Examining Disproved Mathematical Ideas through the Lens of Philosophy

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Mathematics is thought of by many as a flawless field of study. Its emphasis on logic, consistency, and getting a “right answer” gives it the reputation of perfection. Such a view of mathematics, however, would be leaving part of the story untold, one in which disproved ideas contribute to mathematics’ richness and evolution. Over the centuries, some ideas that have been long accepted have later been found flawed. At times, great upset occurred in the mathematical world at the seemingly “un-mathematical” realizations that came with the replacement of past ideas. Even in the state of mathematics today, it would be foolish for us to assume we have outgrown mathematical mistakes. It seems that as long as mathematics and humanity co-mingle, ideas will be continually disproved before being improved in the world of mathematics.

The inherency of these mistakes produces significant effects within mathematical philosophy. The disproof of a theory has repeatedly shattered a former philosophical understanding and opened the door to mathematical advancement. The transformation of the story behind mathematics points to a mathematical philosophy that accounts for both the subject’s perfection and the fallible beings that have shaped it.

The ancient Greek mathematical society of the Pythagoreans significantly contributed to the development of deductive reasoning and abstract mathematics. To them, mathematics was more than a practical tool or even academic pursuit, but was in fact, a lens by which they understood and related to the world around them. Their society hallowed numbers as central to their religion, operating under the dogma, “All things are number,” where “number” referred to the exclusive set of whole numbers (Havil, 12). One of Pythagorean’s fundamental beliefs was the idea of commensurability, which was based on the notion that any two lengths could be compared as ratios with a common measurement. The Greek mathematician Hippasus was perhaps the first to happen upon a contradiction to this belief of universal commensurability among numbers. It is supposed that while studying a square with side length of one unit that Hippasus compared the square's diagonal length to its side length, which was a $\sqrt{2}$ to 1 ratio. To show the incommensurability of these two lengths, a proof by contradiction is contained within Book X of Euclid's *Elements* (Havil, 21-22). We will explore this proof here as well.

Proof:

Consider the square ABCD with diameter AC. Prove that the length of AC is incommensurable with the length of AB. Begin by assuming the opposite: AC is commensurable with AB. This assumption will lead to a contradiction stating that some number is both even and odd.

Firstly, we know that $AC^2 = AB^2 + BC^2$ and since ABCD is a square, we can substitute to get $AC^2 = 2AB^2$. Since we assumed AC is commensurable with AB, AC and AB must create a ratio with whole numbers, which we will call DE:DF in its most reduced form. DE cannot be the unit length of one, since its ratio to DF is the same as AC to AB, where AC is longer than AB. So, DE cannot be both a unit length and greater than the whole number DF. Thus, DE must be a whole number other than one.

Now, we can obtain the following ratios:

$$\frac{AC}{AB} = \frac{DE}{DF}$$

Thus,

$$\frac{AC^2}{AB^2} = \frac{DE^2}{DF^2}$$

Now we already know $AC^2 = 2AB^2$. Thus, DE^2 must equal $2DF^2$ as well to maintain the proportion. We can conclude that DE^2 is even, since 2 must divide into it evenly, and DE must likewise be even.

Now let us designate a midpoint G on the segment DE. We know DE and DF form a reduced ratio. Since DE and DF must be prime with one another and we showed previously that DE is even, DF must be odd. As designated by our segment, DE = 2EG. Therefore, $DE^2 = 4EG^2$. We established already that $DE^2 = 2DF^2$. So $DF^2 = 2EG^2$. This implies that DF^2 is even and thus, DF is even. However, this contradicts our conclusion that DF must be odd. It follows that AC cannot be commensurable with AB. **Q.E.D.**

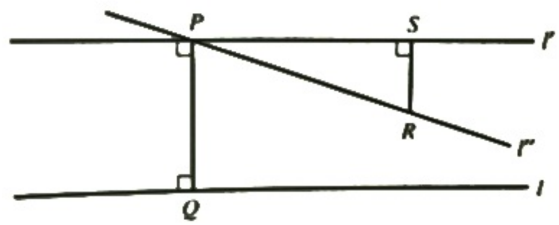
The realization of the falsity of this assertion concerning universal commensurability caused great upset in the ancient mathematical community, and for some time, the idea was forbidden to be spoken of outside the Pythagorean circle. Gradually, the Pythagoreans adjusted their view about mathematics and their resulting worldview to fit the seemingly “illogical” existence of incommensurable numbers. As the faulty theory that all numbers were commensurable was discarded, the Pythagoreans paved the way for future advancements in mathematics, opening the door to the modern day concept of irrationality as well as the revolutionary idea of the transcendental (154). The discovery of the irrationals occurred nearly 2,500 years ago, yet a true understanding of this set of numbers was delayed more than 2,000 after that (1).

The dispensation of the flawed theory of commensurability sparked two important changes in mathematics: it made advancements in mathematics possible and it molded our perspective on mathematics itself. Though perhaps not acknowledged at the time, the discovery of irrational numbers was contributing to an evolution of the philosophy of mathematics. The existence of irrationals seemed to redefine reason, the foundation of mathematics. Mathematics was beginning its journey of morphing into the creature we see today, a paradox of logic and arbitrariness, measurability and infinitude.

Following the disproof of commensurability, we follow the storyline of mathematics to another significant idea soon to be reconsidered: Euclid's famous fifth postulate, or the parallel postulate. The postulate states: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which the angles are less than the two right angles" (qtd. in Franceschetti, 234). Since its appearance in Euclid's *Elements*, mathematicians have struggled with accepting this supposedly self-evident truth and have made attempts to either prove or revise it. One of the mathematicians, who offered revisions for this postulate, was John Playfair, the mathematics chairman of the University of Edinburgh. His became the most popular revision, and it is still contained within our modern textbooks for the study of geometry. It states, "Through a given point, not on a given line, only one parallel can be drawn to the given line" (Burton, 564). Others believed Euclid's fifth postulate needed more than revisions, but actually required proof as a theorem. Among the mathematicians who contributed to the body of proofs, which were one-by-one shown to contain error, we find Proclus of Greece (410-485 AD), Nasir-Eddin of Persia (1201-1274 AD), and John Wallis from England (1616-1703) (Franceschetti, 234).

Let us briefly consider an overview of Proclus' proof:

Proof:



(Image obtained from Burton, 565.)

Consider the lines *l* and *l'* where a point *Q* lies on *l* and a point *P* lies on *l'*. Segment *PQ* is perpendicular to both *l* and *l'*. Lines *l* and *l'* are parallel lines since they have equal alternate interior angles. We can then deduce that any other line *l''* through point *P* will intersect *l*. We can show this by choosing an arbitrary point *R* on *l''* that lies in between *l* and *l'*.

Now draw the perpendicular from *l''* to *l'* with it intersecting *l'* at point *S*. Consider moving the point *R* along *l''* so that its distance from *P* is increasing. Thus, the length of segment *SR* increases as well. Eventually, the length of *SR* will reach and then surpass the distance between *l* and *l'*, that is *PQ*. Thus, *l''* will intersect *l*. Therefore, there is no line except *l'* that goes through *P* and is parallel to *l*. **Q.E.D.**

The downfall of Proclus' proof lies in his assumption that *l* and *l'* are equal distances apart in all places on the lines. Superficially, this seems like a logical assumption. No grounds, however, are provided for this in the preceding four postulates. In fact, Proclus' assumption that the two lines are everywhere equidistant is a subtle form of the parallel postulate itself. Therefore, we see that Proclus has assumed the very thing he has set out to prove (565).

In the 1700s, non-Euclidean geometry was beginning to emerge, though unbeknownst to its developer, Girolamo Saccheri. His book entitled *Euclid Free of Every Flaw* attempted to once more prove the parallel postulate. In effect, he actually showed the postulate's independence from the other postulates of traditional geometry (Franceschetti, 234).

After years of failed proofs, the mathematical public learned the surprising fact that the parallel postulate was not in fact necessary in the logical development of geometry. Georg S. Klugel was among the first mathematicians to assert that Euclid's fifth postulate could not, in fact, be proved, but was rather the result of an observation of the physical world. Johann Heinrich Lambert spurred on by this belief performed more groundwork in the area of non-Euclidean geometry and recognized its possible application to spheres and non-planar surfaces (235).

Non-Euclidean geometry was finally birthed in the 1800s under the work of several mathematicians, including Carl Friedrich Gauss (1777-1855), Nikoali Ivanovich Lobachevsky (1793-1856), and Janos Bolyai (1802-1860). In the article, “Mathematicians Reconsider Euclid's Parallel Postulate,” Donald R. Franceschetti explains that, “[At this point] it was discovered that totally consistent geometries could be obtained by assuming that through a given point either no lines or multiple lines could be drawn parallel to a given line” (235). At last, it became evident that Euclid’s parallel postulate could be neither proved nor disproved.

The expanded understanding of Euclid’s parallel postulate serves as a second case, after the disproof of commensurability, in which major changes are sparked in the mathematical world. Through the disposal of Euclid’s fifth postulate was the development of a new branch of mathematics: non-Euclidean geometry. Along with this development, mathematical philosophy was continuing to be shaped as well, giving more evidence for the influence human understanding exerted on interpreting the absolute truth mathematics offered.

In the midst of such revolutionary advancements in mathematical thought is found the exemplification of seemingly insignificant disproved ideas along the way. These disproved ideas, though likely not revolutionary individually, have impacted the journey of mathematics and the view of this subject as fallible in human history. The fact that theories containing unnoticed errors are sometimes too readily accepted serves as a reminder of how humans, when working with mathematics, exhibit limited scope of understanding and can often be misled by intuition. We might even say that all mathematical theories are flawed in some way, yet are useful for the advancement of mathematical thought, reaching toward more accurate and more enlightened theories ahead. Let us explore one of the many counterexamples that contributed in their own small way to the evolution of mathematics.

For many years, the mathematical community had attempted to prove Fermat's assertion that any number of the form $y = 2^{2^n} + 1$ is prime. Although it had not been formally proved, many accepted Fermat's assertion. Euler, when confronted with the observation presented by Fermat, assumed a skeptical approach, and as a result, found a counterexample:

$$2^{2^5} + 1 = 4{,}294{,}967{,}297$$

In this case, when n=5, we see the resulting number is not prime, since 641 divides evenly into it (Dunham, *Euler: The Master Of Us All*, 7).

While this specific counterexample incited no great change in the philosophy of mathematics, it sparked an interest in Euler for the field of number theory, to which he eventually contributed volumes of work. Dunham writes that "[Euler] plunged into Fermat's work, finding it a source of beauty and endless fascination. Over the course of his career, Euler addressed number theoretic matters of profound importance as well as those of considerably less significance" (7). Euler's counterexample can be examined as one building block among countless others that participate in constructing mathematics as a less than perfect art form that is constantly being refined and built upon.

We now come back to the exploration of more significant disproved mathematical ideas that seem to have shaken and shaped mathematical advancement and philosophy. Next we trace Georg Cantor's study of the infinite. His discoveries defied the universally accepted notions concerning the nature of infinity. Before Cantor, the majority of mathematicians considered infinity to be so abstract that it was almost unprofitable to study it. Thanks to Cantor this idea was dispelled as he showed infinities could be counted, compared, and even deemed complete (Dunham, 252).

From the time of ancient Greece, we see reference to the infinite, perhaps most prominently in Zeno's paradoxes. The first of these, known as the dichotomy, suggests that movement from a point A to a point B is logically impossible because of the infinite divisibility of the distance between any two points. The second, coined by its characters as "Achilles and the Tortoise," emphasizes a similar concept: if runner A begins a race before runner B, the second runner can never overtake the first, no matter how fast the second may be, due to the logical difficulty of infinite regression (Stakhov and Olsen, 367).

Another assumption that existed concerning infinite sets was that given the idea of infinity for the set of rationals and irrationals, there must be hypothetically equal amounts of each within the infinite set of real numbers (Dunham, *Journey through Genius,* 252). It seems the concept of the infinite was little touched over the course of mathematical history before Cantor, possibly because it was considered too mystical for the realm of mathematics. This mathematician, however, who many attribute as having a "creative bent," drastically changed the scene, singlehandedly renovating the idea of the infinite and influencing set theory (Stakhov and

Olsen, 372-373). Dunham contrasts Cantor's belief of the infinite to the previously held idea of infinity, stating, "To Cantor, [the infinite] was a solid, respectable mathematical concept worthy of the most profound intellectual examination" (*Journey Through Genius*, 254). Here we see a seed for change in mathematical philosophy that we will later explore.

The first of Cantor's genius ideas was his implementation of one-to-one correspondence rather than counting to "denumerate" sets of numbers (253). He was thus able to compare the size of infinite sets and determine which infinity was greater in size, or rather possessed greater cardinality (253). The second idea, which Cantor developed and that proved to be helpful for consequent theorems, was the concept of the "actual infinite" rather than just the "potential infinite" (Stakhov and Olsen, 372). The "actual infinite" referred to a "completed" sequence, that is, to all the members of the sequence at once. The other concept of the "potential infinite" describes an infinity that is always developing, each new member of the sequence can be found from the previous term (372). Armed with the radical tools of denumerability found by one-to-one correspondence and the completeness of an infinite set, Cantor was able to prove even more "outrageous" ideas. After showing the denumerability of rationals and the non-denumerability of irrationals, Cantor was able to prove by contradiction the non-denumerability of the set of reals (264).

Similarly, Cantor explored the "exhaustive and mutually exclusive categories" of algebraic and transcendental numbers (265). At this point, transcendental numbers held the reputation of being few and far between, since only a handful of them had been discovered. Still Cantor claimed and then proved that the infinite set of the transcendentals outnumbered the set of algebraic numbers (265). The mathematical world had a hard time accepting this new reality, however (266). Cantor's discovery is a beautiful example of shattering norms of current thought.

He achieved that which was considered impossible and as a result, corrected an inadequacy in previous mathematical theory.

Through exploring each of these disproved mathematical ideas—from Pythagorean commensurability to Euclid's Parallel Postulate to ancient ideas of the infinite—we find support for a distinctive mathematical philosophy. Before expounding upon this particular philosophy, a brief background will be provided for two of the significant philosophical standings of today within mathematics: Platonism and socio-historic constructivism.

According to Reuben Hersh in *What Is Mathematics, Really?,* Platonism is a philosophical perspective that asserts that "mathematical entities exist outside space and time, outside thought and matter, in an abstract realm independent of any consciousness, individual or social" (9). Hersh deems Platonism, the "most pervasive philosophy of mathematics" (9). For many, this philosophy exemplifies the nature of mathematics as absolute truth—definite and unchanging. The mathematics performed by humans is seen as a shadow of the flawless mathematics that persists outside of us. Hersh underscores this philosophy's perspective of the constancy of mathematics, expressing a supposed proponent: "That's what's special about math. There are right answers. Not right because that's what Teacher wants us to believe. Right because they are right" (11).

The second philosophy is defined due to its view of mathematics as a socio-historic construction. Based on this philosophy, Hersh proposes two facts about mathematics:

> 1. Mathematical objects are created by humans. Not arbitrarily, but from activity with existing mathematical objects, and from the needs of science and daily life.
> 2. Once created, mathematical objects can have properties that are difficult for us to discover. (16)

This constructivism sees mathematics as a body of concepts and their relationships as established in human culture throughout history (19). Hersh asserts that "Mathematical knowledge isn't infallible. Like science, mathematics can advance by making mistakes, correcting and recorrecting them" (22). This final remark leads into our consideration of how disproved theories have advanced our understanding of mathematical philosophy.

Mathematics does not reside just in the realm of Platonism nor of a socio-historic constructivism. Rather, mathematics combines the absolute concepts that exist outside of human thinking, as well as social constructions that humans have created throughout history. These previously discussed disproved ideas offer evidence for the adoption of both philosophies, as they shape and strengthen one another.

Beginning with the Pythagoreans, we address the discovery of incommensurable numbers as just that—a discovery. The Pythagoreans had constructed a mathematical world around the concept of commensurable numbers, yet somehow the idea of incommensurability broke through. This event exemplifies the pervasive realm of absolutes that has permeated a mathematical understanding. As a reaction to this norm shattering idea, the Pythagoreans re-shaped their view of number and its properties. Here we see the evidence of Platonism combined with a socio-historic influence as the Pythagoreans reached an external truth through human means of study.

Next, moving to the expansion of study surrounding Euclid's parallel postulate, we find additional evidence of a dual Platonist-Constructivist perspective. Ironically, Euclid built upon the assumptions of "self-evident truths." He believed these mathematical axioms described the reality of the physical world—an exterior mathematical reality. His parallel postulate was later replaced by non-Euclidean axioms, which were deemed to describe the world more accurately.

This discovery of a more accurate mathematical description again supports the argument of Platonism. Despite the presupposition of the parallel postulate, the postulate's consistency with the physical world was logically challenged and, in a sense, disproved. Its consequent replacement with non-Euclidean axioms evidences the existence of an absolute reality. On the other hand, the continued use for traditional Euclidean geometry provides the evidence of a constructivist philosophy. This parallel postulate is a social entity created by humans in relation to other mathematical entities. These Euclidean concepts are still helpful in describing the world and building logical arguments. With respect to Euclid's parallel postulate, Henri Poincaré exemplifies the opposite view of constructivism by stating that "non-Euclidean geometry—dealing with the non-flat surfaces of hyperbolic and elliptical curvatures—proved that Euclidean geometry, the longstanding geometry of flat surfaces, was not a universal 'truth,' but rather one outcome of using one particular set of game rules" (Dekofsky). The fact that non-Euclidean geometry can be linked to the aspects of the physical reality actually reaffirms the Platonist perspective, after all. Yet Poincaré supplies a valuable observation in that the presence of the parallel postulate indicates the handiwork of humans in the mathematical development. These Euclidean and non-Euclidean geometric concepts are equally viable as social constructions, yet one stands as preeminent in its representation of the external mathematical truth.

Next, we discuss disproof through counterexamples, in its contribution to our understanding of a Platonist-Constructivist philosophy. Here we find recurring evidence of how humans actively contribute to mathematical theory, molding it as a socio-historic invention. These counterexamples firstly point to the existence of an outside mathematical truth—one that can be obtained by evidence, argument, and proof. As much as some might like to believe this supports a purely Platonist philosophy, these errors do not just mask the perfect mathematical reality—one that is thwarted by human interference—but also show how mathematics has been worked with and guided by humans. Mathematics as it exists in our world today is marked by human impact. What we accept as true depends on the minds of those before us. This truth is constantly being revised and revisited as we advance in mathematics.

Lastly, Cantor's study of the infinite provides further support of this hybrid mathematical philosophy between Platonism and Constructivism. To Cantor, he was stepping into the realm of perfect forms, being enlightened as to an external reality and bringing it to bear in our limited realm. Within the essay "Cantor's Philosophy on Set Theory," Cantor held two fundamental beliefs: "namely the thesis on specific 'freedom' of mathematics and the thesis that mathematical objects are given to us and not created by us" (Murawski, 18). Throughout mathematical history, we can follow the development of the idea of the infinite. It was built upon by various minds, and Cantor's use of definitions for the actual and potential infinite, directed the future study of the infinite as well. Cantor captures the paradox of the infinite's external reality as it exists outside human comprehension, yet its searchability through study and exemplification in the physical realm:

> [The absolute] overcomes in a certain sense the human ability of comprehension and cannot be described mathematically; on the other hand the transfinite not only fills a substantial space of possibility in God's knowledge but is also a rich and constantly expanding domain of ideal research and—in my opinion—is realized also in the world of created things and does exist to a certain degree and in various relations. (qtd. in Murawski, 23)

Humans bring to bear creative aspects in their interaction with mathematics, yet they are working with something solid and absolute. Cantor was able to study something seemingly abstract in a concrete way. It seems to point to the intersection of these perfect forms with our limited attempts to grasp it.

No single philosophical source fully explains the evolution of this complex field of study. Together the underpinnings of Platonism and socio-historic constructivism, however, greatly clarify the mathematical ontology and pervade the disproved theories that shape the mathematical story. From the Pythagoreans to Cantor, and extending to many discoveries yet to come, mathematics remains a paradox with logical consistency yet human error. The flaws, rather than mar the perfection of mathematics, contribute to its beauty and fingerprint it with human ingenuity.

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