To Infinity and Beyond: A Historical Journey on Contemplating the Infinite

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"The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds." –Georg Cantor

1 Introduction

You are the proud owner of a humongous hotel–so huge, in fact, that it has *infinitely many* rooms. Business seems to be going well because your hotel is at maximum capacity. That's right–you have *infinitely many* guests filling *infinitely many* rooms. One evening, a man enters and asks for a room. For the sake of offering exceptional customer service, you decide to help him out. After thinking on it for a bit, you ask each occupant of your hotel to move to the room number that is one more than their current room number. Thus, the occupant of room #1 moves to room #2, the occupant of room #2 relocates to room #3, and so forth. Now, you have the pleasure of telling your new guest that you have a room available. Relieved yet proud of your work, you return to the front lobby to find that a bus has just arrived. Of course, this is not just any bus; it is an *infinitely large* bus with *infinitely many* passengers. Naturally, they all want rooms in your incredibly full hotel. Again, being the prime example of proper customer service, you try to make it work. This time, you ask every guest to move to the room number that is twice their current room number. Hence, the guest in room #1 moves to room #2, the guest in room #2 moves to room #4, and so on. Now, all the odd-numbered rooms are available, so place your infinitely many new guests in those rooms. You must feel quite accomplished, for you have done the unthinkable [1].

The scenario above is perhaps one of the most popular paradoxes regarding infinity. It was developed in 1924 by German mathematician David Hilbert and is appropriately called Hilbert's Paradox of the Grand Hotel. It is situations, problems, and thoughts such as these that have boggled the minds of mathematicians and philosophers for centuries. The infinite is something that humanity continuously struggles to understand, for it is something we cannot necessarily see, touch, or even fathom. Over the centuries, great thinkers have attempted to make sense of perhaps the most nonsensical concept known to man. Some have succeeded, many have failed, and even more have lost their minds in the attempt to demystify infinity and understand its role in mathematics. Hence, we will explore this journey throughout history and examine the mathematics born out of the pursuit of unraveling the infinite.

2 The Birth of Infinity

Evidence reveals that the first notions of infinity occured in ancient civilizations. It was initially regarded as a philosophical concept by ancient Greeks. Greek philosopher Anaximander (610-546 B.C.) is coined for using the word *apeiron* (ἄπειρον) to describe what is limitless, boundless, and infinite. Apeiron was regarded as a negative word, and Anaximander used it to describe the chaos out of which the universe was born. "Thus, apeiron need not only mean infinitely large, but can also mean totally disordered, infinitely complex, subject to no finite determination"[2]. The idea of something as indescribable as the infinite was beyond the ancient Greeks' understanding, causing them to have a severe disdain for the idea.

Another early appearance of the infinite is found in the works of Zeno of Elea (490-430 B.C.). Zeno is perhaps most well known for his development of his paradoxes, which involve the process of completing infinitely many tasks. He used these to support his claim that continuous time, space, and motion are merely an illusion [13].

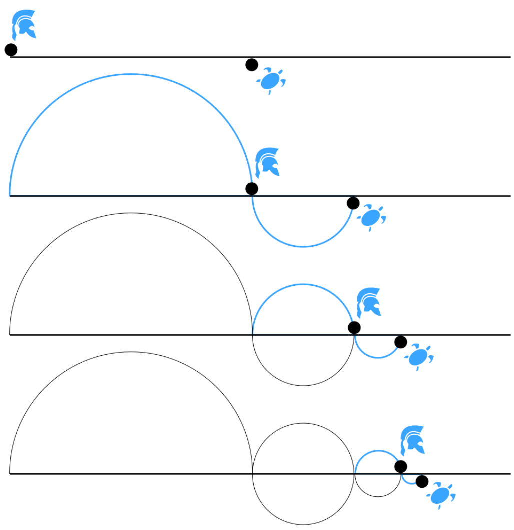

Figure 1: Achilles and the Tortoise [8]

Perhaps the most well known of Zeno's Paradoxes is that of Achilles and the Tortoise. The story goes that Achilles and a tortoise are in a race of sorts in which the tortoise is allowed a head start. Theoretically, in the time it takes for Achilles to reach where the tortoise was when Achilles started, the tortoise has traveled another distance (Figure 1). Should this process continue, Achilles will never pass the tortoise, or even catch up to him for that matter. This is because it would require Achilles to reach an infinite number of places the tortoise reached before conquering him. Zeno's conclusion is, of course, absurd, for it simply does not make sense that a man running at a faster velocity would not be able to pass a slowly-moving tortoise. Paradoxes such as these caused an uproar in philosophical and mathematical communities that would last for centuries, leading them to examine the paradoxes within the context of infinite sums and convergence.

3 Infinity in Greece

3.1 The Horror of the Incommensurable

The Pythagoreans first stumbled upon the concept of infinity when they found $\sqrt{2}$ in their mathematics. Pythagoras and his students believed in "the metaphysic of number and the conception that reality is, at its deepest level, mathematical in nature" [3]. They assigned every item to a natural number and even used numbers to describe the universe around them. The Pythagoreans held an "intuition that any two lengths are 'commensurable' (that is, measurable) by integer multiples of some common unit. To put it another way, they believed that the whole (or counting) numbers, and their ratios (rational numbers or fractions), were sufficient to describe any quantity" [4].

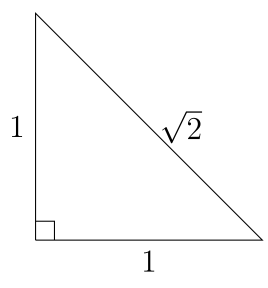

Figure 2: Right Triangle Formed from the Unit Square

One could imagine the horror they experienced when they found irrational numbers, which cannot be represented as a ratio of natural numbers and whose decimal places continue without termination. In other words, they have *infinitely many* decimal places. Using the Pythagorean Theorem, they found that the diagonal of the unit square measured to be $\sqrt{2}$ (Figure 2), which is approximately equal to 1.414213562373095.... They had found a number that could not be written as a ratio and did not fit within their concept of number. They called numbers of the like *incommensurable*, for they could not be assigned a particular finite measure. This completely obliterated everything they believed to be true about numbers, and consequently the world around them.

A Euclidean proof demonstrating the irrationality of $\sqrt{2}$ allows us to examine this particular struggle the Pythagoreans encountered with the incommensurable. Additionally, the proof shows early ideas of describing transfinite numbers, which can be defined as a "denotation of the size of an infinite collection of objects" [5]. Examine Euclid's proof of the irrationality of $\sqrt{2}$ or "the incommensurability of the diagonal of a square with its side" [6].

Theorem 1. *The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number also have their sides commensurable in length. But the squares on straight lines incommensurable in length do not have to one another the ratio which a square number has to a square number; and squares which do not have to one another the ratio which a square number has to a square number also do not have their sides commensurable in length either [7].*

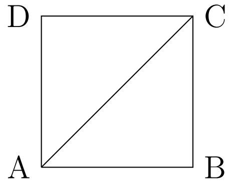

Figure 3: Square $ACBD$ with Diagonal AC

Proof. Examine square $ABCD$, where AC is the diagonal. If diagonal AC and side AB have a common measure λ, then there exist positive integers m and n such that

$$AC = m\lambda \text{ and } AB = n\lambda$$

and whose ratio would be

$$\tfrac{AC}{AB} = \tfrac{m\lambda}{n\lambda} = \tfrac{m}{n},$$

where the common measure λ is divided away and m and n no longer have a common measure. Now examine

$$\tfrac{(AC)^2}{(AB)^2} = \tfrac{m^2}{n^2}.$$

If we apply the Pythagorean Theorem to triangle ABC, then

$$(AC)^2 = AB^2 + AD^2 = 2(AB)^2$$

By substituting in $2(AB)^2$ for $(AC)^2$, we have

$$\begin{aligned} \frac{2(AB)^2}{(AB)^2} &= \frac{m^2}{n^2} \\ 2 &= \frac{m^2}{n^2} \\ 2n^2 &= m^2 \end{aligned}$$

It would follow that m is even. Let $m = 2k$, for any $k \in \mathbb{N}$ such that

$$\begin{aligned} 2n^2 &= (2k)^2 \\ n^2 &= 2k^2 \end{aligned}$$

Thus, we can conclude that n is even. Because both n and m are even, it implies that they have some common measure. This contradicts our statement above that m and n have no common measure, so we can conclude that the diagonal is of incommensurable length [6]. $\square$

3.2 Early Thoughts on the Infinite

3.2.1 Euclid

Euclid of Alexandria was a Greek mathematician of the third century B.C, and he is often called the father of geometry. This comes as no surprise, for his extensive work entitled *Elements* establishes the very foundation of geometry. Also included in *Elements* is a thorough work on number theory. Euclid is known for his astounding use of rigor and proof in his mathematics, and this is well-exhibited in his work. Book IX of *Elements* is on number theory, and in it, Euclid proves that the set of all primes is infinite. However, he does not use this terminology. Proposition 20 reads, "Prime numbers are more than any assigned multitude of prime numbers" [7]. Euclid's phrasing here may imply that he struggled to acknowledge the existence of actual infinity in mathematics. The idea of having an infinite amount of primes may have been an absurdity to Euclid. Nonetheless, he produces a proof that indeed proves the infinitude of the primes:

Proof. Assume that the set of all prime numbers is finite. Then there exists some prime number P that is the greatest of all the primes. Thus,

$$2, 3, 5, 7, ..., P$$

represents the series of all primes. Now, let Q be a number such that

$$Q = (2 \cdot 3 \cdot 5 \cdot 7 \cdots P) + 1.$$

However, Q cannot be prime because when divided by any product of the prime numbers, there will always be a remainder of 1. Thus, Q must not be prime, so there must exist some prime number that divides Q. Again, if Q is divided by any product of primes, there is a remainder of 1, so there needs to be some greater prime such that Q is divisible by it. This contradicts our assumption that P is the greatest prime number. Thus, the set of prime numbers must be infinite. $\square$

3.2.2 Aristotle and Infinity

In the third century B.C, Aristotle also attempted to make sense of the infinite. Being the great thinker he was, Aristotle thought that perhaps the infinite was not as daunting as his superiors found it to be. He found that perhaps there was a role of infinity in the universe. "For instance, it seems possible that time will go on forever; and it would seem that space is infinitely divisible, so that any line segment contains an infinity of points" [2]. However, Aristotle believed that infinity could not be examined in a whole sense, but rather as a potential element. He categorized these as *actual* and *potential* infinity.

3.2.3 Actual vs. Potential Infinity

Aristotle's writings imply that infinity can be expressed two ways: actual and potential. Potential infinity described things that increased or continued without bound. In other words, the potentially infinite is a finite collection which could possibly grow to be infinite in time. Vida Kolar and Tatjana Čadež describe potential infinity in their article on understanding infinity:

> "Potential infinity is related to an ongoing process without an end, for example, counting the natural numbers 1,2,3,4... It is an infinite process of which neither the end nor the last term (of the sequence) can be determined. We can imagine the procedure of acquiring ever new numbers, but we cannot realize it in practice" [13].

Actual infinity, on the other hand, is presently infinite. In other words, its "infinitude exists at some point in time (existing 'all at once')" [13]. Aristotle found that if anyone had a problem with the infinite, it was due to the fact that they were considering *actual* infinity, which is inconceivable for humanity. Aristotle's view on infinity spread to most of the known world and would be adopted centuries later by Galileo Galilei (1564-1642). Galileo went so far as believing that actual infinity had no place in mathematics and should be prohibited in order to "conserve the consistency of our logical reasoning" [13]. The idea of infinity, particularly *actual infinity*, was so bizarre and divergent from what great thinkers like Galileo and Aristotle knew that their solution was to essentially ignore the actual infinite's very existence. Galileo and Aristotle were certainly not alone in this mindset: many great thinkers of their respective times found infinity to be beyond comprehension. It was irrational and illogical, behaving differently than anything they had ever known. This mindset held firmly for centuries–*many* centuries. Astoundingly enough, actual infinity did not begin to receive proper recognition and implementation until 19th century A.D.

3.2.4 Galileo's Struggle with the Infinite

In Galileo Galilei's work, he described the confusion and perplexity of the infinite. He claimed that "infinity and indivisibility are in their very nature incomprehensible to us; imagine then what they are when combined. Yet if we wish to build up a line out of indivisible points, we must take an infinite number of them, and are, therefore, bound to understand both the infinite and the indivisible at the same time" [9]. He described several apparent objections to the supposition of a line with infinitely many indivisible points. He arrived at the fact that if one could split such a line, one of the resulting pieces may be of greater length. Both lines contain infinitely many points, thus "we may have something greater than infinity, because the infinity of points in the long line is greater than the infinity of points in the short line" [9]. Galileo found this to be beyond human comprehension, for he could not fathom an amount greater than infinity. He later stated,

> "This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another" [9].

Galileo also develops a paradox involving the square numbers and their roots. He examines how every square number has a root that it multiplied by itself to retrieve the square. Are there not more non-square numbers than square numbers? If we examine the proportion of squares to non-squares, we see that in the first 100 integers there are 10 square numbers, a ratio of $\frac{1}{10}$. In the first $10,000$ integers, there are 100 square numbers, a ratio of $\frac{1}{100}$. In the first $1,000,000$ integers, there are $1,000$ square numbers, a ratio of $\frac{1}{1,000}$. If this trend continues, which it does, we see the proportion of square to non-square numbers slowly diminish. This should lead one to believe that there are more integers than there are square numbers. However, an individual can square any integer, meaning that one could construct a square number for every integer. Thus, there is necessarily just as many square numbers as there are integers. Galileo believes this to bring about a contradiction, a paradox. He does, however, draw from this scenario that "the attributes 'larger,' 'smaller,' and 'equal' have no place in either comparing infinite quantities with each other or in comparing infinite with finite quantities" [9]. We will later see mathematician Georg Cantor speak against this in his work with the infinite. However, Galileo found that there simply was not an accurate way for humanity to measure or analyze infinity accurately.

4 Modern Examinations of Infinity

4.1 Convergence and Cauchy

French mathematician Augustin-Louis Cauchy (1789-1857) made significant contributions to calculus, particularly in dealing with infinite series. Now, the idea of infinite series is much older than Cauchy; we see infinite series as early as the fourth century B.C. in Zeno's paradoxes. Cauchy's superiors did well in evaluating series, but it was Cauchy that recognized the need to treat infinite series differently. "To Cauchy, the meaning of

$$\sum_{k=0}^{\infty} u_k$$

was more subtle. It required a precise definition in order to determine not only its value but its very *existence*." So Cauchy set out to define he exact value of an infinite sum. He examines this using a sequence of partial sums [10].

"Cauchy introduced the sequence of partial sums

$$S_1 = u_0,$$

$$S_2 = u_0 + u_1,$$

$$S_3 = u_0 + u_1 + u_2,$$

and generally

$$S_n = \sum_{k=0}^{n-1} u_k.$$

Then the value of the infinite series was defined to be the limit of this sequence, that is,

$$\sum_{k=0}^{\infty} u_k \equiv \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} u_k,$$

provided the limit exists, in which case 'the series will be called *convergent* and the limit...will be called the sum of the series'" [10].

What we have just examined is the work that Cauchy completed in order to construct a theory of infinite series. In order to make more sense of the infinite, Cauchy examined something that applies the idea of completing infinitely many tasks. In doing so, he has captured a bit of infinity's behavior in the concept of a limit.

An application of this process in mathematics is the proof that $.999... = 1$. This seems to be a false statement, for our intuition and reason tell us that these are simply two different numbers. However, we can use Cauchy's understanding of the infinite to prove this to be true.

Proof. Let $a = .99999...$. We can rewrite a as a list of partial sums

$$a = .99999... = .9 + .09 + .009 + .0009 + ... = \tfrac{9}{10} + \tfrac{9}{100} + \tfrac{9}{1000} + \tfrac{9}{10000},$$

which can be written as the sum

$$a = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{9}{10^k} = \lim_{n\to\infty} \left(1 - \frac{1}{10^n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{10^n} = 1 - 0 = 1$$

Then, the limit as n approaches infinity is 1, and thus a is also 1.

Hence, .999... = 1 $\square$

4.2 Cantor's Philosophy of the Infinite

Georg Cantor is perhaps one of the most important figures in history who explored the infinite. He is often regarded as the father of set theory, and he is well known for his theory of transfinite numbers. Cantor argued, contrary to many before him, that actual infinity deserved a place in mathematics, and he even developed the mathematics to accommodate it. At this point in time, no one had yet accepted the concept of the actual infinite, thus Cantor was incredibly bold in his pursuit to make sense of the infinite. Mathematicians found that it had no place in their work, and theologians felt that its existence would be a "direct challenge to the unique and absolutely infinite nature of God." Cantor, despite his frequent opposition, persisted and kept working in the face of adversity. In result we have an entire branch of mathematics and ideas that have changed the way we interact with infinity [14].

Cantor was determined to prove that infinity could and should be admitted to the use of mathematics. He knew that many before him had argued against it, and one of their reasons was due to the idea of the annihilation of a number. This is a common proof that essentially showed that summing any number with infinity causes the number to be "annihilated," for it does not affect infinity. Then, $a + b > a$ and $a + b > b$. Suppose b is infinite. Then, $a + \infty = \infty$ for any positive value a. This does not behave the way that positive numbers should, for in this case, $a + b$ is not greater to b. Using infinity in their number system did not make sense or function properly, so they argued it had no place. Cantor found this to be fallacious, for he believed that infinity could not be treated as any other number. Cantor knew what he was up against in taking this on. He read works of Aristotle, Descartes, and Leibniz and familiarized himself with their arguments against actual infinity.

> "These writers had produced the most convincing criticisms known to Cantor of the actual infinite. If he could demonstrate their error in rejecting completed infinities, then he was certain that his transfinite numbers could easily withstand criticism of a similar or less penetrating sort" [14].

In order to understand Cantor's work with the infinite, one must first examine his foundations for set theory and his theory of transfinite numbers. According to Cantor, there exist aggregates, which represent a collection as a whole. We take this today to mean *set*. Each aggregate M is composed of elements m. Thus,

$$M = \{m\}.$$

Cantor continues to explain how these aggregates can be unified as well as examined in terms of their parts or partitions. He states that

> "every aggregate M has a definite 'power,' which we will also call its 'cardinal number.' We will call by the name 'power' or 'cardinal number' of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given." Essentially, Cantor has just established his idea of cardinality, which assigns a number to every set that describes how many elements are in the set. Hence, the cardinality of:
>
> $$M = \{1, 3, 6, 4, 7, 9, \}$$
>
> is 6, for there are six elements contained in set M. The cardinal or power of M would be written as $\overline{\overline{M}}$. Cantor applies this simple structure to sets that are not quite as easy to visualize or even count [15].

In Chapter 6 of his work titled *Contributions to the Founding of the Theory of Transfinite Numbers*, Cantor discussed the concept of the "smallest transfinite cardinal number."

He wrote, "Aggregates with finite cardinal numbers are called 'finite aggregates,' all others we will call 'transfinite aggregates' and their cardinal numbers 'transfinite cardinal numbers.' The first example of a transfinite aggregate is given by the totality of finite cardinal numbers v; we will call this cardinal number 'aleph-zero' and denote it by $\aleph_0$; thus we define

$$\aleph_0 = \overline{\overline{\{v\}}} \ .$$

That $\aleph_0$ is a *transfinite* number, that is to say, is not equal to any finite number μ, follows from the simple fact that, if to the aggregate $\{v\}$ is added a new element e_0, the union-aggregate $(\{v\}, e_0)$ is equivalent to the original aggregate $\{v\}$. Thus we have

$$\aleph_0 + 1 = \aleph_0\text{" [15].}$$

Cantor's work also imply that there is not just one transfinite number but *many* different sizes. This is where Cantor is coined for stating that there are different sizes of infinity. He reveals this same notion in his proof of the uncountability of the real numbers, $\mathbb{R}$. Furthermore, he claimed that some transfinite aggregates are *countable*, while others are *uncountable*. Cantor's obvious method of choice in proving countability is by whether or not he could construct a one-to-one correspondence between two sets or aggregates, one of which would often be the natural numbers, $\mathbb{N}$. Examine his proof for the uncountability of $\mathbb{R}$.

Proof. Let $\mathbb{R}$ be the set of all reals. Assume that the cardinality of $\mathbb{R}$ is equal to that of the natural numbers $\mathbb{N}$. Thus, we can construct a one-to-one correspondence between $\mathbb{R}$ and $\mathbb{N}$ such that for every element in $\mathbb{R}$, there exists exactly one element in $\mathbb{N}$. Now, for the sake of simplicity, examine the interval [0,1]. We can begin listing the elements of each so that we can model this bijection.

$$1 \Longrightarrow .0120340506...$$

$$2 \Longrightarrow .0327938394...$$

$$3 \Longrightarrow .1054930473...$$

$$4 \Longrightarrow .1365789382...$$

$$5 \Longrightarrow .2673649501... \vdots$$

This process would continue until we had each element listed. However, for the sake of argument, let us construct a new number in $\mathbb{R}$ that is comprised of the numbers that form a diagonal in the example above.

$$
\begin{aligned}
1 &\Longrightarrow .[0]120340506...\\
2 &\Longrightarrow .0[3]27938394...\\
3 &\Longrightarrow .10[5]4930473...\\
4 &\Longrightarrow .136[5]789382...\\
5 &\Longrightarrow .2673[6]49501...\\
&\vdots
\end{aligned}
$$

This new number is .03556..., which is clearly not contained in our list. This implies that there is an element in $\mathbb{R}$ that does not have a pre-image in $\mathbb{N}$. Thus, the cardinality of $\mathbb{R}$ must be greater than the cardinality of $\mathbb{N}$, which contradicts what we stated previously. Therefore, the cardinality of $\mathbb{R}$ is necessarily greater than the cardinality of $\mathbb{N}$. $\square$

Thus, Cantor found that there were simply different sizes of transfinite sets. He sought to assign cardinalities to these sets, so he let $\aleph_0$ represent the cardinality of the natural numbers, $\mathbb{N}$, for they were denumerable, or countable. This means that all sets that were denumerable by the natural numbers would also have the cardinality $\aleph_0$. Furthermore, Cantor let $\aleph$ represent the cardinality of the real numbers, $\mathbb{R}$. Thus, $\aleph > \aleph_0$. Cantor also defined a concept known as a *power set*. He was able to prove that the power set of any set is strictly larger than the original set, denoted by

$$\overline{\overline{\wp(M)}} > \overline{\overline{M}}.$$

Today, we know this as Cantor's Theorem, and its importance lies in the implications it holds: there exists a hierarchy of transfinite cardinalities. Cantor showed that "$\aleph = \overline{\overline{\wp(M)}}$ and suggested that there are no cardinal numbers between $\aleph_0$ and $\aleph$, a conjecture known as the continuum hypothesis" [16].

Cantor certainly advanced the way mathematicians and philosophers alike viewed mathematics. Albeit his views were not generally accepted until after his death, Cantor's development of set theory, philosophy of the infinite, inspiration for calculus speaks to the apparent significance of his work.

Cantor's ideas were rejected by most of his contemporaries and even the Christian church. In their time, his mathematical ideas were so counter-intuitive and revolutionary that they just could not be seen as rational solutions to the problem of infinity. The church found his views on the actual infinite to indicate pantheism, for they felt it implied that something other than God could be ultimately infinite. Cantor, being a Christian himself, denied this claim and tried to demonstrate how learning more about the infinite helps learn more about God. He even felt that God had revealed the ideas of the infinite to him. Even so, Cantor could not handle the massive amounts of rejection he faced. Later in his life, he fell ill to depression and spent periods of time in and out of the sanatorium. Cantor spent his final year of life there, where he would die in early 1918 [14].

4.3 Richard Dedekind

Richard Dedekind was another nineteenth-century mathematician who is known to have made significant contributions to transfinite mathematics. Much of his work reinforces the work of Cantor, for we see Dedekind's work echo the very ideas Cantor fought so hard to defend. An example of Dedekind's work is his definition of similarity. Dedekind claimed that two sets were similar if one could construct a one-to-one correspondence between them. This resonates Cantor's idea of cardinality, namely using the process of constructing one-to-one correspondences to examine set size. Dedekind uses his definition of similarity to define an infinite set, which he defines to be a set that is "similar to a proper part of itself." In other words, two sets are similar if it is equinumerous to one of its proper subsets [17].

Take the natural numbers $\mathbb{N}$, for example. $\mathbb{N}^3$ and $\mathbb{N}$ are equinumerous, for every element in the natural numbers has a cube. $\mathbb{N}^3 \subset \mathbb{N}$ because every element of $\mathbb{N}^3$ is in $\mathbb{N}$, but there are some elements in $\mathbb{N}$ not in $\mathbb{N}^3$, as demonstrated below.

Dedekind is also known to have admired the work of Georg Cantor and supported his ideas concerning transfinite numbers. It is here that we perhaps see significant develop-

$$
\begin{aligned}
\mathbb{N} &\longrightarrow \mathbb{N}^3\\
1 &\longrightarrow 1\\
2 &\longrightarrow 8\\
3 &\longrightarrow 27\\
4 &\longrightarrow 64\\
5 &\longrightarrow 125\\
6 &\longrightarrow 216\\
7 &\longrightarrow 343\\
&\vdots
\end{aligned}
$$

Figure 4: Mapping $\mathbb{N}$ onto $\mathbb{N}^3$

ment in thought on infinity.

5 Conclusion

Mathematicians have been positively perplexed by the infinite for centuries. The idea of the infinite contradicts everything we find to be true about nature, logic, intuition, and reason, for it behaves in a way that cannot be easily described. It is a subject that demands bold ideas, creative thinking, and struggle. "From time immemorial, the infinite has stirred men's emotions more than any other question. Hardly any other idea has stimulated the mind so fruitfully. Yet, no other concept needs clarification more than it does" [18]. The infinite causes us to stretch our imaginations, stumble in the dark, and sometimes arrive at beautiful, revolutionary ideas. In their search for the infinite, many mathematicians have stumbled upon remarkable ideas and theories. The development of infinity reminds us as humans to not be afraid of the vast unknown and to not shy away from the bigger ideas. Thinking about infinity has caused us to think bigger about mathematics, and mathematics has become all the more beautiful in result.

References

[1] Huemer, M. (2016). Approaching infinity. New York: Palgrave Macmillan.

[2] Rucker, R. (1982). Infinity and the Mind: The Science and Philosophy of the Infinite. Retrieved from

https://math.dartmouth.edu/ matc/Readers/HowManyAngels/InfinityMind/IM.html

[3] Pythagoreanism. (2016). In Encyclopædia Britannica. Retrieved from https://www.britannica.com/topic/Pythagoreanism

[4] Incommensurables. (2016). In Encyclopædia Britannica. Retrieved from https://www.britannica.com/topic/Incommensurables-1688515

[5] Transfinite Number. (n.d.). Retrieved November 22, 2016, from https://www.britannica.com/topic/transfinite-number

[6] Burton, D. M. (1985). The History of Mathematics: An Introduction. Boston: Allyn and Bacon.

[7] Euclid's Elements, Book X. (n.d.). Retrieved November 20, 2016, from http://aleph0.clarku.edu/ djoyce/java/elements/bookX/bookX.html

[8] Grandjean, M. https://commons.wikimedia.org/w/index.php?curid=39999636

[9] Galilei, G. (1954). Dialogues concerning two new sciences. New York: Dover Publications.

[10] Dunham, W. (2005). The Calculus Gallery: Masterpieces from Newton to Lebesgue. Princeton, NJ: Princeton University Press.

[11] Lay, Steven R. Analysis: with an Introduction to Proof. Englewood Cliffs, NJ, Prentice Hall, 1990.

[12] Salmon, Wesley C. "Space, Time and Motion: A Philosophical Introduction." The Philosophical Review, vol. 86, no. 4, 1977, p. 560. doi:10.2307/2184569.

[13] Kolar, V. M., & Čadež, T. H. (2011). Analysis of factors influencing the understanding of the concept of infinity. Educational Studies in Mathematics, 80(3), 389-412. doi:10.1007/s10649-011-9357-7

[14] Dauben, J. W. (1979). Georg Cantor: His mathematics and philosophy of the infinite. Cambridge, MA: Harvard University Press.

[15] Cantor, G. (1952). Contributions to the founding of the theory of transfinite numbers. New York: Dover Publications.

[16] "Set Theory." Encyclopedia Britannica Online, Encyclopedia Britannica, www.britannica.com/topic/set-theory.

[17] Dedekind, R. (1963). Essays on the Theory of Numbers. New York: Dover Publications.

[18] Hilbert, D. (n.d.). On the Infinite. Retrieved November 22, 2016, from https://math.dartmouth.edu/ matc/Readers/HowManyAngels/Philosophy/Philosophy.html