract considerations relating to the symbols of Mr. Cantor" [Bai98, p. 36].

While he and Lebesgue display a greater comfort than Borel with the set theoretic tools they adopt, none share in the exuberance for set theory displayed by Cantor and his other early followers.

Effectiveness

Effectiveness was a key mantra for the French analysts. For instance, Borel writes that his proof of the Heine-Borel Theorem, while more complicated than other proofs, has the chief virtue that it demonstrates that "one can choose *effectively* a limited number of intervals" covering the given interval [Bor94, p. 43; emphasis mine]. In *Lessons*, he is concerned to give an "effective" demonstration of the existence of sets having uncountable cardinality, and praises Charles Hermite's (1822-1901) demonstration of the transcendence of e: "It was, in effect, the first *effective* example, if one may so speak of it, of a transcendental number; that is to say, the first example of a transcendental number defined in a simple way by analysis and not only by arithmetical series" [Bor98, p. 25].[1]

Similarly, Baire extols the interest of "effectively demonstrating the existence of functions belonging to different classes" [Bai98, p. 71]. Lebesgue, for his part, draws much of the justification for the project of his 1905 *On the Analytic Representation of Functions* from Baire's failure to do just that. The construction of explicit, concrete examples was to be preferred to the cardinality-based arguments he had adduced to show the existence of a Lebesgue measurable set that was not Borel, and that Baire had adduced to show that there was a function not in any function class.

What, exactly, was meant in this period by "effectiveness"? Borel's *Lessons* provides some indication:

> "It seems to us that this is an axiom which must be admitted in the same way as the axiom of Archimedes, and in a very general manner; It is in any case certain that this proposition is not dubious, by the words, 'any function,' one understands a function which can be effectively defined, that is to say such that one can, by a limited number of operations, calculate, with a given approximation, its value for a given value of the variable" [Bor98, p. 117].

In parallel, Baire remarks that the advantage of his system of classification of discontinuous functions is that they grow "more and more complicated, but [are] always capable of being tethered in a very precise manner to the continuous functions" [Bai98, p. 70]. Indeed, it is this connection—borne out in a finite number of steps by the fact that, since no decreasing sequence of ordinals can be of infinite length—that Lebesgue used to such effect in *On Analytically Representable Functions*.

Effectiveness, in short, meant the potential to actually be carried out explicitly and concretely. The standard was not a metaphysical one, but rather heuristic, and determined in reference to past mathematical practice: Hermite's proof of the transcendence of e was effective because it involved something seemingly more specific and natural, i.e., the limit of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \tag{3.1}$$

[1] Borel is contrasting Hermite not only with Liouville's construction, which he found unnatural, but also the set theoretic proofs, which, with regard to effectiveness, were even more objectionable. Of course, Borel invented one of the set theoretic proofs, in which one shows that the algebraic numbers have measure zero. This is a typical example of the kind of dissonance with which Borel in particular contended.

than the more anonymous Liouville numbers; likewise, arguments based upon arbitrary functions and cardinalities lacked the meat of demonstrations based upon what was more easily recognizable as a calculation. Of course, effectiveness was, for different individuals, a virtue in varying degrees: for Borel, it was law—arbitrary discontinuous functions were totally inadmissible, as one had no procedure for calculating when two were the same [Bor98, p. 125]. For Lebesgue, it was merely a signpost, pointing the way toward richer mathematics. Indeed, Lebesgue criticizes the stringency of the requirements placed by Borel upon functions, noting that they are too restrictive even to admit the second level of the Baire hierarchy:

> "In general, a calculation is illusory if it is assumed that two passages are performed successively at the limit, unless the second is related to a uniformly convergent sequence. Now, it is such a calculation that one would have to make to calculate $\chi(C)$,[2] C being given later by its decimal digits, that is to say by a series" [Leb05, p. 206].

Nevertheless, the value was a shared one, and played a signal role in shaping their mathematical endeavors.

Applicability

The final value of the French analysts, both explicit and tacit in their work, is applicability. A piece of mathematics rose to the level of a *contribution* when it shed light on well-established mathematical problems. Thus, we find Borel, in his thesis, self-consciously attempting to "see what importance [his results on discontinuous functions] might have for applications, notably in mathematical physics" on the basis of the fact that "no demonstration has ever been given that one can apply Taylor's formula to the functions one encounters in physics" [Bor94, p. 3]. For Baire, set theory was an indispensable tool for the study of functions:

> "[A]ny problem relative to the theory of functions leads to certain questions relative to the theory of sets; and it is to the extent that these latter questions are advanced or can be advanced that it is possible to resolve more or less completely the given problem" [Bai98, p. 121].

Lebesgue, perhaps more so than any of the others, sees with acuity the importance of set theory to matters of fundamental importance in mainstream mathematics. His principal object in his thesis is "to give definitions as general and precise as possible of some of the numbers one looks at in Calculus: a definite integral, the length of a curve, and the area of a surface" [Leb07, p. 1212]. Indeed, of the three, his contribution to mainstream mathematics is perhaps the most profound, having utterly reshaped the field of real analysis [HL02, p. 3]. But it was Borel who summed up the attitude best when he wrote some years later, reflecting on set theory's increasing formality and distance from ordinary mathematical concerns, that

> "From the day set theory stops being metaphysical and becomes practical, the new ideas may produce a flowering of beautiful results... Maybe from this profusion of formal logic, which appears as a construction without any basis, one day some useful idea will come" [GK09, qtd., p. 63].

[2] Here, χ denotes the indicator function of the rational numbers.

The ordinals and cardinals

Nowhere are these values more striking than in the French analysts' treatment of transfinite arithmetic. For Cantor, the ordinals were isolated from "*transsubjective* or *transient* reality" of which one's ideas of numbers form a part by virtue of occupying a completely determinate place in one's understanding [Can05, §8.1]. For the French analysts, this explanation of what the ordinal numbers are must have seemed like empty speculation with no place in mathematics. It is no surprise then that they were consequently loathe to use the ordinal numbers: even ten years after the publication of his thesis, Lebesgue felt the need to apologize for his use of the ordinals in the second edition of *Lessons on Integration and the Search for Primitive Functions*[3] [HL02, p. 5].

Far more strongly opposed was Borel, who, in the appendix to *Lessons*, criticizes Cantor's transfinite numbers for being subject to antinomies and constructed in a circular manner. Cantor had posited three principles of generation for ordinal numbers. The second of these was that "if any defined sequence of integers is put forward of which no greatest exists ... a new number is created, which is thought of as the limit of those numbers" [Can05, §11.2]. It was on this basis that Cantor deduced the existence of ω_1. Both this and the construction of the second number class as a *completed* entity concern Borel. He writes,

> "On the other hand, as [the second principle's] application [to a countable sequence in the second number class] leads only to a countable set, to which it is *again* applicable, there is an antinomy which ... cannot be resolved except by attributing a sense to the word *transfinitely* and admitting, consequently, that in applying the theorem *transfinitely* on will have a set ... of cardinality greater than the first" [Bor98, p. 121].

Of course, Borel thinks to do so is totally illegitimate: the only meaning one can assign, *a priori*, to the word "indefinitely" is "as often as there are whole numbers" [Bor98, p. 122]. One cannot assign it the meaning "as often as there are numbers of the second number class," as the second number class is assumed to be the result of "indefinitely" constructing larger numbers of the second number class [Bor98, p. 122].

For this reason, Borel denies the existence of $\aleph_1$—and is presumably why he did not attempt, as Baire, to stratify his measurable sets. He feels secure in this renunciation, because of a kind of working mathematician's continuum hypothesis: the cardinality of the continuum and the cardinality of the natural numbers "suffice for the applications we have in view" [Bor98, p. 20].

But some ordinal numbers are indispensable for his analytic task. Borel's solution is to construct all of the ordinal numbers one might need in analysis in terms of concrete, familiar mathematical objects: functions. Given a function $\varphi(n)$ growing faster than n,[4] Borel notes that

$$\lim_{n\to\infty}\frac{\varphi(n)}{n} = \lim_{n\to\infty}\frac{\varphi(\varphi(n))}{\varphi(n)} = \ldots = \lim_{n\to\infty}\frac{\varphi_m(n)}{\varphi_{m-1}(n)} = \ldots = \infty \tag{3.2}$$

and

$$\lim_{n\to\infty}\frac{n}{\varphi(n)} = \lim_{n\to\infty}\frac{\varphi(n)}{\varphi(\varphi(n))} = \ldots = \lim_{n\to\infty}\frac{\varphi_{m-1}(n)}{\varphi_n(n)} = \ldots = 0 \tag{3.3}$$

[3] *Leçons sur l'Integration et la Recherche des Fonctions Primitive.*

[4] Borel actually construes the functions as functions of a real variable; this presentation is merely simpler.

where $\varphi_{m+1}(n) = \varphi(\varphi_m(n))$. Thus, the functions n, $\varphi(n)$, $\varphi_1(n)$, …, $\varphi_m(n)$, … can be ordered by their rate of growth in exactly the same manner as the natural numbers. Now, the function $\psi(n) = \varphi_n(n)$ has the property that

$$\lim_{n\to\infty} \frac{\psi(n)}{\varphi_m(n)} = \infty \quad \lim_{n\to\infty} \frac{\varphi_m(n)}{\psi(n)} = 0 \tag{3.4}$$

Thus $\psi(n)$ stands in relation to the ordering n, $\varphi(n)$, $\varphi_1(n)$, …, $\varphi_m(n)$, … exactly as ω, in Cantor's system, stands in relation to 1, 2, …, m, … The function $\varphi(\psi(n))$ can go proxy for $\omega + 1$, and similarly, one can build a countable collection of functions having the order type of any countable ordinal. Moreover, this construction—effected by means of a theorem of Paul David Gustav du Bois-Reymond (1831-1889)—as opposed to Cantor's second principle of generation, "is not a postulate; it is a *mathematical fact* that does not rest upon any *a priori* consideration" [Bor98, p. 121]. It has the additional advantage that "its power is much more limited [than Cantor's principle]; it carries within itself its bounds [*principe d'arrêt*], for it is not applicable except as far as the set already obtained is countable" [Bor98, p. 121].

Lebesgue employs the same gambit in *On the Analytically Representable Functions*, writing,

> "I wish to say why the application, used in this classification, of the transfinite numbers does not raise, in my opinion, any difficulty. If one studies the growth of functions and if, having characterised the growth of x^n by n, one notes that e^x grows faster than x^n, one could feel the desire to characterize this new growth rate by a new symbol, ω. No one will see any inconvenience… Besides, the classification of Mr. Baire… can, as the theory of the growth of functions, provide a solid base for the theory of transfinite numbers" [Leb05, pp. 142-3].[5]

It is in this manner that the French analysts rid themselves—or attempt to rid themselves—of anything appertaining to Cantor's infinite not absolutely necessary for the study of analysis.

3.3.2 The late period: Poincaré and Zermelo

Poincaré and real mathematics

Borel, Baire, and Lebesgue were influenced a great deal by Poincaré. Already in 1894, Poincaré had begun to lay out his philosophical views on mathematics, praising intuition—intended in a Kantian sense—as the foundation of mathematics, and the source of mathematical meaning and progress. His opponents were the Logicists, a group of mathematicians who sought to remake mathematics purely on the basis of logic. To Poincaré, this was totally unnacceptable: "Pure logic could never lead us to anything but tautologies; it could create nothing new; not from it alone can any science issue" [Poi05a, §3].

[5] There is actually something slightly more problematic about what Lebesgue is proposing than the quotation above indicates. Recall that it is in *On the Analytically Representable Functions* that Lebesgue proves that the Baire classes corresponding to different ordinals are actually different. To do so, he actually requires the existence of the ordinals for which he is here proposing the Baire classes could replace.

Poincaré's opposition was informed, in large part, by a perceived disconnect between the practice of logic and the practice of mathematics:

> "If you are present at a game of chess, it will not suffice ... to know the rules for moving the pieces. That will only enable you to recognize that each move has been made conformably to these rules, and this knowledge will truly have very little value. Yet this is what the reader of a book of mathematics would do if he were a logician only" [Poi05a, §8].

The rapidly developing methods of formal logic, and its accompanying suite of esoteric symbols, seemed, to Poincaré, to miss the essence of mathematics.

Poincaré's view was a deeply psychologistic one [Gol88, p. 63]. Poincaré construed mathematics as constructive mental activity, undergirded by those truths of which one is immediately convinced [Gol88, p. 63]. In this, he did not differ from, for instance, Cantor or Dedekind, but he bitterly opposed them. The antinomies of set theory, Poincaré proposed, were merely a byproduct of mathematics which had become too disconnected from the realities of mathematical practice: while the logicians struggled to resolve the paradoxes, Poincaré maintained that "True mathematics, where one does not wallow in the actual infinite, is not in question" [Poi05b, §13]. Nonetheless, Poincaré recognized the importance of set theory; like Borel, he merely thought that the Cantorians had erred in forgetting that "There is no actual (given complete) infinity... It is true that Cantorism has been of service, but this was when applied to a real problem whose terms were precisely defined, and when we could advance without fear" [Poi05b, §15].

Zermelo's principle and the mature formulation of the traditionalist view

Poincaré's objections, though they began in the 1890's, crescendoed only after the turn of the century. They reached their loudest in 1905, following the publication in 1904 of *Proof that every Set can be Well-Ordered*, a short paper by Ernst Friedrich Ferdinand Zermelo (1871-1953). In it, Zermelo provides a proof of what Cantor considered a "fundamental law of thought of great consequence" [Zer67c, qtd., p. 139]: that every set can be brought into the form of a well-ordered set. Zermelo's proof depends upon what would become known as *Zermelo's Principle* or *the Axiom of Choice*: given any collection of non-empty sets, one can simultaneously choose from each a single element. This seemingly innocuous, even self-evident, statement soon caused a stir, drawing the criticism of such eminent mathematicians as Peano, Borel, and Poincaré [Poi05b, §13], [Zer67b, p. 186].

Zermelo's goals were quite different from those of the French analysts. "Set theory," he wrote, "is that branch of mathematics whose task it is to investage mathematically the notions 'number', 'order', and 'function' ... and to develop thereby the logical foundations of arithmetic and analysis" [Zer67a, p. 200]. The antinomies had threatened the theory, but by grounding the sets in explicit axiomatics, one could "retain all that is valuable in ... the entire theory created by Cantor and Dedekind" [Zer67a, p. 200].

But the traditionalists, who viewed set theory as playing quite a different role in mathematics, did not agree, and rejected his axiom of choice. Their reasons for doing so evinced Poincaré's influence. Borel, one of Zermelo's quickest critics, saw the situation in characteristic fashion:

> "It seems to me that the objections that one can raise against [Zermelo's proof] apply equally well against any reasoning where one supposes an *arbitrary choice* to be made a non-denumerable infinity of times; such reasonings are outside the realm of mathematics" [BBLH05, p. 1077].

But he was not without his detractors. Jacques Salomon Hadamard (1865-1963) wrote to Borel soon after, posing the following objection:

> "What is certain is that Zermelo provides no method to carry out *effectively* the operation which he mentions, and it remains doubtful that anyone will be able to supply such a method in the future. But the question posed in this way (the effective determination of the desired correspondence) is none the less completely distinct from the one that we are examining (does such a correspondence exist?). Between them lies all the difference, and it is fundamental, separating what Tannery calls a *correspondence* that can be *defined* from a correspondence that can be *described*" [BBLH05, p. 1078].

Foreseeing the potential importance of their conversation, Borel forwarded the letter to Baire and Lebesgue. What followed was an profoundly revealing exchange of letters, in which each of the French trio revealed the distillation his ideas had undergone in a few short years. While Borel still did not believe in the uncountable, Baire, who had proudly advanced knowledge of "arbitrary functions" only a few years before now went further than Borel, maintaining that all infinities, even countable ones, are merely "in the realm of *potentiality*... [D]espite appearances, in the last analysis everything must be reduced to the finite" [BBLH05, p. 1080]. Lebesgue, for his part, intoned that "I do not grant ... any validity to the argument showing that a set which is not finite has a denumerable subset. Although I seriously doubt that a set will ever be named which is neither finite nor infinite, it has never been proven to my satisfaction that such a set is impossible" [BBLH05, p. 1083].

In their discussions, one question emerges as fundamental: what is required to prove existence in mathematics? All three object on the basis that, in the sense they find meaningful, one may not even succeed in choosing a *single* element from an arbitrary set. In Lebesgue's words,

> "I use the word *to choose* in the sense of *to name* ... So as to convey more clearly the difficulty that I see, I remind you that in my thesis I proved the existence ... of sets that were measurable but were not Borel-measurable. Nevertheless, I continued to doubt that any such set could ever be named. Under these conditions, would I have the right to base an argument on this hypothesis ... even though I doubt that anyone could ever name one? Thus I already see a difficulty with the assertion that 'in a determinate M' I can choose a determinate m'' " [BBLH05, p. 1082].

The crux of the matter is that without a *procedure*—what they might see as a determinate mental procedure—for distinguishing a single element in a set from all others, one cannot concretely effect such a choice.

Of course, Lebesgue and Baire, at least, are sensitive to the apparent contradiction that Hadamard points out between their current positions with their earlier works. Borel, for instance, implicitly called on the axiom of choice in his 1894 proof of the Heine-Borel theorem. But they rebuke Hadamard's criticism that, in focusing so pointedly on definitions, "existence is a fact like any other" independent of the way it was proven [BBLH05, p. 1084]. Borel, in an elegant summary of the traditionalist ethos, closes the conversation with the following words:

> "I prefer not to write alephs. Nevertheless, I willingly state arguments equivalent to those which you mention, without many illusions about their intrinsic value, *but intending them to suggest other more serious arguments*... One may wonder what is the real value of these arguments that I do not regard as absolutely valid but that still lead ultimately to effective results ... They have a value analogous to certain theories in mathematical physics, through which we do not claim to express reality, but rather to have a guide that aids us, by analogy, in predicting new phenomena which must be verified" [BBLH05, p. 1086].

3.3.3 The end of the French school

Borel, Baire, and Lebesgue's seminal works in analysis between 1894 and 1905 recast analysis in the image of set theory; no longer was it possible to ignore set theory as mere philosophical speculation, without mathematical substance. But just as quickly as the new techniques were ushered in, interest in them—at least in France—dried up. By 1904, Lebesgue had produced, in more or less consummate form, the solution to the problem of the Fundamental Theorem of Calculus promised in his thesis, as well as invaluable tools for analysis, such as the Dominated Convergence Theorem [HL02, p. 6]. Fubini's Theorem as well as early applications in potential theory and Fourier analysis vindicated the new conception of integral and measure: Plancherel proved his eponymous theorem in 1910 using Lebesgue's integral, providing, in some sense, a complete solution to the century old problem [HL02, p. 7], [HS05]. Lebesgue's interest in set theory began to wane around 1910, in part due to a fear that his contributions had been too systematic: "reduced to general theory mathematics would be a beautiful form without content. It would die quickly, as many branches of our science have died just at the time when general results seemed to guarantee them a new activity" [HL02, qtd., p. 10].

Borel, for his part, had begun to turn more and more toward applied branches of mathematics—especially probability and game theory—and politics [HL02, p. 9]. Though closely connected intellectually, Lebesgue and Borel's relationship had always suffered from a certain strain. Borel, whose grandfather had been a rich wool merchant and whose wife was the daughter of the celebrated French mathematician Paul Emile Appell (1855-1930)—fueling the facetious observation, common at the time, that "genius is transmitted to sons-in-law" [GK09, p. 42]—belonged to a milieu in which the more modest Lebesgue was not always at home [GK09, pp. 63-4]. Lebesgue was a Dreyfusard; Borel, a firm supporter of the military [HL02, p. 9]. Their friendship took a blow in 1912 when Borel published a paper trivializing Lebesgue's work and worsened considerably when Lebesgue was placed under his command in World War I [HL02, p. 8], [GK09, p. 64]. By 1917, their friendship had evaporated, destroyed by a bitter priority dispute, and for many years neither had returned to the set theory of his youth [GK09, p. 64]. Baire, for his part, languished in provincial teaching posts, unable to research because of a mental illness aggravated by his lack of recognition [OR00].

Before long, no new set theory was issuing from its three former stars. Borel, describing the evolution of his attitude toward set theory spoke in some sense for all three when he wrote: “Like many young mathematicians, I had been immediately captivated by the Cantorian theory; I don’t regret it in the least, for that is one mental exercise that truly opens up the mind” [GK09, qtd., p. 40]. Nevertheless, important questions remained. An eager group of investigators would soon discover that Lebesgue’s *On Analytically Representably Functions* had resolved less than it claimed. Indeed, even its principal achievement—the effective construction of a non-Borel measurable set—was subject to doubt on the basis of its dependance upon the existence of $\aleph_1$. With a whole world of sets now intermediate between the extremes of perfect and arbitrary sets, the possibility of revitalizing the program of research that had led to the Cantor-Bendixson theorem emerged. A new group of mathematicians who shared in many of the French analysts values—especially the priority of analysis and the importance of effective constructions—would vigorously pursue these matters.

Bibliography

[Bai98] René Baire. *Sur les Fonctions de Variables réelles.* Gauthier-Villars, Paris, 1898.

[BBLH05] E. Borel, R. Baire, H. Lebesgue, and J. Hadamard. Five letters on set theory. In William Ewald, editor, *From Kant to Hilbert: A Source Book in Mathematical Logic*, volume II, pages 1077–1086. Clarendon Press, Oxford, 2005. Translated by George Bruce Halstead.

[BF67] Cesare Burali-Forti. A question on transfinite numbers. In Jean Van Heijenoort, editor, *From Frege to Godel: A Source Book in Mathematical Logic*, pages 104–11. Harvard University Press, Cambridge, 1967. Translated by Jean van Heijenoort.

[Bor94] Émile Borel. *Quelques points sur la théorie des fonctions.* Gauthier-Villars, Paris, 1894.

[Bor98] Émile Borel. *Leçons sur la théorie des fonctions. (Principes de la théorie des ensembles en vue des applications à la théorie des fonctions).* Gauthier-Villars, Paris, 1898.

[Can52] Georg Cantor. *Contributions to the founding of the theory of transfinite numbers.* Dover Publications, Inc., New York, N. Y., 1952. Translated, and provided with an introduction and notes, by Philip E. B. Jourdain.

[Can05] Georg Cantor. Foundations of a general theory of manifolds: a mathematico-philosophical investigation into the theory of the infinite. In William Ewald, editor, *From Kant to Hilbert: A Source Book in Mathematical Logic*, volume II, pages 881–920. Clarendon Press, Oxford, 2005. Translated by William Ewald.

[GK09] Loren Graham and Jean-Michel Kantor. *Naming infinity.* The Belknap Press of Harvard University Press, Cambridge, MA, 2009. A true story of religious mysticism and mathematical creativity.

[Gol88] Warren Goldfarb. Poincaré against the Logicists. In *History and philosophy of modern mathematics (Minneapolis, MN, 1985)*, Minnesota Stud. Philos. Sci., XI, pages 61–81. Univ. Minnesota Press, Minneapolis, MN, 1988.

[HL02] G. T. Q. Hoare and N. J. Lord. 'intégrale, longueur, aire' the centenary of the Lebesgue integral. *The Mathematical Gazette*, 86(505):3–27, 2002.

[HS05] N. Hungerbühler and M. Schmutz. Michel Plancherel. In *The MacTutor History of Mathematics archive*. School of Mathematics and Statistics, University of St Andrews, 2005.

[Leb05] Henri Lebesgue. Sur les fonctions représentables analytiquement. *Journal de Mathématiques Pures et Appliquées*, 1(505):139–216, 1905.

[Leb07] Henri Lebesgue. Integral, length, area. In Stephen Hawking, editor, *God Created the Integers*, pages 1212–1253. Running Press, Philidalphia, 2007.

[OR00] J. J. O'Connor and E. F. Robertson. René-Louis Baire. In *The MacTutor History of Mathematics archive*. School of Mathematics and Statistics, University of St Andrews, 2000.

[Poi05a] Henri Poincaré. Intuition and logic in mathematics. In William Ewald, editor, *From Kant to Hilbert: A Source Book in Mathematical Logic*, volume II, pages 1012–1020. Clarendon Press, Oxford, 2005. Translated by George Bruce Halstead.

[Poi05b] Henri Poincaré. Mathematics and logic: III. In William Ewald, editor, *From Kant to Hilbert: A Source Book in Mathematical Logic*, volume II, pages 1052–1071. Clarendon Press, Oxford, 2005. Translated by George Bruce Halstead.

[Zer67a] Ernst Zermelo. Investigations in the foundations of set theory I. In Jean Van Heijenoort, editor, *From Frege to Godel: A Source Book in Mathematical Logic*, pages 199–215. Harvard University Press, Cambridge, 1967. Translated by Stefan Bauer-Mengelberg.

[Zer67b] Ernst Zermelo. A new proof of the possibility of a well-ordering. In Jean Van Heijenoort, editor, *From Frege to Godel: A Source Book in Mathematical Logic*, pages 183–98. Harvard University Press, Cambridge, 1967. Translated by Stefan Bauer-Mengelberg.

[Zer67c] Ernst Zermelo. Proof that every set can be well-ordered. In Jean Van Heijenoort, editor, *From Frege to Godel: A Source Book in Mathematical Logic*, pages 139–41. Harvard University Press, Cambridge, 1967. Translated by Stefan Bauer-Mengelberg.