The Assumptive Attitudes of Western Scholars

Regarding the Contributions of Mathematics from India:

Assessing *yukti-s* from the *Yuktibhāṣā* of Jyeṣṭhadeva

Rye Ledford

University of Missouri – Kansas City

India experienced three distinct periods of mathematical and astronomical growth [4, p. 1]. The last period, before the modern era, saw a rise to the beginnings of calculus and an exploration of mathematical analysis. Mādhava of Sangamagrāma (c. 1340-1425) founded a school in this last period, the Nila school [6, p. 481]. Approximately two-hundred years later a student of the school, Jyeṣṭhadeva, contributed an incredible mathematics and astronomical treatise on a palm-leaf [12, p. xxxiv]. Both mathematicians and the many others who were part of the Nila school made important advancements in mathematics and astronomy, but India has been historically ignored by western scholars who allowed assumptive attitudes regarding their contributions to form their opinions [5, p. 311]. The history of the Nila school and its student, Jyeṣṭhadeva, provides grounds that the attitudes regarding Indian advancements were unfounded.

The Nila River in western India, known today as Bharathappuzha, and its shores have facilitated intellectual growth for thousands of years [6, pp. 385-507]. In the 13th century the area surrounding the Nila River came under the protection and patronage of the kings of the Zamorin dynasty which brought peace to the area [4, p. 257]. With this peace and prosperity, the traditional pupil-teacher relationship thrived, and a school of mathematics was born along its shores during the 14th century. It has been given the name the Kerala school in some texts, which ignores its geographic location and the importance of the river itself. Kerala is a state that stretches the length of the western shores of the Indian peninsula, but Nila River is situated within a small portion of the Kerala state. Many of the mathematicians and astronomers that are considered part of this school lived and worked within the illams or manas (traditional compounds or estates) along the river that extends into mainland India. Thus, it is more appropriately referenced as the Nila school in many modern works. It will be referenced as the Nila school below. [4, p. 259]

The birth of the Nila school began with Mādhava of Sangamagrāma (c. 1340-1425). His life is shrouded in obscurity and his only existing works are regarding astronomy [5, p. 380]. He is credited with providing the groundwork for the Gregory (1638-1675) series for arctangent, the Leibniz (1646-1716) series for π, and the infinite series for circular and trigonometric functions during his life [5, p. 419]. Though much of his work is presumed lost or never written, his pupils provided a wide array of works and treatises that expanded on his discoveries. These concepts were presented hundreds of years before they appeared in Europe, yet India's mathematicians and their corresponding works have been largely ignored. Western mathematicians have brushed them aside as cheap copies of Grecian work, that lack formal proof to substantiate them. American Mathematician C.B. Boyer writes: [1, p. 194]

> "Indian Mathematics is frequently described as 'intuitive,' in contrast to the stern rationalism of Greek geometry."

This attitude is not rare. Morris Kline, another great American mathematician, also held a similar view. [8, p. 190]

> "As our survey indicates, the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than the deductive patterns. Their name for mathematics was *gaṇita,* which means "the science of calculation." There is much good procedure and technical facility, but no evidence that they considered proof at all. They had rules, but apparently no logical scruples. Moreover, no general methods or new viewpoints were arrived at in any area of mathematics."

Both Kline and Boyer wrote their respective works, *Mathematical Thought from Ancient to Modern Times* in 1972 and *The History of Mathematics* in 1968. A source book, completed in

1985 listed 285 published Indian works in mathematics and astronomy completed between the 12th century and the 19th century [13]. It is estimated that 100,000 ancient Sanskrit manuscripts exist regarding mathematics and astronomy with only 95 of these having been translated into European languages [9, p.1]. Thus, the pool of works available to either scholar at the time of their respective publications was small, diminishing their ability to take a full appreciation of the mathematics from India. Efforts on the part of western scholars have been minimal until the recent millennium in which reliable translations and trustworthy source books have been produced. In 2008, Srinivas wrote: [15, p. 214]

> "A major reason for our lack of comprehension, not merely of the Indian notion of proof, but also of the entire methodology of Indian mathematics, is the scant attention paid to the source-works so far."

There are two Sanskrit words used for mathematical arguments. The more popular word, *upapatti*, translates into English as "proving right" or "resulting" and is often used when referring to what is considered an informal style western proof. The earliest mathematical argument that is considered an *upapatti* is within the *Bhasya* of Govindasvāmin (c. 800) which is a commentary on Bhāskara I's (c. 600-680) *Mahābhāskarīya* [14, p. 215]. Bhāskara II (1114-1185) wrote: [14, pp. 228-229]

> "Without the knowledge of the *upapatti-s*, by merely mastering the *gaṇita* (calculational procedures) described here, … a mathematician will not have any value in the scholarly assemblies; without the *upapatti-s* he himself will not be free of doubt (niḥsaṃśaya)."

Upapatti-s have been observed by modern scholars since C.M. Whish (1795-1833) when his article, published in 1834 post-mortem, titled 'On the Hindu Quadrature of the Circle, and the infinite Series of the proportion of the circumference to the diameter exhibited in the four

S'astras, the *Trantra Sangraham*, *Yucti Bhasha*, *Carana Padhiti*, and *Sadratnamala'* included translations of portions of the mentioned works and defended their use of proofs [16]. His translations and subsequent discoveries of the mentioned works went relatively unnoticed and his promise to publish more on the subject was broken as he passed shortly before the article above was published.

The word y*ukti* is a more precise term, which conveys "reasoning", "an argument", or "correctness of," and usually follows a formal proof. Kṛṣṇa Daivajña in a 16th century commentary wrote: [14, pp. 218-219]

> "How can we state without proof (*upapatti*) that twice the product of two quantities when added or subtracted from the sum of their squares is equal to the square of the sum or difference of those quantities?
>
> $[a^2 + b^2 + 2ab = (a + b)^2$ and $a^2 + b^2 - 2ab = (a - b)^2]$
>
> That it is seen to be so in a few instances is indeed of no consequence… as it is possible that one would come across contrary instances (v*yabhicāra*) also. Hence it is necessary that one would have to provide a proof (*yukti*) for the rule…"

The term *yukti* is becoming increasingly more common as more precise translations of expositions and commentaries are published. Divakaran wrote in 2018: [4, p. 404]

> "As if to advertise these newly recognized virtues, there is a subtle shift in the terminology as well, the old *upapatti* making way gradually for the term *yukti."*

A highly regarded treatise that contains *upapatti-s* and *yukti-s* is the recently translated *Yuktibhāṣā* (Rationales in Mathematical Astronomy). It was otherwise written in 1530 by Indian mathematician and astronomer Jyeṣṭhadeva [14, p. 218]. The modern English translation by Sarma (1919-2005) was published post-mortem in 2008 and is titled the *Ganita-Yukti-Bhāsā*

[12]. It is uncertain why Sarma chose this new name, but the treatise is known in both Sanskrit and Malayālam as the *Yuktibhāṣā* and will be referenced as such here [4, p. 295]. Extraordinarily little is known about Jyeṣṭhadeva's life, including the period in which he lived which is estimated as between 1475 and 1575 [10, p. 606] or 1500 and 1610 [12, p. xxi]. His greatest work, the *Yuktibhāṣā*, was incorrectly credited to a different author [12, pp. xxiv]. Evidence of the true author was on a palm-leaf manuscript in Malayālam. It contained sound evidence crediting Jyeṣṭhadeva as the author. This palm-leaf reads as a chronology and commentary of the lineage of the Nila school, and is housed at the Oriental Institute, Baroda, Ms. No. 9886 and gives the order of students after Vāṭaśśeri Parameśvara (lived between 1360 and 1460). Parameśvara's pupil and son was Vāṭaśśeri Dāmodara (fl. 1450) who in turn was the mentor to Nīlakaṇṭha Somayājī (lived between 1443 and 1544). Nīlakaṇṭha followed the teacher-pupil tradition and was the mentor to Jyeṣṭhadeva [5, p. 420]. The palm-leaf gives for each mathematician a mention of their works, in which it states that Jyeṣṭhadeva is the author of *Yuktibhāṣā* [12, p xxxvii]. The teacher to Parameśvara, not stated on the palm-leaf manuscript but known to us, was Mādhava of Sangamagrāma (c. 1340-1425), who founded the Nila school.

The claim in the *Yuktibhāṣā* is that the principal ideas are based on Mādhava's mathematics and an expansion of Nīlakaṇṭha's *Tantrasaṅgraha* (c. 1501), a work that revised the Aryabhatan model for planet interiors and provided a close approximation of the equation of the center. Better approximations would not appear until Johannes Kepler (1571-1630) [5, p. 421].

The *Yuktibhāṣā* is a remarkable work. Modern translations divide it into two sections, the first on mathematics and the second regarding astronomy [12]. There are seven chapters in the mathematical section of the *Yuktibhāṣā.* The focus here is on chapter six, titled Circle and Circumference. The sixth chapter begins with a simple but highly effective *yukti* of the so-called

Pythagorean Theorem, which was well known in India. This knowledge dates to Bhāskara I and his commentary on Aryabhaṭa's treatise, the Āryabhaṭīya (c. 629), which is the oldest known mathematical prose in Sanskrit [7, p. 10].

Translator notes are inside <angle brackets>. My commentary is inside [square brackets]. Any accompanying illustrations are mine.

6.1 $\textit{Bhujā}^2 + \textit{Koṭi}^2 = \textit{Karṇa}^2$ [12, pp. 45-46]

[Theorem]

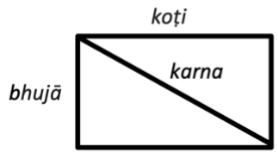
It is explained here <how>, in a rectangle, the sum of the squares of a side and of the height is equal to the square of the diagonal

> [Also referred to as the cut-and-move proof, similar to Nīlakaṇṭha Somayājī's, the mentor of Jyeṣṭhadeva, though his work has not been reliably translated. *Bhujā, koṭi,* and *karṇa* are common variables assigned when working with geometry. *Karṇa* often represents the hypotenuse of a triangle.]

[Theorem statement in symbols]

Now, the square of a length is the area of a square having <that length> as its side. In a square or in a rectangle, the diagonal < *karṇa*> is the straight line [line segment] drawn from one corner to its opposite corner through its centre.

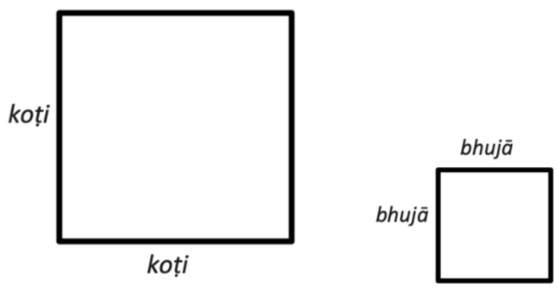
In a rectangle, the *koṭi* stretches lengthwise on two lateral [horizontal] sides. The two vertical-sides called *bhujā* will be shorter, as presumed here. [*bhujā* < *koṭi*].



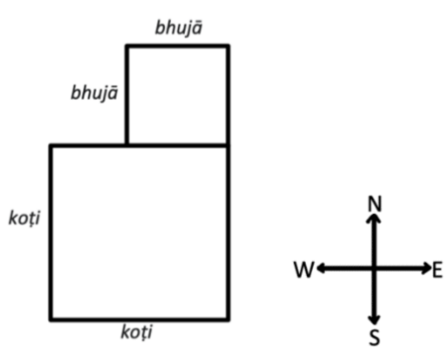
It is the diagonal [*karṇa*] of such a rectangle that is sought to be known.

[Proof]

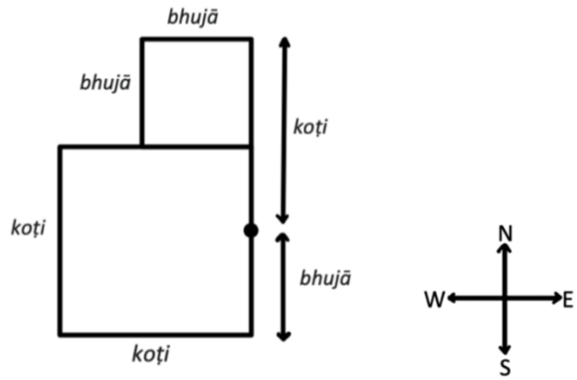
Now, draw a square <with its side> equal to the *koṭi* and another equal to the *bhujā*. Draw, in this manner, two squares.



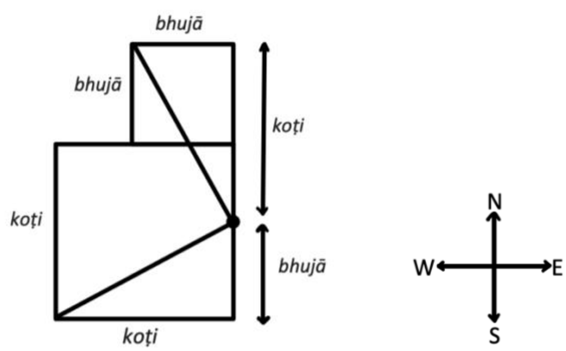
Let the *bhujā* -square [smaller square] be on the northern side and the *koṭi*-square [larger square] on the southern side in such a way that the eastern side of both the squares fall on the same line; and in such a manner that the southern side of the *bhujā* -square falls on the northern side of the *koṭi* -square. This <northern> side <of the *koṭi* -square> will be further extended in the western-side than the *bhujā* <since it is longer>.



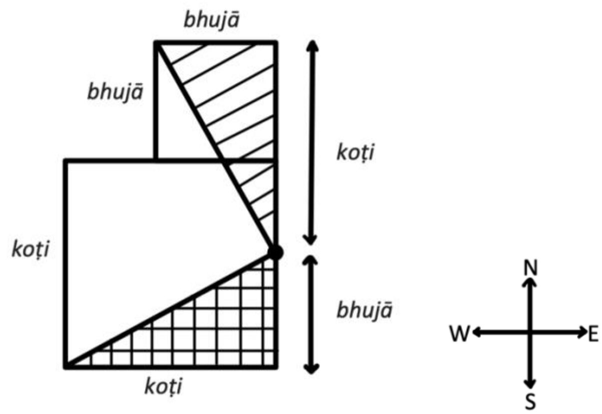
From the north-east corner of the *bhujā* square, measure southwards a length equal to the *koṭi* and mark the spot with a point. From this <point> the <remaining> line towards the south will be of the length of the *bhujā.*



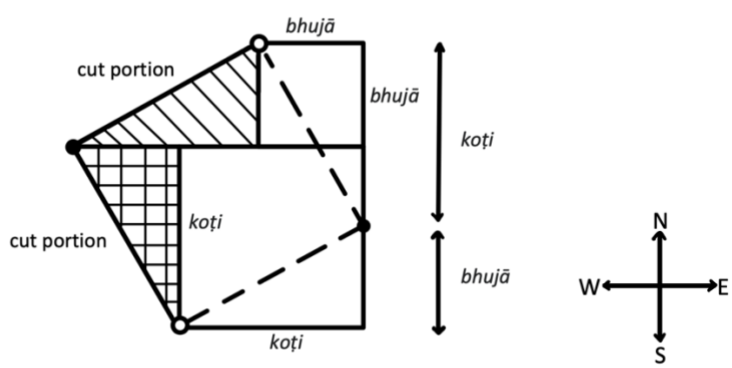
Then cut along the lines starting from this point towards the south-west corner of the *koṭi* -square and the north-west corner of the *bhujā* -square. Allow a little clinging at the two corners so that the cut portions do not fall away. [Jyeṣṭhadeva would have imagined this figure as constructed, from paper or fabric, thus he uses terms such as “cut”, “clinging”, and similar language below.]



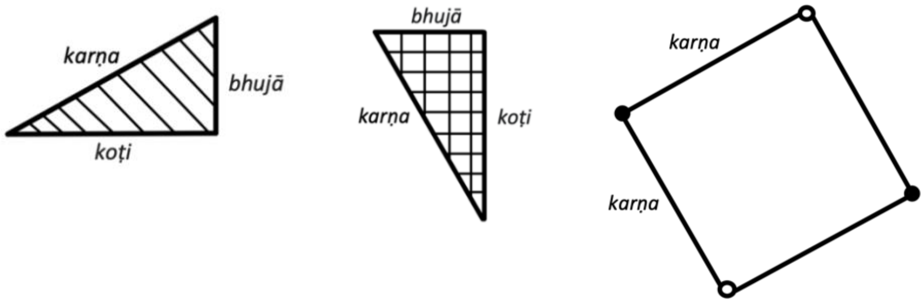
Now break off the two parts <i.e., the [congruent] triangles> from the marked point,



turn them round [to] alongside the two sides of the bigger <i.e., *koṭi*> square, so that the corners of the triangles, which met at that point earlier, now meet in the north-west direction, and join them so that the cut portions form the outer edges. [Imagine the northern right triangle rotated clockwise around the northwest corner of the *bhujā*-square and the southern right triangle rotated counterclockwise around southwest corner of the *koṭi*-square to their new positions.]



The figure formed thereby will be a square and the sides of this square will be equal to the *karṇa* associated with the <original> *bhujā* and *koṭi*.



Hence it is established [by the "cut-and-move proof"] that the sum of the squares of the *bhujā* and *koṭi* is equal to the square of the *karṇa* [$bhuj\bar{a}^2 + ko\d{t}i^2 = karna^2$] and it also follows that if the square of one of them is deducted from the square of the *karṇa*, the square of the other will be the result. This is to be understood in all cases.

[End of proof]

Note that a Euclidean geometry proof would require verification that the "figure formed thereby will be a square" by referencing Euclid I-32 that "In any triangle, … the three interior angles of the triangle are equal to two right angles." But mathematics in India had a different point of view. Srinivas summarizes: [14, pp. 231-232]

> "1. The Indian mathematicians are clear that results in mathematics… cannot be accepted as valid unless they are supported by *yukti* or *upapatti*. It is not enough that one has merely observed the validity of a result in a large number of instances…
>
> 4. In the Indian mathematical tradition the *upapatti-s* mainly serve to remove doubts and obtain consent for the result among the community of mathematicians.
>
> 7. …There was apparently no attempt to present the *upapatti-s* as a part of the deductive axiomatic system… there was no attempt at formalization of mathematics."

This proof illustrates that the use of *yukti-s* (formal proofs) existed in India since at least 1530 and that mathematicians of the time were aware of geometry and its many uses. This *yukti* uses an ancient Indian practice of cut-and-paste or in this case, cut-and-move. Documentation shows that this physical style of written instructions existed in India since the *Baudhayana-Sulbasutra* (c. 800-600 BCE) [6, pp. 387-392]. The same text provides a problem, 2,300 years before the *Yuktibhāṣā*, that reads: [6, p. 389]

> "The areas [of the squares] produced separately by the length and the breadth of a rectangle together equal the area [of the square] produced by the diagonal. This is observed in rectangles having sides 3 and 4, 12 and 5, 15 and 8, 7 and 24, 12 and 35, 15, and 36."

While this unproven generalization from six examples, without an *upapatti* is simple, it illustrates that there had been an interest in geometry for thousands of years and Indians had developed their own understanding of mathematics outside of Greek influence. This proof given above is the beginning of one of the most interesting chapters in *Yuktibhāṣā,* and provides contrast to the statement of Kline in his chapter of Indian Mathematics; [8, p 189]

> "They offered no geometric proofs; on the whole they cared little for geometry."

Chapter six also includes passages on *Sankalita* (Summation of series). The sum of natural numbers was known at least since Āryabhaṭa I (476-550). The only surviving work authored by him, the *Āryabhaṭīya,* contains the sections, "Sum of Series Formed by Taking Sums of Terms of an Arithmetical Progression" and "Sums of series Formed by Taking Squares and Cubes of Terms of an Arithmetical Progression." Neither section contains an *upapatti* as they are examples of problems and their solutions [2, pp. 37-38]. Other mentions of arithmetic progression occurred within Brahmagupta's (b. 598) *Brahama Sphuta Siddanta* (628), Mahavira's (9th century) *Ganitha Sara Sangraha* (850), and Narayana Pandita's (14th century) *Gaṇita Kaumudi* (1356) but do not include *yukti-s* [3, p. 265]. Nīlakaṇṭha gave geometrical demonstrations involving progressions but his student, Jyeṣṭhadeva, built upon his predecessor's knowledge of summation, improved upon them and provided *yukti-s.* While reading these *yukti-s* it is to be noted the repetition within each one.

> "The reiteration of the key idea is a measure of importance Nīlakaṇṭha [Jyeṣṭhadeva's mentor] attached to getting it across, a pedagogic device *Yuktibhāṣā* also uses to good effect." [4, p. 270]

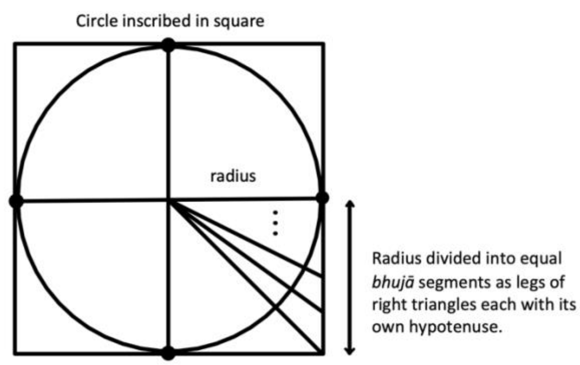
6.4 *Sankalita:* Summation of series [12, p. 61]

Now is described the methods of making the summations <referred to in the earlier sections>. At first, the simple arithmetical progression <*kevala-sankalita*> is described $[1 + 2 + \cdots + n]$. This is followed by the summation of the products of the equal numbers <squares> $[1^2 + 2^2 + \cdots + n^2]$...

6.4.1 *Mūla-sankalita:* [Finite] Sum of natural numbers $[1 + 2 + 3 + \cdots + n]$ [12, pp. 61-62]

Here, in this *mūla -sankalita* <basic [finite] arithmetical progression>, the final *bhujā* [variable] is equal to the radius.

[The context for the use of the words "radius" and "hypotenuses" here is described in 6.3.1, p. 49, and illustrated there by the following figure, although this "context" does not influence the argument below.]



The term before that will be one segment <*khanda*> less. The next one will be two segments less.

[This finite arithmetical progression of *n* terms has the form

$radius - (n-1), \ldots, radius - 2, radius - 1, radius$

where "*radius*" may not be a natural number length.]

Here, if all the terms <*bhujā -s*> had been equal to the radius, the result of the summation would be obtained by multiplying the radius by the number of *bhujā -s*.

[If the *n* terms were all "*radius*", then $radius + \cdots + radius \; = \; n \cdot radius$.]

However, here, only one *bhujā* is equal to the radius. And, from that *bhujā*, those [vertical legs in the figure] associated with the smaller hypotenuses are less by one segment each, in order. Now, suppose the radius to be the same number of units as the number of segments to which it has been divided, in order to facilitate remembering <their number> [$radius = n$ units, so the sum above if "all the terms <*bhuja-s*> had been equal to the radius" is $n + \cdots + n = n \cdot n$]. The number associated with the penultimate [next to last] *bhujā* will be less by one <from the number of units in the radius> [meaning $n-1$]; the number of the next one, will be less by two from the number of units in the radius [$n-2$]. This reduction <in the number of segments> will increase by one <at each step>. The last reduction will practically be equal to the measure of the radius [the last reduction $= n-1 \approx n = radius$], for it will be less only by one segment. In other words, when the reductions are all added, the sum thereof will practically <*prāyeṇa*> be equal to the summation of the series from 1 to the number of units in the radius; it will be less only by one radius length.

[Step 1: the reduction in the number of segments is 1 from n units to $n-1$ units;

Step 2: the reduction in the number of segments is 2 from n units to $n-2$ units;

…

Step $n-1$: the reduction in the number of segments is $n-1$ from n units to 1 unit.

So, since $1+2+\cdots+(n-1) = (1+2+\cdots+n) - n$ we have

$(sum\ of\ reductions) = summation - n$.]

Hence, the summation will be equal to the product of the number of units in the radius with the number of segments plus one, and divided by 2.

[Since the reductions $1,\ 2,\ \ldots,\ n-1$ equal the remaining units out of n , namely

$n-1,\ n-2,\ \ldots,\ 1$ listed in reverse, starting from above we have

$n \cdot n = n + \cdots + n$

$= [1+(n-1)] + [2+(n-2)] + \cdots + [(n-1)+1] + n$

$= (first\ reduction + remaining) + \cdots + (last\ reduction + remaining) + n$

$= (sum\ of\ reductions) + (sum\ of\ remaining) + n$

$= (sum\ of\ reductions) + (sum\ of\ reductions) + n$

$= 2(sum\ of\ reductions) + n$

$= 2(summation - n) + n$

$= 2(summation) - n.$

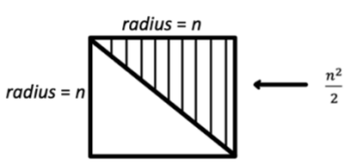
$n \cdot n + n = 2(summation)$

$\frac{n \cdot (n+1)}{2} = summation.$]

The summation of all the *bhujā -s* of the different hypotenuses is called *bhujā -sankalita*.

[How to approximate a very large sum of natural numbers]

Now, the smaller the segments, the more accurate <*sūkṣma*> will be the result. Hence, do the summation also by taking each segment as small as an atom <*aṇu*> [infinitesimal]. Here, if it <namely, the *bhujā* or the radius> is divided into *parārdha* [a variable] <a very large number> parts [*n* large], to the *bhujā* obtained by multiplying by *parārdha* add one part in *parārdha* and multiply by the radius and divide by 2, and then divide by *parārdha* [a variable]. For, the result will practically be the square of the radius divided by two. In order that the number might be full, it is divided by *parārdha*. Thus, if the segments are small, only one small segment shall have to be added to get the summation. Hence, not adding anything to <the units in> the *bhujā,* if it is multiplied by the radius and divided by 2 it will be *bhujā-saṅkalita* when it has been divided into extremely small segments. Thus, the square of the radius divided by 2 will be the *saṅkalita* when the segment <*bhujā -khaṇḍa* into which the *bhujā* or the side of the square is divided> is very small. [For large n , $\frac{n(n+1)}{2} \approx \frac{n^2}{2}$. In the previous circle figure see, the lower right square:



]

Jyeṣṭhadeva continues his work on the summations in the *Yuktibhāṣā* with expansions on the summation of squares, then a work on summations of third and fourth powers, and finishes the section with an invaluable proof on the general principle of summations $1^k + \cdots + n^k$ which in Sanskrit is *Samaghata-sankalita.*

6.4.2 *Varga-saṅkalita:* Summation of squares [12, pp. 62-64]

Now is explained the summation of squares <*varga-saṅkalita* >.

> [We suppose again, as above, "the radius to be the same number of units as the number of segments to which it has been divided." So, radius = n units going forward.]

Obviously, the squares of the *bhujā-s,* which are summed up above [the sum of natural numbers], are the *bhujā-s* each multiplied by itself $[1^2, 2^2, \dots, (n-2)^2, (n-1)^2, n^2]$. Here, if the *bhuja-s* $[1,2,\ \dots,\ n\ ]$ which are all multipliers had all been equal to the radius $[n\ ]$, their sum, <*saṅkalita* derived above>, multiplied by the radius would have been the summation of their squares $[n^2 + \cdots + n^2 = (n + \cdots + n) \cdot n]$. Here, however, only one multiplier happens to be equal to the radius, and that is the last one $[n = radius\ ]$. The one before that will have the number of segments one less than in the radius $[n-1\ ]$. <Hence> if that, <i.e., the second one>, is multiplied by the radius $[(n-1) \cdot n]$, it would mean that one multiplied by the penultimate [next to last] *bhujā* $[(n-1) \cdot 1]$ would have been the increase in the summation of the squares. Then <the segment> next below is the third $[(n-2) \cdot n]$. That will be less than the radius by two segments $[n-2\ ]$. If that is multiplied by the radius $[(n-2) \cdot n]$, it will mean that, the summation of the squares will increase by the product of the *bhujā* by two <segments> $[(n-2) \cdot 2]$.

> [If the correct summation of the squares
>
> $1^2 + 2^2 + \cdots + (n-2)^2 + (n-1)^2 + n^2$
>
> was replaced by the larger sum
>
> $[1 + 2 + \cdots + (n-2) + (n-1) + n] \cdot n,$
>
> the increase in the penultimate term would be
>
> $(n-1) \cdot n - (n-1)^2 = (n-1)\big(n - (n-1)\big) = (n-1) \cdot 1$
>
> = "one multiplied by the penultimate *bhujā*".

The increase in the "next below term" would be

$(n-2)\cdot n-(n-2)^2=(n-2)\big(n-(n-2)\big)=(n-2)\cdot 2$

= "product of the *bhujā* $[n-2]$ by (two) segments."]

In this manner, the summation in which the multiplication is done by the radius $[n\,]$ <instead of the *bhujā-s*> would be larger than the summation of squares by terms which involve the successively smaller *bhujā-s* multiplied by successively higher numbers. If <all these additions> are duly subtracted from the summation where the radius is used as the multiplier, the summation of the squares <*varga- saṅkalita*> will result.

$[[1+2+\cdots+(n-2)+(n-1)+n]\cdot n$

$-(n-1)\cdot 1-(n-2)\cdot 2-\cdots-2\cdot(n-1)-1\cdot(n-1)$

$=[1+2+\cdots+(n-2)+(n-1)+n]\cdot n-[(n-1)\cdot n-(n-1)^2]$

$-[(n-2)\cdot n-(n-2)^2]-\cdots-[2\cdot n-2^2]-[1\cdot n-1^2]$

$=n^2+(n-1)^2+(n-2)^2+\cdots+2^2+1^2.]$

[The modern formula is $1^2+\cdots+n^2=\frac{1}{6}(n)(n+1)(2n+1)$, which for large n approximates to $\frac{1}{6}(n)(n)(2n)=\frac{1}{3}n^3$.]

…

6.4.4 *Samaghāta-saṅkalita:* General principle of summation [12, pp.65-67]

Now, the square of the square <of a number> is multiplied by itself $[(n^2)^2\cdot n\ =\ n^5]$, it is called *sama-pañca-ghāta* <number multiplied by itself five times>. The successive higher order summations are called *sama-pañcādi-ghāta-saṅkalita* <and will be the summations of powers of five and above>. Among them if the summation < *saṅkalita*> of powers of some order is multiplied by the radius, then the product is the summation of summations <*saṅkalita -saṅkalita*

> of the <powers of the> multiplicand <of the given order>, together with the summation of powers <*sama-ghāta- saṅkalita*> of the next order.

[For *radius* = n and order = 5, so "next order" = 6, since

$$(n-1)^5 \cdot n \;=\; (n-1)^5 \cdot [(n-1)+1] \;=\; (n-1)^6 + 1 \cdot (n-1)^5$$

$$(n-2)^5 \cdot n = (n-2)^5 \cdot [(n-2)+2] = (n-2)^6 + 2 \cdot (n-1)^5$$

…

$$2^5 \cdot n = 2^5 \cdot [2+(n-2)] = 2^6 + (n-2) \cdot 2^5$$

$$1^5 \cdot n = 1^5 \cdot [1+(n-1)] = 1^6 + (n-1) \cdot 1^5$$

we see that, adding vertically,

$$(1^5 + 2^5 + \cdots + (n-1)^5 + n^5) \cdot n$$

$$= [1^6 + 2^6 + \cdots + n^6]1 \cdot (n-1)^5 + 2 \cdot (n-2)^5 + \cdots + (n-2) \cdot 2^5 + (n-1) \cdot 1^5$$

$$= [1^6 + 2^6 + \cdots + n^6] + (n-1)^5 + (n-2)^5 + \cdots + 2^5 + 1^5$$

$$+(n-2)^5 + \cdots + 2^5 + 1^5$$

$$\cdots$$

$$+2^5 + 1^5$$

$$+1^5.]$$

Hence, to derive the [approximate] summation of the successive higher powers: Multiply each summation by the radius. Divide it by the next higher number [power] and subtract the result from the summation got before. The result will be the required summation to the higher order.

[The example above gives

$$1^6 + \cdots + n^6 = [1^5 + \cdots + n^5] \cdot n - [1^5 + \cdots + (n-1)^5]$$

$$-[1^5 + \cdots + (n-2)^5]$$

…

$$-[1^5 + 2^5]$$

-1^5.

Recalling the approximation $1+\cdots+n=\frac{n(n+1)}{2}\approx\frac{n^2}{2}$ for large *n* generalized to

$1^m+\cdots+n^m\approx\frac{n^{m+1}}{m+1}$, we have here the approximation

$$1^{k+1}+\cdots+n^{k+1}\approx\left[\frac{n^{k+1}}{k+1}\right]\cdot n-\left[\frac{(n-1)^{k+1}}{k+1}+\frac{(n-2)^{k+1}}{k+1}+\cdots+\frac{2^{k+1}}{k+1}+\frac{1^{k+1}}{k+1}\right]$$

$$\approx\frac{1}{k+1}\cdot\left[n^{k+2}-\frac{(n-1)^{k+2}}{k+2}\right]$$

$$\approx\frac{1}{k+1}\cdot\frac{1}{k+2}\cdot[(k+2)\cdot n^{k+2}-n^{k+2}]$$

$$=\frac{1}{k+1}\cdot\frac{1}{k+2}\cdot(k+1)\cdot n^{k+2}$$

$=\frac{n^{k+2}}{k+2}$ as desired.]

Thus, divide by two the square of the radius. If it is the cube of the radius, divide by three. If it is the radius raised to the power of four, divide by four. If it is <the radius> raised to the power of five, divide by five. In this manner, for powers rising one by one, divide by numbers increasing one by one. The result will be, in order, the [approximate] summations of powers of numbers <*sama-ghāta-saṅkalita* >. Here, the basic summation is obtained from the square, the summation of squares from the cube, the summation of the cubes from the square of the square. In this manner, if the numbers are multiplied by themselves a certain number of times <i.e, raised to a certain degree> and divided by the same number, that will be the summation of the order one below that. Thus <has been stated> the method of deriving the summations of <natural> numbers, <their> squares etc. [$1+\cdots+n\approx\frac{n^2}{2}, 1^2+\cdots+n^2\approx\frac{n^3}{3}, 1^3+\cdots+n^3\approx\frac{n^4}{4}, \ldots$]

"Now [in 2010], regarding the notion of proof and reasoning, it may be noted that Kerala mathematicians placed considerable emphasis on providing an elaborate exposition of various results, by discussing their reasoning, supported by several numerical illustrations and various kinds of proofs in algebraic and geometrical backgrounds that were necessary for the benefit of all kinds of students. Their exposition often started from the elementary level and was presented in an instructive form that could be easily followed and understood by all." [17, p. 162]

These y*ukti-s* from the *Yuktibhāṣā* allow a thought-provoking window into the Nila school and the culture in India that surrounded it. Jyeṣṭhadeva and his predecessors cared for mathematics that impacted daily life, such as trigonometry to help define the movements of celestial bodies and their importance in religion, but also emphasized on that which was theoretical and did not pertain to necessary actions made by ordinary people, such as the summation of series outlined above. The callous view western scholars adopted regarding mathematics from India was unfounded, growing from preconceived notions and a lack of material to ingest. The scenery has changed. With reliable translations published and fresh interest in the field the assumptive attitudes of the west regarding mathematics from India will continue to change.

Works Cited

1. Boyer, C.B., Merzbach, U.C. (1968). *A History of Mathematics*, 3rd ed. John Wiley & Sons, Inc.
2. Clark, W.E. (1930). *The Āryabhaṭīya of Āryabhaṭa*. The University of Chicago press.
3. Deshpande, S.M. (2017). "Basics of Development of Integration and Series in India." International Journal of Mathematics And its Applications.
4. Divakaran, P.P (2018) *The Mathematics of India: Concepts, Methods, Connections.* Springer.
5. Joseph, George G. (2000) *The Crest of the Peacock,* 3rd ed. Princeton University Press.
6. Katz, V. J. (2007). *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook.* Princeton University Press.
7. Keller, Agethe. (2012). "Dispelling Mathematical Doubts. Assessing the Mathematical Correctness of Algorithms in Bhâskara's Commentary on the Mathematical Chapter of the Âryabhatîya." Cambridge University Press, pp.487-508.
8. Kline, Morris (1972). *Mathematical Thought from Ancient to Modern Times.* Oxford University Press.
9. Krishnamachari, Suganthy (2020). *Decoding manuscripts on mathematics.* The Hindu.
10. Puttaswamy (2021). *Mathematical Achievements of Pre-Modern Indian Mathematicians.* Elsevier.

11. Raju, C.K. (2015). "Eternity and Infinity: The Western Misunderstanding of Indian Mathematics, and Its Consequences for Science Today*",* Volume 14, #2. Center for Studies in Civilization, New Delhi.
12. Sarma, K.V. (2008). *Ganita-Yukti-Bhasa.* Springer.
13. Subbarayappa, B.V., Sarma, K.V. (1985). *Indian Astronomy: A Source Book.* Nehru Centre.
14. Srinivas, M.D. (2005). *Contributions to the History of Indian Mathematics.* Hindustan Book Agency, New Delhi.
15. Srinivas, M.D. (2008). *The Methodology of Indian Mathematics and Its Contemporary Relevance.*
16. Whish, C.M. (1834). "On the Hindu Quadrature of the Circle, and the Infinite Series of the Proportion of the Circumference to the Diameter Exhibited in the Four S'astras, the *Tantra Sangraham, Yucti Bhasha, Carana Padhati*, and *Sadratnamala*." Transactions of the Royal Asiatic Society of Great Britain and Ireland.
17. Yadav, B.S. (2010). *Ancient Indian Leaps into Mathematics.* Birkhauser