Theories on the Origins of the Sexagesimal System

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Civilization has come a long way from its cradles in ancient Mesopotamia—from the primitive farms along the Euphrates to the endless fields of the Prairies; from the small city-states of Uruk and Ur to the empires of Spain and Britain; from the chaotic clusters of mud huts and ziggurats to the grid-like towers of Manhattan and Dubai, human civilization has indisputably progressed in the past 7000 years—though such advancements were neither linear nor straightforward. Likewise, mathematics has evolved a great deal—from everyday computations to Euclidean geometry to real analysis to graph theories. While much scholarship has documented various segments of the evolution, its beginnings in Mesopotamia more than 5000 years ago were inadequately studied. Scholars of the medieval and early modern periods, despite practicing the legacies of the ancient Near East, had little knowledge of even the existence of the Sumerians, Akkadians, and Babylonians. Only in 1900 were the Babylonian progenitors of modern mathematics reintroduced to the world. From 1927 to 1939 was the heyday of research on Mesopotamian mathematics, and after that, only sporadic efforts, most notably by Powell and Friberg, were made on said topic (Robson 2008, p. 6-7).

According to mathematician Robert Creighton Buck, one asks three questions about any artifacts of ancient mathematics: what their properties are, what their original purpose is, and what they tell us about the culture (Buck 1980, p. 5). While modern scholars have deciphered numerous cuneiform texts of ancient Mesopotamia that broadened our understanding of Babylonian mathematical knowledge, these findings have failed to answer the second question in a critical manner. If we were to treat the sexagesimal number system, on which most Babylonian mathematical texts since the third millennium B.C.E. were based, as an artifact, what would be its original purpose? In other words, what was the origin of the sexagesimal system?

The inadequacy to answer this question was not due to scholarly negligence: “In the history of science,” remarked Dr. Buck, “one expects neither theorems nor rigorous proofs. The subject is replete with conjectures and even speculations.” (Buck 1980, p. 5) There have been numerous theories attempting to explain the origins of the sexagesimal system, but most have not been convincing. This paper will examine the existing theories and argue that the divisibility of the number 60 was the most important reason leading to the adoption of the sexagesimal number system in ancient Mesopotamia.

The sexagesimal number system, despite its unusually large base, was simple to use. Only two distinct symbols were utilized—that of 1 and 10—to express any positive number. Numbers 1 through 59 were expressed decimally, by repeating the “1” and “10” symbols; for example, the number 42 would be expressed by 4 “10”s and 2 “1”s. The number 60—as well as its n^{th} and $\left(\frac{1}{n}\right)^{th}$ powers ($n \in \mathbb{N}$)—is denoted identically to the number “1”; for instance, the number 420 would be expressed by 7 “1”s, as 420 is 7 times 60 (Mazur 2014, p. 13-15).

Historian of mathematics Marvin Powell has divided the inquiry of the sexagesimal system’s origin into two parts—the origin of the number 60 as a base and the origin of a place-value system (Powell 1972, p. 5). The latter is trivial, as the place-value system possesses qualities superior to its alternatives. The place-value system does not require many distinct symbols to express large numbers. For instance, the Hindu-Arabic numerals contain merely ten distinct symbols, yet are capable of expressing numbers as large as imaginable. On the other hand, the non-place-value Egyptian numerals require as many distinct symbols for a number as there are (non-zero) digits in said number; to express a 100-digit number would require 100 distinct symbols, which would make the number difficult to read (Mazur 2014, p. 17). Likewise, place-value number systems do not require additional symbols to express increasingly fine fractions. It's also easier to multiply and divide large numbers using the place-value Hindu-Arabic system than it is using the non-place-value Roman numerals. Given the advantages of the place-value system, its adoption in ancient Babylon was thus an overwhelmingly likely consequence of its invention—a fact that could be circumstantially demonstrated by the dominance of place-value systems in the modern day. The binary computer systems, the decimal Hindu-Arabic numerals, and the sexagesimal division of temporal units (i.e., 60 seconds in a minute) are all examples of place-value number systems used in the modern day. Unfortunately, we do not know of the genius who first devised such a system (Powell 1972, p. 17).

The origin of the base 60 was a more intriguing inquiry. A study referenced by Merzbach and Boyer[1] found that, among the hundreds of Native American tribes surveyed, a third had used a decimal base, another third adopted a quinary or quinary-decimal system, while about 10 percent had employed a vigesimal system. The origins of these systems were obvious—counting fingers. Other number bases commonly used were 2 and 3, which Merzbach and Boyer conjectured to be more primitive number systems than the decimal, quinary, and vigesimal ones (Merzbach and Boyer 2011, p. 4). They did not explain how the base 60 came to be in Mesopotamia.

[1] They did not provide in-text citations.

Of the many theories concerning the origins of the base 60, Thureau-Dangin's was a convincing one—until his linguistic understanding of the ancient Sumerian language was found inaccurate. He conjectured that the Sumerians, like most cultures analyzed by Merzbach and Boyer, had experienced phases of using quinary and decimal bases. With the later base 6 being unable to wholly replace the earlier base-10, a mixed-based sexagesimal system was thus created, as the Sumerians treated 60 as the greater (compared to 1) unit (Thureau-Dangin 1939, p. 101-105).

Thureau-Dangin's evidence was the poorly understood Sumerian language. He discovered that the Sumerian words for the numbers 7 (imin) and 9 (ilimmu) had meant 5 plus 2 and 5 plus 4 respectively. Thus, he hypothesized a proto-quinary stage of Sumerian numeration. Likewise, he found that the etymological roots of the Sumerian words for 30 (ušu), 40 (nimin), and 50 (ninnu) were 3 times 10, 20 times 2, and 40 plus 10 respectively, leading to his belief of a proto-decimal stage (Thureau-Dangin 1939, p. 102).

However, the number 10 served as a poor base, due to its inability to divide by 3. 6 or 12, as Thureau-Dangin remarked, would serve as a better unit. However, despite the "common tendency to take 6 or 12 as a new unity," such as in the dozen, a base 6 or 12 numeration system does not exist in human history. The tradition of finger counting was too prominent to be replaced—even in Mesopotamia, as evident by the preservation of the unique symbol for the number 10. Thus, Thureau-Dangin reasoned, the sexagesimal system must have been established before the Sumerians counted to 100, as it eliminates the necessity for 60 to assume the place of 100 (Thureau-Dangin 1939, p. 102-104).

Thureau-Dangin's theory was not only logically meandering and highly speculative, but was also founded on false premises. His thesis was founded on the erroneous reading of the Sumerian language. Contrary to his belief, the Sumerian words for 1 and 60 were not identical: the word for 1 was "diš" while the word for 60 was "geš" (Powell 1972, p. 7-8). If the Sumerians did view 60 as a greater 1, there is no known linguistic evidence for it. Moreover, a set of proto-Elamite tokens, which according to Friberg's interpretation expressed the number 324 in decimal form and predated the widespread adoption of sexagesimal cuneiforms, has dated the ancient Mesopotamians' grasp of numbers above 100 prior to their widespread adoption of sexagesimal numeration, which is in direct contradiction to Thureau-Dangin's conjectures (Friberg 2019, p. 186).

As with Thureau-Dangin, Powell also attempted to find the origins of the sexagesimal system linguistically, though the two scholars have arrived at different conclusions. Powell agreed with Thureau-Dangin that Sumerian numeration began as a proto-quinary system. However, Powell denied the existence of a proto-decimal core in the sexagesimal system. He argued that while the number 10 "functions in Sumerian as a multiplier, it is never multiplied." (Powell 1972, p. 8) The number 30 (ušu), he argued, should be etymologically interpreted as ten threes (eš-u), not three tens (u-eš) as in the modern decimal system. Curiously, 40 (nimin) and 50 (ninnu) were written as two twenties (niš-min) and two twenties plus ten (niš-min-u) (Powell 1972, p. 9).

While Powell rejected Thureau-Dangin's theory, he could not provide a convincing alternative of his own. His analysis of a Sumerian dialect suggested that the number 3 might have played a role in the origins of the sexagesimal system, though he could not formulate a cohesive thesis detailing how. He simply maintained that the problem could be solved once the etymological roots of the Sumerian word for 60 were unearthed (Powell 1972, p. 9-10).

Beyond its incapacity to arrive at a definitive conclusion, Powell's linguistic approach also featured a fatal logical error. In order to prove, as Powell hypothesized, that the sexagesimal number system was the creation of the Sumerian language, one must chronologically place the latter ahead of the former. Incidentally, in order to prove that the Sumerian language had influenced their number system, one must find traces of the base 60—for instance, the identity between the words for 1 and 60, as Thureau-Dangin mistakenly believed—within the language itself. This is paradoxical: the language could not have predated the number system while featuring said system; any new evidence that could suggest the latter implication would necessarily date the origins of the sexagesimal system to no later than the era in which the evidence was manufactured, leading to the negation of the former implication. Interestingly, it is also impossible to prove the influence of the sexagesimal system on the Sumerian language—i.e. the inverse of Powell's hypothesis—due to the archeological necessity to recognize archaic concepts (e.g. the sexagesimal number system) through symbolic remains (e.g. written languages). In other words, we cannot prove a people's knowledge of the sexagesimal system without finding their written records of said system. Therefore, since neither Powell's hypothesis nor its inverse could be proven, it is impossible to establish a causational relationship between the philology of the ancient Sumerian language and the origins of the base 60.

Many other theories concerning the origins of the base 60 have also failed. Julca's superficial attempt to connect the geometric properties of circles and the number 6 was based on no historical evidence. In fact, his 2007 paper did not feature the word "cuneiform" at all.

(Julca 2007). Theories concerning ancient astronomy are easily refuted by the fact that sexagesimal mathematics had predated its application in measuring the sky by more than a millennium (Joseph 2010, p. 172). Neugebauer's conjecture regarding metrology's influence on the number system has misjudged the relative importance of a unit of measurement and the number system—the latter should have modified the former, not vice versa (Neugebauer 1952, p. 19-20).

The theory of divisibility, however, remained probable; that the divisibility of the number 60 by 2, 3, and 5 had led to its adoption as the number base. First proposed by Theon of Alexandria in the 4^{th} century A.D. and referenced by later authors such as Thureau-Dangin, the theory is appropriately supported by historical evidence (Powell 1972, p. 6; Thureau-Dangin 1939, p. 102).

By around 2050 B.C.E. there were two parallel systems of fractions in ancient Mesopotamia—the unit fractions and the sexagesimal fractions. The former, commonly used in metrology, had been in place since at least 2500 B.C.E. while the latter had developed as a corollary of the sexagesimal number system. While the unit fractions remained commonly used after 2050 B.C.E., their important functions of computation and conversion have been replaced by the sexagesimal fractions (Robson 2008, p. 76-77).

Before the introduction of the sexagesimal fractions, messy metrological notations were commonplace. For instance, there was an expression that read " $\frac{1}{3}$ mina 1 shekel and a 4^{th} part of washed silver," which translates to $\frac{85}{4}$ shekels. Another expression read "its area is 10 minus 1 sar, minus a 4^{th} part," which translates to $\frac{35}{4}$ sar. These expressions made it difficult to convert between different units of measurement. Thus, after sexagesimal fractions were invented around 2050 B.C.E. it quickly took over as the "mediator" between units of measurement, allowing for easier comparisons (Robson 2008, p. 77-78). For instance, the aforementioned " $\frac{1}{3}$ mina 1 shekel and a 4th part of washed silver" would thus be simplified to [21, 15] shekels. The divisibility of the number 60 has made conversions to and from unit fractions whose denominators are factors of 60 straightforward.

The introduction of sexagesimal fractions also made computations much simpler. For example, adding one-half with one-third would yield five-sixths, which prior to 2050 B.C.E. would require a special symbol as was the case in ancient Egypt (Robson 2008, p. 77). The exact algorithm an ancient Mesopotamian would have used to compute that expression is unknown, but it could hardly have been simpler than adding 30 with 20, which yields 50, the expression for five-sixths in sexagesimal notation. Division is also greatly simplified using sexagesimal notations. To divide by a number, the ancient Babylonians simply had to multiply by its reciprocal (Joseph 2010, p. 142). For instance, to divide by 5 would be to multiply by 12 (and to adjust the "decimal point" accordingly, though decimal points do not exist in ancient Mesopotamia). However, this shortcut only works for numbers whose reciprocal is a nice sexagesimal fraction. Hence the divisibility of the number base becomes critical: a reciprocal table found on tablet Plimpton 322 featured 25 distinct pairs of "nice" reciprocals for numbers 1 through 59. (Robson 2002, p. 21) With a reciprocal table as such, it was thus possible to divide large numbers with ease using sexagesimal numbers.

Lastly, the ancient Mesopotamian algorithm for solving quadratic equations relies on the use of reciprocals, and by extension, the divisibility of the base 60. While the standard form of quadratic equations in the modern days is $x^2 + bx + c = 0$, in ancient Babylon it was $x - \frac{1}{x} =$

d.[2] They have found an algorithm for solving these "igi-igibi" problems, which were commonly used as practice "puzzles" for students—"find half of *d*, square it, add 1, take the square root, and then add and subtract half of *d*." (Buck, p. 12) Even if taking squares and square roots was possible using unit fractions, it was certainly simpler in sexagesimal fractions.

The peculiarity of the sexagesimal system has fascinated scholars since at least the 4th century C.E. While there are numerous hypotheses on the puzzling choice of 60 as a number base, none had been conclusive. Theon of Alexandria argued that the divisibility of the number 60 was the primary reason it was used as the base. Thureau-Dangin misinterpreted some key evidence and was led to believe that it was the compromise between a decimal and a trinary number system. Powell insisted that the origin of the base 60 was buried in the philology of the ancient Sumerian language itself, though he had failed to unearth such evidence. Others have looked to astronomy, metrology, and geometry for possible explanations—mostly in vain. This paper has, through refuting competing hypotheses and scrutinizing historical evidence, arrived at a conclusion similar to that of Theon's: that the divisibility of the number 60— its ability to allow for simpler fractions, calculations, and the extraction of square roots— was the primary reason the number 60 was adopted as the base of the ancient Mesopotamian numeration system.

While the divisibility theory on the origin of the sexagesimal system appeared to be the most probable, it nevertheless raised a few important questions. Perhaps the most important question was why stop at 60—why not adopt a base-420 system to allow for greater divisibility?

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[2] Quadratic equations in Ancient Babylon were, of course, written using complete sentences instead of mathematical symbols.

And at what point does the burden of using a large base outweigh the marginal benefit of an additional divisor it may have? Other important questions concerned its legacy. Why did the ancient Greeks, who were aware of the sexagesimal number system (hence their partial adoption of sexagesimal fractions), not use them to express whole numbers? How and why did the need for a divisible base and a fraction-friendly number system subside? These questions, when adequately answered, could not only further solidify the divisibility theory, but also grant us a greater understanding of our Mesopotamian ancestors' mathematical wisdom.

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