It Appears That Four Colors Suffice: A Historical Overview of the Four-Color Theorem

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Certainly any mathematical theorem concerning the coloring of maps would be relevant and widely applicable to modern-day cartography. As for the Four-Color Theorem, nothing could be further from the truth. Kenneth May, a twentieth century mathematics historian, explains that "books on cartography and the history of map-making do not mention the four-color property, though they often discuss various other problems relating to the coloring of maps.... The four-color conjecture cannot claim either origin or application in cartography" [WIL2]. In addition, there are virtually no other direct applications of this theorem outside of mathematics; however, mathematicians still continue to search for an elegant proof to this day. In *Map Coloring, Polyhedra, and the Four-Color Problem,* David Barnette guarantees that these efforts have not been wasted. He points out that many advances in graph theory were made during the process of proving the Four-Color Theorem. These advances provide justification for the 150+ years spent proving a theorem with rather useless results [BAR]. In this paper, the historical progress of the Four-Color Theorem will be examined along with the work of some of its contributors.

Without doubt, the Four-Color Theorem is one of the few mathematical problems in history whose origin can be dated precisely. Francis Guthrie (1831-99), a student in London, first posed the conjecture in October, 1852, while he was coloring the regions on a map of England. He noticed that he could color the map using only four colors in such a way so that no two countries sharing a common border receive the same color. Guthrie wanted to know if this were true of all maps and could it be proven mathematically. He explained his conjecture in a letter to his younger brother Frederick, who was a student at University College in London. Frederick could not solve his brother's problem and he forwarded it to his instructor Augustus DeMorgan (1806-71), a famous mathematician of the nineteenth century. DeMorgan was very impressed with the conjecture and, although he could not solve it himself, he presented the problem to his colleague and friend Sir William Rowan Hamilton (1805-65), the inventor of quaternions, in a famous letter dated October 23, 1852.[1] This final stage in the "Four-Color Conjecture chain letter" marks the birth of the Four-Color Theorem.

[1] Part of this letter is reprinted in [FRI].

Though it is now accepted that Francis Guthrie began the quest to show that four colors suffice, some incorrectly claimed it originated with German mathematician and astronomer August Ferdinand Möbius (1790-1868) in 1840. Möbius, who is best known for his Möbius strip, asked his geometry class to solve the "problem of the five princes," knowing that there is no solution. The problem had been posed in several different forms, but it can be abbreviated to state the following: Find an arrangement of five neighboring regions so that each region shares a border with the other four. If such an arrangement exists, then coloring each region with a different color would require five colors, providing a counter-example to the Four-Color Theorem. Since such an arrangement can be proven impossible, some claimed that the Four-Color Theorem was a trivial result. However, there is a flaw in the logical reasoning used to make this claim. For instance, it is true that if there is a map with five neighboring regions, then the Four-Color Theorem is false. It follows that if the Four-Color Theorem is true, then there is no map with five neighboring regions. However, it is illogical to state that if there is no map with five neighboring regions, then the Four-Color Theorem is true. In other words, the impossibility of a map with five neighboring regions is a necessary, but not sufficient, condition in proving the Four-Color Theorem. So, verification of this condition, which is quite simple, does not prove the theorem. This fallacious proof flourished in the late 1800's when German Geometer Richard Baltzer published it in a paper. Belief in this proof floated around until 1959, whereby geometer H. S. M. Coxeter finally exploited the error. Since then, Guthrie has been credited as the originator of the problem [WIL2].

Prior to delving into any more history, it is imperative to understand the theorem with regards to mathematics. Though the Four-Color Theorem is most commonly regarded as a graph theory topic, several other areas of mathematics are also applicable. In their joint work *Every Planar Map is Four Colorable,* Kenneth Appel and Wolfgang Haken claim that "so many areas of mathematics have been involved in various attempts to prove the Four-Color Theorem that it would be impossible to discuss them all here" [APP]. This paper will approach the theorem from a graph theory perspective. To do so, it is important to begin with a few definitions.[2]

1. **Graph** - A collection of vertices and edges such that each edge is associated with exactly two vertices, its endpoints.
2. **Loop** - An edge whose endpoints are equal.
3. **Adjacent Vertices** - Two vertices that serve as endpoints for the same edge.
4. **Degree of a vertex** - The number of edges incident on a given vertex.
5. **Planar Graph** - A graph with no loops that can be drawn in the plane in a way such that no edge crosses another.

[2]Even the most basic graph theory terms are not standardized; therefore, various textbook definitions may differ slightly from these definitions provided.

6. k-**colorable** - A graph is k-colorable if each of its vertices can be colored using k colors such that no two adjacent vertices share the same color.

Having defined these terms, the theorem can now be stated.

Theorem 1 (Four-Color Theorem) *Every planar graph is 4-colorable.*

The translation from graph theory to cartography is readily made by noting that each vertex can represent a country on a map and an edge joining two vertices can represent a boundary line between two neighboring countries. Robin Thomas further simplifies this correlation using perhaps a more visual approach. He suggests that "for each country we select a capital (an arbitrary point inside that country) and join the capitals of every pair of neighboring countries. Thus we arrive at the notion of a plane graph . . . " [THO] [3]

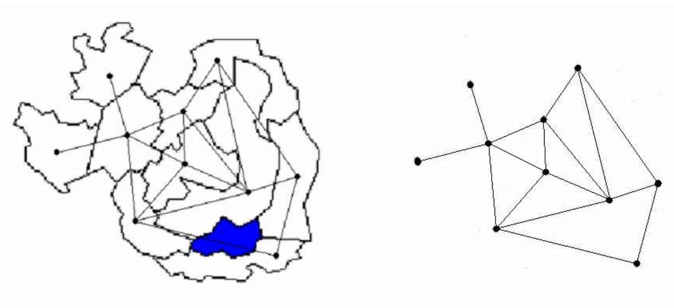

Figure 1: *The correlation between the cartography interpretation and the graph theory interpretation*

The problem received hardly any attention from the time it was conjectured until July 13, 1878, when Arthur Cayley (1821-95), English mathematician and lawyer, reignited enthusiasm in the four-color conjecture by placing an inquiry in the mathematical section of the Royal Society. His inquiry asked if anyone had formed a solution to the conjecture yet. This curiosity sparked the interests of mathematicians and immediately placed the four-color conjecture back into their sights. It did not take more than a year for a purported solution to surface and the complete proof was published in the *American Journal of Mathematics.* Sir Alfred Bray Kempe (1849-1922), London barrister and mathematician, was the author of this proof and, although he did not know it at the time, this proof would become one of the most famous proofs in the history of mathematics.[4]

[3]For consistency's sake, the graph theory terms *vertex* and *degree* will be used to refer to the cartography terms *country* and *number of neighbors,* respectively. In some instances authors may not have used these exact terms, but their ideas and implications are preserved in using the synonymous graph theory translation.

[4]Kempe's proof can be found in chapter 1 of David Barnette's *Map Coloring, Polyhedra, and the Four-Color Problem* [BAR].

In his *American Journal of Mathematics* paper [KEM], Kempe introduces the problem, makes a few remarks about Cayley's contributions, and then explains:

> *Some inkling of the nature of the difficulty of the question, unless its weak point be discovered and attacked, may be derived from the fact that a very small alteration in one part of a map may render it necessary to recolour it throughout. After a somewhat arduous search, I have succeeded, suddenly, as may be expected, in hitting upon the weak point, which proved an easy one to attack . . . How this can be done I will endeavour – at the request of the Editor-In-Chief – to explain* [WIL2].

This quotation exhibits the surprisingly immense magnitude of this problem. From here, Kempe proceeds to divide his paper into three main sections, one of which provides a brief understanding of the structure of the proof. In this section, he states and proves a principle that is vital to his approach. He finds that "every map drawn on a simply connected surface must have a district with less than six boundaries." In our graph theory terms, the theorem can be stated as follows:

Theorem 2 *Every planar graph must have a vertex with degree less than six.*

Using this result, Kempe proposed the following scheme for coloring all graphs:

1. Locate a vertex with degree five or less (such a vertex must exist according to the theorem above).

2. Cover this vertex with a patch of the same shape but slightly larger.

3. Extend all the boundaries that meet this patch and join them together at a single point within the patch. This has the effect of deleting the vertex and all edges incident on it, thus decreasing the number of vertices by one.

4. Repeat the above procedure with the new graph, continuing until there is just one vertex remaining: the whole graph is now said to be *patched out.*

5. Color the single remaining vertex with any of the four colors.

6. Reverse the above process, stripping off the patches in reverse order, until the original graph is restored. At each stage, color the restored vertex with any available color until the entire graph is colored with four colors.

Upon reading this approach, the obvious question is how do we prove that it is possible to complete step six using only four colors? Using Kempe chains of course! It is within this question and answer that Kempe took great leaps in graph theory [WIL2].

A Kempe chain is a "chain" of vertices that are colored with two alternating colors. Though it is cumbersome to explain this precisely, Kempe chains can be most easily understood through a concrete example. If our map (graph) is the

United States, then New York (Blue), Pennsylvania (Red), Ohio (Blue), Indiana (Red), and Illinois (Blue) is a Blue-Red Kempe chain; however, California cannot be added to this Kempe chain because it shares no border with any of these five states.

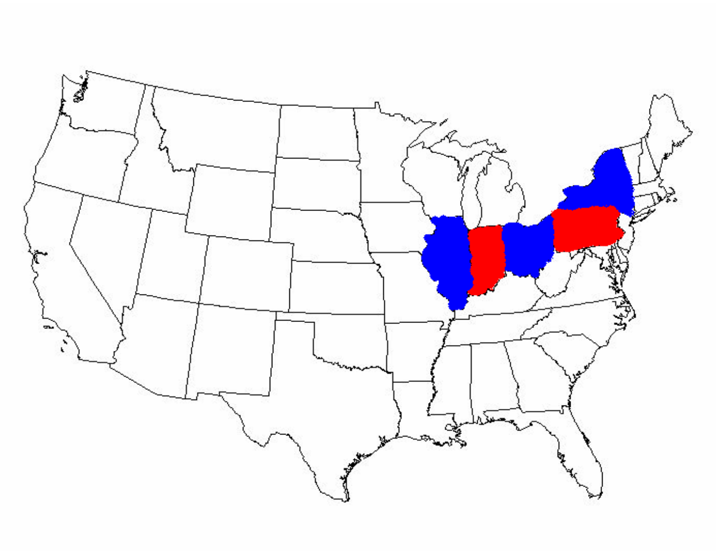

Figure 2: *An concrete example of Kempe chains.*

These Kempe chains served as the backbone of Kempe's proof of the four-color theorem, which is rather easy to understand with a basic knowledge of graph theory. He separates each stage of step six above into several cases and he proceeds to verify each of these cases using his Kempe chains. In addition to his proof of the four-color theorem, he included some remarks and special cases that were overlooked by those who had previously published on the topic. Though there were other proposed proofs of the time, namely those written by Baltzer (1885) and Peter Guthrie Tait (1880), Kempe was given credit as the one who proved the four-color theorem. Until . . .

In 1890, a man named Percy John Heawood (1861-1955) found a flaw in Kempe's proof. As mentioned earlier, Kempe's proof is one of the most famous proofs in the history of mathematics, and it is the fallacy in it that makes his proof so famous. It is quite a tragedy that Kempe is remembered only for his error and not his successes, because he was regarded as a fine mathematician by his contemporaries. Additionally, many of Kempe's results and ideas were used by mathematicians in the twentieth century to make great advances on the four-color theorem and in graph theory. Even furthermore, it is important to note that it took eleven years for mathematicians to discover his flaw; therefore, it is safe to say that the mistake was rather subtle. Nevertheless, Kempe will always be most noted for having the fallacious proof to the four-color theorem [WIL2].

Heawood provided a counterexample to Kempe's proof[5], but this counterexample did not contradict the Four-Color Theorem, it only contradicted Kempe's method of proving the Four-Color Theorem. Kempe tried to prove that when restoring his patched vertices, he could always recolor the graph using only four colors. He only had two cases to prove because Theorem 2 guarantees that each patched vertex has degree less than six and those cases with degree three or less are trivial. Kempe's proof for the case of degree four is completely valid; however, Heawood found the flaw in his proof for the case of degree five. Though Heawood presented this error in Kempe's method, he confessed that he did not have a valid proof for the Four-Color Theorem and that his article was not constructive. But, Heawood's article was not entirely unproductive as he did extract the Five-Color Theorem from Kempe's work. Based on its name, it is painfully obvious that the Five-Color Theorem states that *every planar graph is 5-colorable.* It turns out, not surprisingly, that the Five-Color Theorem is much easier to prove than the Four-Color Theorem. In addition, Heawood made progress on other pertinent topics within the subject and this promoted future advances by prominent mathematicians in the twentieth century [FRI].

Even though Heawood rekindled the four-color flame in mathematics when he wrote his article, very little advancement was achieved in the years immediately following. In the twentieth century, the four-color obsession shifted from being mostly a British quest to include many American mathematicians, such as George Birkhoff, Oswald Veblen, Philip Franklin, Hassler Whitney, and others. These American mathematicians contributed greatly to the theorem as the four-color problem narrowed itself to achieving one task using two principles inherent in Kempe's proof, namely *unavoidable sets* and *reducible configurations.* Both of these concepts can be understood using the foundations laid above in the explanation of Kempe's proof [WIL2].

Theorem 2 stated above best explains the idea of an unavoidable set. If A is a set containing all vertices with degree less than six, then A is an unavoidable set in any planar graph. This is because the vertex set in any planar graph must intersect A, meaning the graph *cannot avoid* A. Since it has already been established that proving Kempe's flawed "degree five" case completes the proof of the four-color theorem, mathematicians sought unavoidable sets that are extensions of A in an effort to make this task easier [WIL2].

In order to comprehend reducible configurations, it is important to understand the concept of a *minimal counter-example* or *minimal criminal.* Using a proof by contradiction, assume that the Four-Color Theorem is false with the intention of arriving at an absurdity. If the Four-Color Theorem is false, then there exists a graph that is not 4-colorable, i.e., one that requires five colors. It follows that there must exist such a graph with a minimum number of vertices, a minimal criminal. More precisely, a minimal criminal is not 4-colorable, but all graphs with fewer vertices are 4-colorable [WIL2]. A reducible configuration is any arrangement of vertices that cannot exist in a minimal criminal. Examples

[5]Heawood's counterexample can be found in *The Quarterly Journal of Pure and Applied Mathematics* (1890) [HEA].

of such configurations are vertices with degree four or less. If a graph contains a reducible configuration, then any coloring of the rest of the graph with four colors can be extended, after necessary recoloring, to color the entire graph with four colors. However, as mentioned earlier, Kempe was unable to effectively prove the "degree five" case and this case was the only thing standing between him and a correct proof. So, most of the twentieth century efforts poured into proving this theorem are targeted at finding an unavoidable set of reducible configurations. Such a discovery proves the Four-Color Theorem as follows: Since the set is unavoidable, each graph must contain at least one of its elements, namely the reducible configurations. None of these configurations can be contained in a minimal criminal; therefore, no graph can be a minimal criminal. Narrowing the theorem down to this single conquest sounded promising, but initial expectations gradually diverged from actual progress as mathematicians sweat over this problem for more than just a few years [WIL2].

As mentioned above, progress was stagnant after Heawood's article in 1890. The next sparks of advancement came with Oswald Veblen (1880-1960), a famous geometer from Princeton. In a presentation given to the American Mathematical Society on April 27, 1912, Veblen, using concepts from Heawood's number theory approach, explained the problem in the form of linear equations over a finite space. Veblen's work did not support any great advances toward a proof, but he influenced a few of his pupils to attack the Four-Color Theorem, and their contributions far exceeded his own. The two most notable of these are George David Birkhoff (1884-1944) of Harvard and Philip Franklin (1898-1965) of MIT. Birkhoff, one of America's first prominent mathematicians, analyzed Kempe's work and improved upon it. Though Birkhoff did not prove the all-important degree five case mentioned above, he did find certain large reducible configurations. Franklin expanded upon the work of Birkhoff, and in 1920 he proved that the Four-Color Theorem holds true for all graphs with 25 or fewer vertices.[6] The methods of Birkhoff were utilized by many mathematicians between 1913 and 1950 to improve the number and size of reducible configurations, and though their efforts were certainly not wasted, by 1950 the set of all known reducible configurations was dwarfed by that which was necessary to produce an unavoidable set. In fact, at this time the only improvement in Franklin's lower bound was a jump from 25 to 35 [APP]. The progression went as follows:

1920	Philip Franklin	25
1926	C. N. Reynolds	27
1936	Philip Franklin	31
1938	C. E. Winn	35
1968	O. Ore and J. Stemple	40

So, as of 1968, any counter-example to the Four-Color Theorem must have at

[6]Books actually differed on the year and number of vertices for which Franklin proved this. These other claims include 1920 for the year and 22 for the number of vertices.

least 41 vertices [RIN].

While the aforementioned quest for reducible configurations continued, there was a simultaneous mission to seek out unavoidable sets. The goal was to increase the size and quantity of unavoidable sets and to increase the complexity of reducible configurations, hoping that somewhere they would meet each other to form an unavoidable set of reducible configurations. So, concerning unavoidable sets, the objective was to find an unavoidable set containing some sort of transformation of the all-important degree five case, because the degree five case itself had not been proven reducible. Paul Wernicke, a German mathematician, was the pioneer for attempting to construct larger unavoidable sets. In May, 1903, Wernicke published a paper which proved that any graph containing no vertices of degree four or less must contain at least one of the following:

1. Two adjacent vertices, each of degree five

2. Two adjacent vertices, one of degree five and one of degree six.

This result got the ball rolling for finding unavoidable sets. Philip Franklin's contributions actually stemmed from his work on finding larger unavoidable sets. He wrote his doctoral thesis on map coloring and in 1920 he presented a portion of this thesis to the National Academy of Sciences. Franklin's results extended those of Wernicke to form a larger unavoidable set. Franklin proved that any graph that contains no vertices of degree four or less must contain at least one of the following:

1. A vertex of degree five adjacent to two other vertices of degree five.

2. A vertex of degree five adjacent to one vertex of degree five and one vertex of degree six.

3. A vertex of degree five adjacent to two vertices of degree six.

Additional unavoidable sets were produced by Henri Lebesgue (1875-1941), who is better known for Lebesgue integrals used in analysis. The year before Lebesgue died he wrote a paper on the counting formula he used to construct these unavoidable sets. Most recent advances in this area were achieved using a modern approach called *discharging* [WIL2].

The method of discharging was invented by Heinrich Heesch (1906-95), a mathematician from the University of Hanover who began his work on the Four-Color Theorem in 1936. According to Kenneth Appel and Wolfgang Haken, Heesch may have been "the first mathematician (after Kempe) who publicly stated a belief that the Four-Color Conjecture could be proved by finding an unavoidable set of reducible configurations" [APP]. Heesch further claimed that the set would contain about ten thousand configurations would have certain size restrictions on each of them. This seemed overwhelming to mathematicians of the time, but the birth of high-speed computers made such a proof possible. Heesch proceeded by systematizing reducibility and he originated "D-reducibility," a concept that he proved and implemented using a computer program. If a program could prove that a configuration is D-reducible, then the configurations is certainly a reducible one. Thus, D-reducibility implies reducibility. He later improved the principle of D-reducibility and called it C-reducibility. The computers come into play by testing the configurations for such reducibility. The complexity of a configuration is denoted by its *ring-size,* which is the number of vertices "surrounding" or "encapsulating" it. At the time, computers were only powerful enough to test figures with ring-size less than 12, which was conjectured to be insufficient for the proof [FRI].

In 1964, Heesch needed to adapt his algorithm to computer programming and he did this with the help of Karl Dürre, who taught high school at the time. Heesch and Dürre collaborated to create quite a stir in the mathematical world with their new method of attempting to prove the four-color theorem by computer. On November 23, 1965, the first test was run on a CDC 1604A computer at the Institute of Technology in Hannover, Germany. By today's standards, the technological setting was primitive, but at the time this was quite a breakthrough. They first tested a configuration that was already known to be reducible in order to verify the accuracy of their program. The following month, for the first time ever, a configuration of ring-size nine whose reducibility had been previously unknown was tested for reducibility with a computer. This sounded extremely promising; however, they were severely bounded by computing capacity. In the late 1960's, Heesch and Dürre had made several trips to the United States to work with American mathematicians on much more powerful computers, and these trips resulted in breakthroughs, but the problem still remained unsolved. Heesch finally returned to Hannover with aspirations of finishing the proof of the Four-Color Theorem alone. However, due to disagreements with the German research community and lack of funding, Heesch was denied the resources and computing power necessary to complete the project. It is quite a shame, but he and his ideas just happened to be in the wrong place at the right time [FRI].

Heesch did, however, influence Wolfgang Haken, who had attended one of his lectures and gained interest in the problem. As a young student at the University of Kiel in Germany, Haken began to tackle the three main unsolved problems in mathematics: the knot problem, the Poincarè conjecture, and the Four-Color Theorem. He solved the first and almost solved the second, which still has no solution today. After working diligently on the Poincarè conjecture and proving 198 of its 200 cases, Haken, who was at the University of Illinois at the time, conceded and decided to address the third problem of this triumvirate. Haken touched base with Heesch and they had met a few times in the United States to test Dürre's programs, as mentioned above. Haken and Heesch traded results and worked distantly together while Heesch was back in Germany. In the early 1970's, once the funding became an issue for Heesch, Haken began to wave the white flag and, in a lecture in Illinois, he declared that "the computer experts have told me that it is not possible to go on like that. But right now I'm quitting. I consider this to be the point beyond which one cannot go without a computer" [WIL2]. It was at this point that one wearied soldier's pessimism was outweighed by another young soldier's optimism. Upon hearing this statement by Haken, Kenneth Appel, a mathematician and computer programmer at the University of Illinois, insisted that he and Haken could finish this journey. Haken was intimidated by his own ignorance in the field of computer science, since a large portion of the proof would involve programming. He was reassured that Appel would handle all the computer science implementation and Haken was revived by this enthusiasm; therefore, they pressed onward.

Their work consisted largely of finding unavoidable sets first, then proceeding to check for reducibility. The estimate that some of these unavoidable sets would be of ring-size as high as 16 created a problem for Appel and Haken, and it is easy to see why. The following chart lists the number of possible colorings for configurations of ring-sizes 6 through 14:

ring-size	6	7	8	9	10	11	12	13	14
colorings	31	91	274	820	2461	7381	22,144	64,430	199,291

This chart suggests that for a configuration of ring-size 14, it is necessary to consider no fewer than 199,291 different colorings of the surrounding ring of vertices. It is not terribly difficult to see that testing a single configuration of ring-size 16 for reducibility would require a colossal amount of computer power. So, in the summer of 1974, the two sought help from the University of Illinois computer science department and it was there that they found graduate student John Koch. They gave Koch the task of economizing the reducibility tests and, after working out some kinks in their system, he and Appel did just that. After hitting yet another brick wall the following year, Haken, while on vacation, had an epiphany that led to an enormous breakthrough with their method. This breakthrough enabled the restriction of all configurations to ring-size 14 or less [WIL2].

In the summer of 1976, after several hundred pages of scrutinizing details and over 1200 hours of output on a powerful computer, the four-color conquest finally came to a halt. Shortly after testing the final configuration for reducibility, Appel celebrated the success by etching the statement 'Modulo careful checking, it appears that four colors suffice' onto the department's blackboard. Appel and Haken presented their proof to a group of mathematicians at a meeting in Toronto. Shortly after this presentation, they polished and published the proof, which is lengthy but basically boils down to checking 1,936 cases for reducibility. Since each of these cases required a computer to perform up to 500,000 logical operations, the validity of the proof still remains in question to this day. Many mathematicians do not consider it a proof because it is virtually impossible to verify the proof without a computer [BUR]. Regardless of all the skepticism, it was indeed commemorated as a valid proof. The University of Illinois mathematics department's postal meter adopted the slogan 'four colors suffice.'

The Haken-Appel proof did not completely end research on the four-color theorem. As mentioned, many mathematicians did not believe the proof and tried to find ways to improve their methods. This was accomplished several times between 1976 and the present. It was found that of the 1,936 configurations, 102 of them were redundant, thus reducing the number to 1,834. This number was later reduced to 1,482, but still nothing to convince many mathematicians. Then, in 1996, Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas made a rather large improvement in the proof, reducing the unavoidable set to only 633 configurations [WIL1]. Though their methods strongly resemble those of Appel and Haken and are actually improvements of them, Robertson, Sanders, Seymour, and Thomas currently have the most efficient proof of the Four-Color Theorem.

Over the past 150 years, the Four-Color Theorem certainly had a strong impact on the mathematical society. From Guthrie's conjecture all the way to the most recent proof by Robertson, Sanders, Seymour, and Thomas, the theorem took mathematicians on quite a ride, and many great things have resulted from this useless theorem. Some mathematicians still do not believe the solution proposed by Robertson, Sanders, Seymour, and Thomas actually constitutes a proof, because it cannot be verified by hand. David Burton elaborates:

> *It cannot be ruled out that a short and convincing proof of the conjecture may yet be found, but it is just as conceivable that the only valid proofs will involve massive computations requiring computer assistance. If this is the case, we must acknowledge that a new and interesting type of theorem has emerged, one having no verification by traditionally accepted methods. Admitting these theorems will mean that the apparently secure notion of a mathematical proof is open to revision.* [BUR]

Indeed, we may never find an elegant proof to the four-color theorem. But, as always, only time will tell . . .

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