# Sangaku: The Mathematical Art of the Edo Period

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The Edo period in Japan, also known as the Tokugawa period, lasted from 1603 to 1868 [2, p. 2]. The period was named after the Tokugawa shogunate that governed Japan from Edo Castle in the eastern city of Edo (modern-day Tokyo) [2, p. 3]. A shogunate was a form of military government in Japan that existed from the 12th to the 19th century. It was led by a shogun, who was essentially the military ruler of Japan, while the emperor was a ceremonial figurehead. The shogunate system decentralized power, with regional warlords, or daimyo, holding considerable authority over their domains. In the late 1630s, the Tokugawa shogunate began implementing a foreign policy known as Sakoku [10, p. xv] (鎖国, literally “Locked Country”), which remained in place until 1853, when Japan’s era of seclusion was put to an end by Commodore Matthew Perry [7, p. 175]. While not a policy of absolute isolation (Japan still participated in trade with nations such as China and Korea [2, p. 5]), the people of Japan were largely isolated from the rest of the world. However, this isolation did not stifle cultural and educational development. Amidst the peace and economic prosperity of the Edo period, Japanese art, culture, and education flourished. A remarkable example of this cultural and educational efflorescence is the phenomenon of sangaku, intricate mathematical tablets that exemplify the intersection of Japan's rich cultural heritage and its commitment to intellectual growth.

Sangaku (算額, literally “Calculation Tablet”) are wooden tablets, on which are colorful paintings of intricate geometry puzzles, along with inscriptions in Kanbun, “which used Chinese characters and essentially Chinese grammar, but included diacritical marks to indicate Japanese meaning. Kanbun played a role similar to Latin in the West and its use on sangaku would indicate that whoever set down the problems was highly educated.”[2, p. 9]. However, these tablets were not mere academic tools, but also served as a religious practice in Japan. Sangaku tablets were brought to Shinto shrines and Buddhist temples to be left as offerings to the gods, a tradition that has its roots in Japan's native religion, Shintoism, where wooden tablets depicting horses were often left at shrines as offerings [2, p. 8]. It was not just mathematicians creating these sangaku tablets, "the inscriptions on the tablets make clear that whole classes of students, children, and occasionally women dedicated sangaku."[2, pp. 9-10].


A sangaku solved by an 11 year old boy. Hisashi Okamoto gives us the translation:

"Suppose the fan in the figure is 1/3 of the complete circle. As is indicated in the figure, one east circle, two west circles, two south circles and two north circles are constructed. Suppose the diameter of the south circle is given. What, then, is the diameter of the north circle?

Answer:

Multiply the diameter of the south circle by ( sqrt(3072) + 62 )/193.

You have the diameter of the north circle.

Meiji 6th ( 1873 ), December

Kinjiro Takasaka 11 years old, a student of Master Shoryu Yamazaki."

Hisashi Okamoto's comment: "There is no indication to which circles are east, which are west, ... Judging from the answer, the red one is north and the white one is south." [8]

Sangaku problems are known for their concise presentation, often featuring only the problem statement, the solution, and occasionally a formula, omitting the detailed derivation. To solve these problems, Japanese mathematicians, known as wasanka, would apply traditional mathematical methods, or wasan, to arrive at the solution. Their solutions are characterized by a blend of elegance and simplicity. An example of such a solution is shown below [4, p. 3]:

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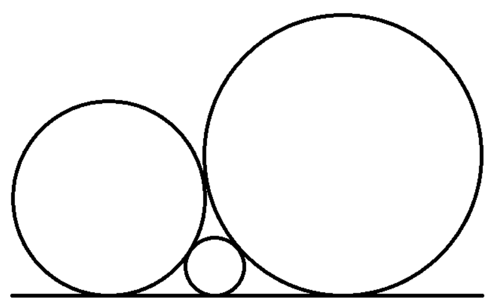
## Problem I: The Three Tangent Circles Problem



*Given three circles tangent to each other and to a straight line, express the radius of the middle circle via the radii of the other two.*

First solution, utilizing a special case of Descartes' (1596-1650) theorem of kissing circles [6, p. 338]:

This problem can be solved trivially by using Descartes' theorem of kissing circles: For four circles that are tangent to each other at six distinct points, with radii $r_i$ and curvatures $k_i = \frac{1}{r_i}$ for $i = 1, 2, 3, 4$:

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2).$$

Descartes stated this theorem in 1643 without proof in his letters to Princess Elisabeth of Bohemia.

Thinking of the straight line as a circle of infinite radius, and thus no curvature, we set $k_3 = 0$:

$$(k_1 + k_2 + k_4)^2 = 2(k_1^2 + k_2^2 + k_4^2)$$

We solve for $k_4$, which we define as the curvature of the middle circle:

$$k_1^2 + k_1k_2 + k_1k_4 + k_2k_1 + k_2^2 + k_2k_4 + k_4k_1 + k_4k_2 + k_4^2$$

$$= 2k_1^2 + 2k_2^2 + 2k_4^2$$

$$2k_1k_2 + 2k_1k_4 + 2k_2k_4 = k_1^2 + k_2^2 + k_4^2$$

Move all terms to the right side and write in terms of $k_4$:

$$k_4^2 - 2(k_1 + k_2)k_4 + k_1^2 + k_2^2 - 2k_1k_2 = 0$$

Using the quadratic formula on this quadratic in $k_4$,

$$k_4 = \frac{1}{2}[2(k_1 + k_2) \pm \sqrt{4(k_1 + k_2)^2 - 4(k_1^2 + k_2^2 - 2k_1k_2)}]$$

$$= k_1 + k_2 \pm \sqrt{k_1^2 + 2k_1k_2 + k_2^2 - k_1^2 - k_2^2 + 2k_1k_2}$$

$$k_4 = k_1 + k_2 \pm 2\sqrt{k_1k_2}$$

Recall that, by definition, $k_i = \frac{1}{r_i}$:

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} \pm 2\sqrt{\frac{1}{r_1r_2}}$$

$$= \frac{r_1 + r_2 \pm 2\sqrt{r_1r_2}}{r_1r_2}$$

$$= \frac{(\sqrt{r_1})^2 + (\sqrt{r_2})^2 \pm 2\sqrt{r_1r_2}}{r_1r_2}$$

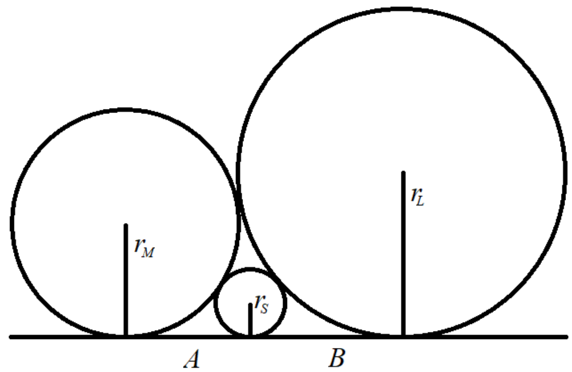
$$= \frac{(\sqrt{r_1} \pm \sqrt{r_2})^2}{r_1r_2}.$$

$r_4 = \frac{r_1r_2}{(\sqrt{r_1} + \sqrt{r_2})^2}$, the radius of the middle circle.

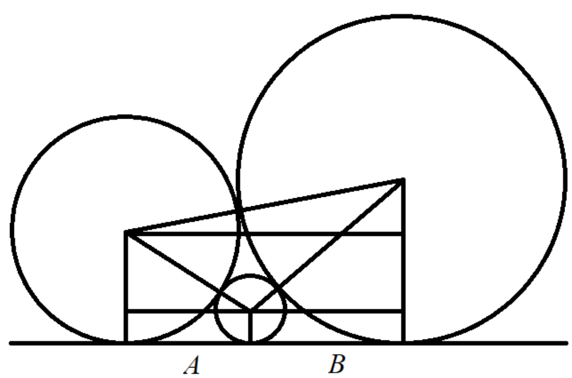
However, due to Japan's isolation, Descartes' theorem was unknown. Instead, Japanese mathematicians made use of another familiar theorem, the Pythagorean theorem, though it was known to the Japanese as the Gougu theorem of Chinese origin. We will now proceed to solve this problem using the Pythagorean theorem.

Second solution, based on the solution given by Dr. Alexander Bogomolny [1]:

We begin by connecting the center of each circle to the straight line beneath them, denoting the radius of the large circle as $r_L$, the radius of the mid-sized circle as $r_M$, the radius of the smallest circle as $r_S$, and denote the line segments lying between the intersection points as $A$ and $B$ respectively:

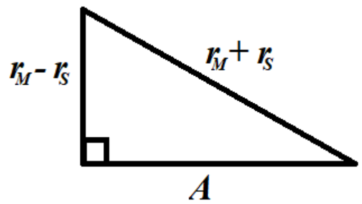



Then, we draw lines connecting the centers of each circle, as well as a horizontal line connecting the center of the mid-sized circle to the vertical line extending from the center of the large circle, and finally, add a line parallel to the previous, running through the center of the smallest triangle:

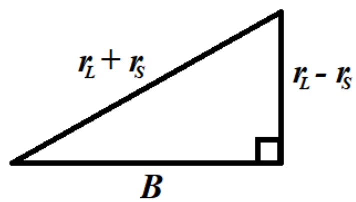



Notice we now have three right triangles,

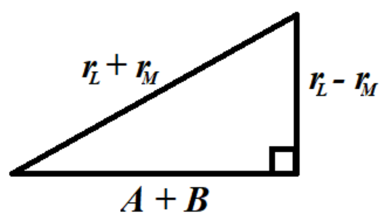
$$r_M - r_S,\ A,\ r_M + r_S$$

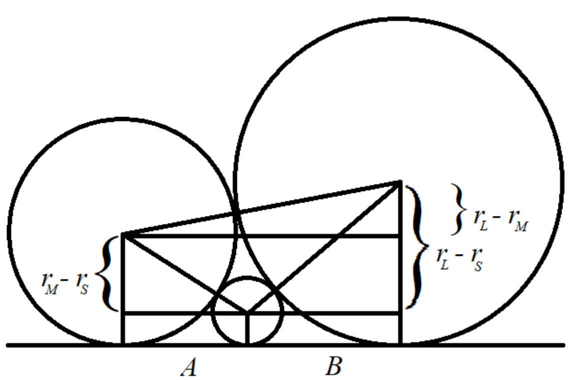



$$r_L - r_S,\ B,\ r_L + r_S$$




$$r_L - r_M,\ A + B,\ r_L + r_M$$




By the Pythagorean theorem applied three times:

First right triangle:

$$(r_M - r_S)^2 + A^2 = (r_M + r_S)^2$$

$$A^2 = 4r_M r_S$$

$$A = 2\sqrt{r_M r_S}$$

Second right triangle:

$$(r_L - r_S)^2 + B^2 = (r_L + r_S)^2$$

$$B^2 = 4r_L r_S$$

$$B = 2\sqrt{r_L r_S}$$

Third right triangle:

$$(r_L - r_M)^2 + (A + B)^2 = (r_L + r_M)^2$$

$$(A + B)^2 = 4r_M r_L$$

$$A + B = 2\sqrt{r_M r_L}$$

Now form the following equation:

$$A + B = A + B$$

$$2\sqrt{r_M r_S} + 2\sqrt{r_L r_S} = 2\sqrt{r_M r_L}$$

$$\sqrt{r_M r_S} + \sqrt{r_L r_S} = \sqrt{r_M r_L}$$

$$\sqrt{r_S}(\sqrt{r_M} + \sqrt{r_L}) = \sqrt{r_M r_L}$$

$$\sqrt{r_S} = \frac{\sqrt{r_M r_L}}{\sqrt{r_M}+\sqrt{r_L}}$$

$$r_S = \frac{r_M r_L}{\left(\sqrt{r_M}+\sqrt{r_L}\right)^2}$$

Notice that this solution is identical to the solution found utilizing Descartes' theorem.

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An 1824 sangaku presented at a Shinto shrine in Gunma prefecture, depicting the Three Tangent Circles Problem in the middle [5]. The text is written in Kanbun, but the Takasaki-shi Board of Education (高崎市教育委員会) translated the three problems into modern Japanese [5]. My translation, based on their translation of the Three Tangent Circles problem is as follows:

> "In the diagram, the diameter of the large circle is 3 shaku 6 sun [about 43 inches].
> When the diameter of the between [in size] circle is 9 sun [about 10.7 inches],
> how much is the diameter of the small circle?"

Possibly the most infamous sangaku problem was the Gion shrine problem, first proposed on a 1749 sangaku at Kyoto's Gion (now Yasaka) shrine by Tsuda Nobuhisa (1718-1787)

[2, p. 250]. On the sangaku, Tsuda demonstrated a solution in the form of a 1024 degree equation, which was potentially intended to be solved using sangi (calculation rods) [3, p. 4]. A brief description of Tsuda's solution was recorded by Saito Mitsukuni (c.1750) on a sangaku that he hung at the Zenkoji Temple in Nagano Prefecture in 1815 [3, p. 4], and was subsequently transcribed into the travel journal of itinerant mathematician Yamaguchi Kanzan (d.1850) [3, p. 4]. A solution in the form of a 46 degree equation was allegedly found by a mathematician named Nakata, however details are scarce [9, p. 198]. In 1774, Ajima Naonobu (1732-1798), also known as Ajima Chokuyen [9, p. 195], a student of the Seki school of Edo, was able to famously deduce a solution in the form of a 10th degree equation [9, p. 198], which is contained in a manuscript titled Kyoto Gion-gaku toujutsu. Hirayama and Matsuoka published an annotated version of the original manuscript in 1966. Following this, Naoi Isao performed a modern mathematical analysis of Ajima's solution, and attempted to fully reconstruct Ajima's solution based on the original manuscript [3, p. 4]. Due to the length and complexity of Naoi's recreation of Ajima's solution, what follows will be a summary of Ajima's result, as it was provided by Hidetoshi Fukagawa and David Clark, along with specific portions of Naoi's result for clarification.

All comments contained in [square brackets] are mine.

All pictures, other than the first two, are mine. [3, pp.4-7]

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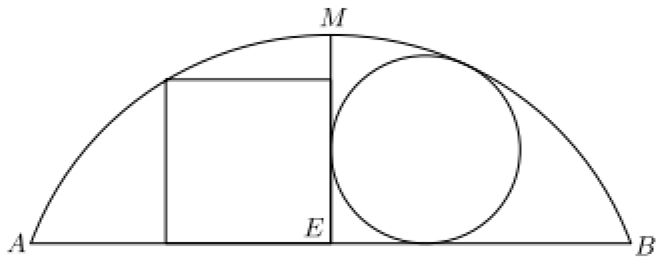
## Problem II: The Gion Shrine Problem (Summarized Result)

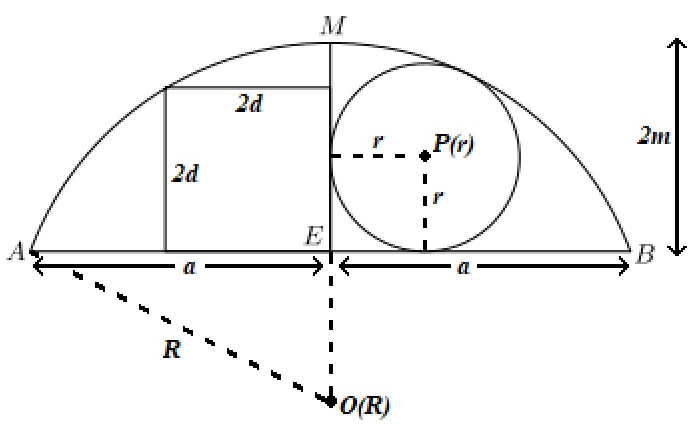


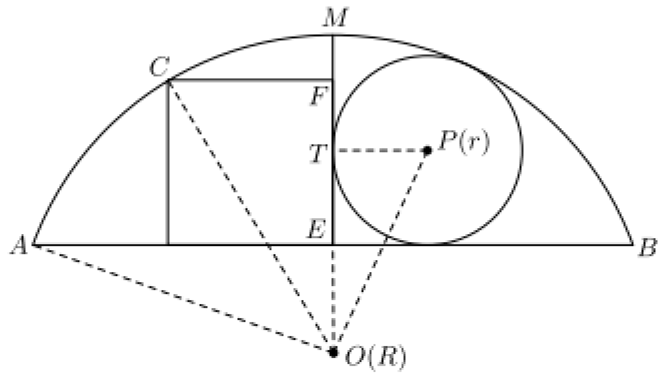

*Consider the circular segment of radius R shown in the figure. A smaller circle of radius $r$ and a square of side $2d$ are arranged as in the figure, tangent at the segment's bisector, with $ME = 2m$ and $AB = 2a$. If $p = a + m + d + r$ and $q = \frac{m}{a} + \frac{r}{m} + \frac{d}{r}$ are given, find $a$ in terms of $p$ and $q$.*

[From Naoi's result [3, pp. 6-7]:]

[$P(r) =$ center of the small circle of radius $r$,

$O(R) =$ center of the circular segment of radius $R$.]

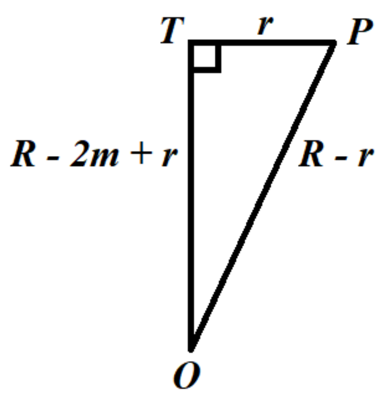





From the above figure, in [right] triangle $OPT$ we have [by the Pythagorean theorem]:

$$OP^2 = OT^2 + TP^2$$

[Note that since the distance dist($O$, through $P$ to the circular segment) $= R$ and dist($P$, circular segment) $= r$, we have $OP = R - r$, $OT = OE + ET = (MO - ME) + r = R - 2m + r$, and $TP = r$.]




which implies:

$$(R - r)^2 = (R - 2m + r)^2 + r^2$$

or

$$4R(m - r) = (2m - r)^2 \qquad (1)$$

$$[(R - r)^2 = (R - 2m + r)^2 + r^2$$

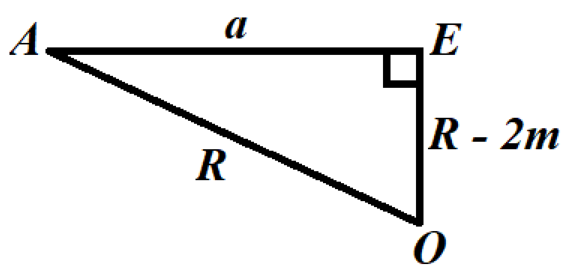
$$R^2 - 2Rr + r^2 = R^2 + 4m^2 + 2Rr - 4Rm - 4mr + 2r^2$$

$$4Rm - 4Rr = 4m^2 - 4mr + r^2$$

$$4R(m - r) = (2m - r)^2]$$

In [right] triangle $OAE$ we have [by the Pythagorean theorem]:

$$OA^2 = OE^2 + EA^2$$




which implies:

$$R^2 = (R - 2m)^2 + a^2$$

or

$$4Rm = a^2 + 4m^2 \qquad (2)$$

$$[R^2 = (R - 2m)^2 + a^2$$

$$R^2 = R^2 - 4Rm + 4m^2 + a^2$$

$$4Rm = a^2 + 4m^2]$$

And in [right] triangle $OCF$ we have [by the Pythagorean theorem]:

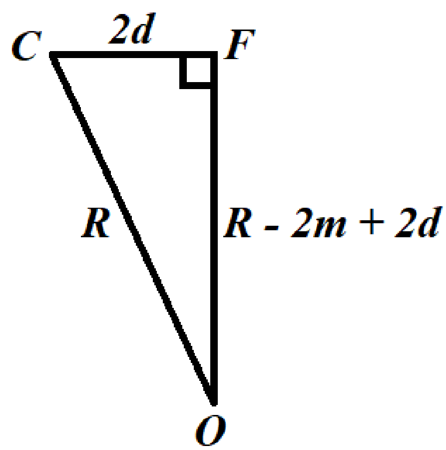
$$OC^2 = OF^2 + FC^2$$

[Note that $OC = r$, $OF = MO - MF$

$= R - (ME - FE)$

$= R - (2m - 2d)$

$= R - 2m + 2d$, and $FC = 2d$.]




which implies:

$$R^2 = (R - 2m + 2d)^2 + 4d^2$$

or

$$R(m - d) = (m - d)^2 + d^2 \qquad (3)$$

$$[R^2 = (R - 2m + 2d)^2 + 4d^2$$

$$R^2 = R^2 - 4Rm + 4Rd + 4m^2 - 8md + 8d^2$$

$$4Rm - 4Rd = 4m^2 - 8md + 8d^2$$

$$R(m - d) = (m - d)^2 + d^2]$$

Canceling [eliminating] $R$ from (1) and (2) gives us:

$$m(a + r)(a - r) = a^2r$$

[ Solve for $4R$ in (2):

$$4Rm = a^2 + 4m^2$$

$$4R = \frac{a^2+4m^2}{m}$$

Substitute $\frac{a^2+4m^2}{m}$ for $4R$ in (1):

$$4R(m - r) = (2m - r)^2$$

$$\left(\frac{a^2+4m^2}{m}\right)(m - r) = (2m - r)^2$$

$$(a^2 + 4m^2)(m - r) = m(2m - r)^2$$

$$a^2m - a^2r + 4m^3 - 4m^2r = 4m^3 - 4m^2 + mr^2$$

$$a^2m - mr^2 = a^2r$$

$$m(a^2 - r^2) = a^2r$$

$$m(a + r)(a - r) = a^2 r]$$

Setting $s = p - a$, we have that:

$$s = m + d + r$$

[From the statement of the problem,

$$p = a + m + d + r$$

$$p - a = m + d + r$$

$$s = m + d + r]$$

$$amrq = m^2 r + ar^2 + amd$$

[From the statement of the problem,

$$q = \frac{m}{a} + \frac{r}{m} + \frac{d}{r}$$

$$(amr)q = (\frac{m}{a} + \frac{r}{m} + \frac{d}{r})(amr)$$

$$amrq = \frac{m}{a}(amr) + \frac{r}{m}(amr) + \frac{d}{r}(amr)$$

$$amrq = m(mr) + r(ar) + d(am)$$

$$amrq = m^2 r + ar^2 + amd]$$

With that as background, we can now proceed to Fukagawa and Clark's summary [3, pp. 4-6]:

After identifying the Pythagorean relationships for three right triangles, Ajima conducts a series of algebraic manipulations, during which he defines the following variables in terms of $p$, $q$, and $a$:

$$s = p - a$$

$$b_2 = a(3 + q)$$

$$b_3 = s + aq$$

$$c_0 = 8s^2 + 34as - a^2 - 9a^2q$$

$$c_1 = -32s^2 - 55as + 17asq + 27a^2 + 36a^2q$$

$$c_2 = -15s^2 + 22as + 17asq + 18a^2 + 14a^2q$$

$$c_3 = 10a + 12aq - 11s$$

$$d_1 = a(c_0 + c_1)$$

$$d_2 = b_2c_0 + c_2a$$

$$d_3 = b_3c_0 - c_3a^2$$

$$d_4 = b_3c_0 + 3c_2a - c_3a^2$$

$$d_5 = 3c_3a + c_0$$

$$d_6 = b_2c_2 - b_3c_1 - d_5a$$

$$d_7 = c_1 - b_2c_3$$

$$d_8 = c_2 - b_3c_3$$

Ajima then ultimately arrives at the following equation, with a degree of 10 in $a$:

$$(c_3 d_1 a + c_0^2 a)(d_5 d_6 + d_4 d_7) + (c_3^2 a^2 + c_0 d_8)(d_3 d_4 - d_2 d_6)$$

$$+ (c_0 c_2 a^2 + d_1 d_7 a)(c_2 d_5 + c_1 d_7)$$

$$= (c_3 d_7 a^2 + c_2 d_8 a)(c_2 d_2 - c_1 d_3) + (c_2 c_3 a^2 - c_0 d_7 a)(d_3 d_5 + d_2 d_7)$$

$$+ (d_1 d_8 - c_0 c_3 a^2)(c_1 d_6 - c_2 d_4).$$

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Ajima Naonobu lived before Charles Lutwidge Dodgson (1832–1898), better known as Lewis Carroll. This equation seems to have been derived from a determinant calculation showing similarities to Dodgson's condensation method for determinant computation, which he described in his work from 1867 [3, p. 6]. And due to Japan's closed-country policy at the time, it is unlikely that Ajima was aware of Pierre-Simon Laplace's (1749–1827) similar results [3, p. 6]. Although Ajima is credited with using this method, it is possible that other Japanese mathematicians, such as Kurushima Yoshihiro (d.1757), had independently discovered it earlier. Hirayama and Matsuoka (1966) mention Kurushima as a potential earlier innovator before Ajima.

It is worth noting, however, that Ajima's contributions to wasan far exceed his solution to the Gion shrine problem. Ajima made great contributions to yenri (円理, literally "Circle Logic"), a calculus-like system developed in the Edo period, in that he took equal divisions of the chord as opposed to the arc, which simplified the work significantly [9, p. 201]. Ajima was able to find the area of a segment of a circle bound by two parallel lines and in the equal arcs occupied by them, that is, inscribed area $ABCD$ in the following figure. Chord $c$ of the arc is divided into $n$ equal parts [9, pp. 201-203].

All comments contained in [square brackets] are mine.

All pictures, other than the first one, are mine.

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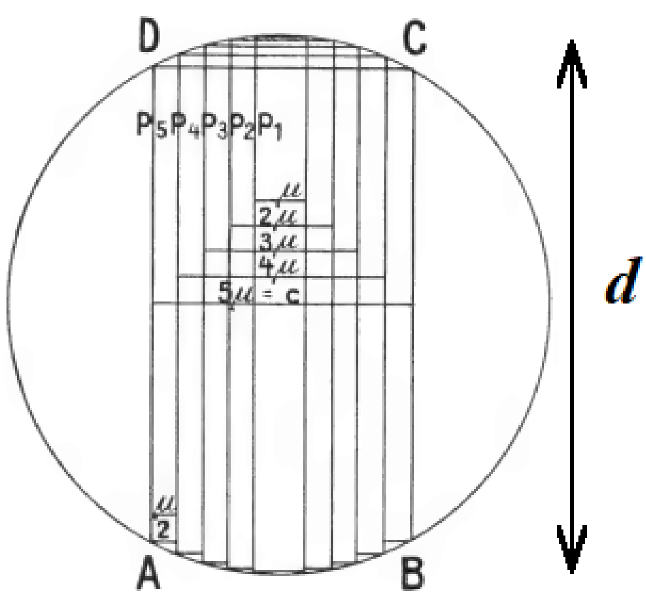
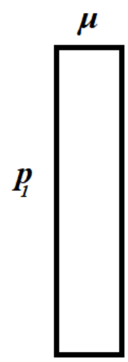
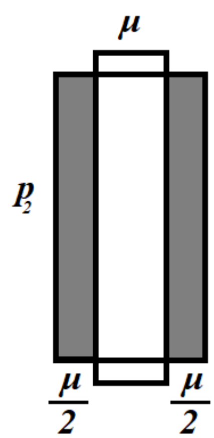


[Where




total area $= p_1\mu$,

then




$$\text{total area} = p_1\mu + p_2\left(2 \cdot \frac{\mu}{2}\right)$$

$$= p_1\mu + p_2\mu,$$

and so on,

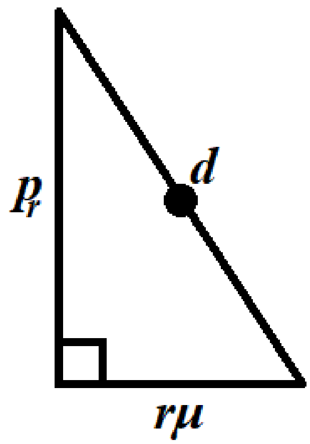
$$\text{until total area} = \sum_{r=1}^{n} p_r\mu.]$$

From the figure, it is apparent that [in the $n^{th}$ case] :

$$\mu = \frac{c}{n}$$

[and by the Pythagorean theorem,]

$$p_r^{\ 2} = d^2 - (r\mu)^2$$




where $p_r$ is [length of the] the $r^{th}$ parallel from the diameter $d$.

Ajima then expands $p_r$, without explaining his process (evidently that of the tetsujutsu [Tetsujutsu Sankai (1722), a paper written by Takebe Katahiro (1664-1739), containing a method to calculate a binomial series expansion] ), and obtains:

$$p_r = d[1 - \frac{1}{2}(\frac{r\mu}{d})^2 - \frac{1}{4\cdot 2}(\frac{r\mu}{d})^4 - \frac{3}{4\cdot 3\cdot 2}(\frac{r\mu}{d})^6 - \cdots]$$

[Note that the above series contains an error. The fourth term is missing a factor of $\frac{1}{2}$.

$$p_r^{\ 2} = d^2 - (r\mu)^2$$

$$= d^2\left(1 - (\frac{r\mu}{d})^2\right).$$

$$p_r = d(1 - (\frac{r\mu}{d})^2)^{\frac{1}{2}}$$

Binomial expansion:

$$= d[1 - \frac{1}{2}(\frac{r\mu}{d})^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(\frac{r\mu}{d})^4 - \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(\frac{r\mu}{d})^6 - \cdots]\,]$$

$$= d[1 - \frac{1}{2}(\frac{r\mu}{d})^2 - \frac{1}{8}(\frac{r\mu}{d})^4 - \frac{3}{48}(\frac{r\mu}{d})^6 - \frac{15}{384}(\frac{r\mu}{d})^8 - \cdots]$$

Summing for $r = 1, 2, 3, \ldots, n$, and multiplying by μ we have the following series:

$$\sum_{r=1}^{n} p_r\mu = d\mu[n - \frac{1}{2}(\frac{\mu}{d})^2\Sigma r^2 - \frac{1}{8}(\frac{\mu}{d})^4\Sigma r^4 - \frac{3}{48}(\frac{\mu}{d})^6\Sigma r^6 - \frac{15}{384}(\frac{\mu}{d})^8\Sigma r^8 - \cdots]$$

$$= d\mu[n - \frac{1}{2}\cdot\frac{1}{6}(\frac{\mu}{d})^2(2n^3 + 3n^2 + n)$$

$$- \frac{1}{8}\cdot\frac{1}{30}(\frac{\mu}{d})^4(6n^5 + 15n^4 + 10n^3 - n)$$

$$- \frac{3}{48}\cdot\frac{1}{42}(\frac{\mu}{d})^6(6n^7 + 21n^6 + 21n^5 - 7n^3 + n)$$

$$- \frac{15}{384}\cdot\frac{1}{40}(\frac{\mu}{d})^8(10n^9 + 45^8 + 60n^7 - 42n^5 + 20n^3 - 3n) - \ldots]$$

[For instance, note that

$$\Sigma r^2 = \sum_{r=1}^{n} r^2 = 1^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(2n^3 + 3n^2 + n).]$$

Now substituting for μ its value $\frac{c}{n}$, and letting $n$ approach ∞, all terms with $n$ in the denominator approach 0 as a limit, and the limit to which the required [inscribed] area approaches is:

$$\text{area} = d[c - \frac{1}{6}\cdot\frac{c^3}{d^2} - \frac{1}{40}\cdot\frac{c^5}{d^4} - \frac{3}{336}\cdot\frac{c^7}{d^6} - \frac{15}{3456}\cdot\frac{c^9}{d^8}$$

$$- \frac{105}{43240}\cdot\frac{c^{11}}{d^{10}} - \frac{945}{599040}\cdot\frac{c^{13}}{d^{12}} - \cdots]$$

Ajima later developed a method for computing volumes by double integration. The method was developed to solve the problem of finding the volume cut from a cylinder by another cylinder. His result is given by his pupil Kusaka Sei (1764-1839) in his manuscript, the Fukyo Sampo (1799), with no explanation [9, p. 204]. It is even reported in the literature that Aijima had studied the spiral of Archimedes, although under a different name.

Over time, Japan was exposed to some Western mathematics through China and the Dutch, including concepts like logarithms and the ellipsograph. However, traditional Japanese mathematics persisted until the Meiji government's push for modernization in the late 19th century. In 1872, the Meiji leaders mandated the teaching of Western mathematics in schools, leading to the decline of wasan despite some resistance [2, p. 24]. While the loss of wasan may not be significant from a purely mathematical perspective, its demise represents a loss of aesthetic beauty in the problems and methods it offered. Similarly, the practice of hanging sangaku also declined after the fall of the Tokugawa shogunate, with only a fraction of the original tablets surviving today. While some enthusiasts continued to create and display sangaku into the 20th century, particularly in the Toyama prefecture where tablets were discovered as late as 2005, the tradition faded over time [2, p. 9]. Despite their eventual decline, the legacy of wasan and sangaku remains as a testament to Japan's rich mathematical heritage and aesthetic appreciation for mathematical beauty.

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