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The Evolution of the Circle Method in Additive Prime Number Theory

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1 Introduction

The study of prime numbers first began with Euclid in approximately 300 BC. In his monumental *Elements* he laid the foundation of the Fundamental Theorem of Arithmetic and established the basic multiplicative nature of primes. However, ever since their discovery, any attempt to understand the additive nature of primes has been confounded. As a result, two landmark problems have risen as representatives of the efforts in the field: the strong and weak Goldbach Conjectures. This paper will primarily examine the development of the circle method and its use to address the weak conjecture. It will establish basic results in Prime Number Theory and then trace the evolution of the Circle Method as it has been adapted in efforts to prove the weak conjecture. This paper posits that the development of the proof of the weak conjecture is mirrored by development in the bounds of primes in arithmetic progression.

1.1 Goldbach's Conjectures

The two Goldbach conjectures first came about in a letter from Christian Goldbach and Leonhard Euler between June 7th, 1742. Translated, the conjecture is stated,

> Every number N which is a sum of two primes is a sum of as many primes including unity as one wishes (up to N), and that every number > 2 is a sum of three primes.

In particular, Golbach considered 1 as a prime in his original formulation and had a very different perspective. Euler then brought up in his reply on June 30th that the first statement, that if N is the sum of two primes, then it is a sum of as many primes as one wishes, followed from a previous observation by Goldbach that every even number is the sum of two primes [1, Chapter XVIII]. The problem is thus now split into two parts, the first of which is Goldbach's Strong Conjecture, also known as his Even Conjecture or his Binary Conjecture.

Conjecture 1.1 (Strong Goldbach Conjecture)**.** *Every even number greater than or equal to four can be the sum of two primes.*

Similarly, the other part is his Weak Conjecture, also known as his Odd Conjecture or his Ternary Conjecture, is simply described

Conjecture 1.2 (Weak Goldbach Conjecture)**.** *Every odd number greater than or equal to seven can be the sum of three primes.*

Early efforts were largely unsuccessful. Many statements were made without proof. One such statement was made by Euler in 1780 where he said every number of the form $4n + 2$ is the sum of two primes of the form $4k + 1$ and verified this for all $4n + 2 \leq 110$. Other statements described properties related to Golbach's problems. For instance, in 1879 F. J. E. Lionett proved that, for x the number of representations of $2a$ as the sum of two odd primes, y the number of representations of $2a$ as the sum of two distinct odd composite numbers, z the number of odd primes $< 2a$, and $q = \lfloor \frac{a}{2} \rfloor$, that $q + x = y + z$. In relating this back to Goldbach's problem, he stated it was probable that there are some cases where $q = y + z$, hence that $x = 0$, though he gave no support to the statement [1, Chapter XVIII].

1.2 The Distribution of Primes

The lack of major results before the development of the circle method in the efforts against Goldbach's problems can be roughly attributed to a lack of "handle" on which the mathematicians could grasp. This began to change as what is now known as the Prime Number Theorem was developed. This section provides a brief history of the development of the Prime Number Theorem and related theorems for arithmetic progressions to create a framework by which results of the circle method may be understood. For a more in-depth treatment [2] and [7] are invaluable.

In particular, mathematicians had no way of estimating the number of primes from unity to some number. This function, denoted $\pi(x)$ is of great importance when attempting to solve problems of prime numbers. The first steps towards estimating $\pi(x)$ were made by A. M. Legendre culminating in 1808 with the conjecture of the approximation

$$\pi(x) \approx \frac{x}{\log x + A(x)}. \tag{1}$$

where $\lim_{x\to\infty} A(x) = 1.80336\ldots$. He also conjectured about primes in arithmetic progression, where $\pi_{q,a}(x)$, for relatively prime k and l, gives the number of primes less than x of the form $qn + a$,

$$\pi_{k,l}(x) \to \infty \text{ as } x \to \infty. \tag{2}$$

Essentially, that there will be infinitely many prime numbers of any such appropriate form. This was later proved by Dirichlet. The proof is outside the scope of this paper, but is treated nicely in [2].

In contrast to Legendre's approximation for $\pi(x)$, C. F. Gauss asserted that

$$\pi \approx \mathrm{Li}(x) = \int_2^x \frac{dt}{logt} \tag{3}$$

was a better approximation for $\pi(x)$. He came to this logarithmic integral as a good approximation for the number of primes up to x as the result of computations which he conducted in his spare time. He would count primes in intervals of 1000 when he has "an idle quarter of an hour" [2, Appendix B], eventually compiling a study up of the distribution of primes up to 3000000. He suspected that the number of primes in an interval $[a, b)$ was approximately

$$\int_a^b \frac{dx}{\log x}. \tag{4}$$

His computations seemed to corroborate this suspicion. Gauss had performed these computations between 1792 to 1793, but have never shared his results until a correspondence with an astronomer Enke in 1849. It was in this correspondence that Gauss was made aware of the work of Legendre in the subject. Gauss observed that, though Legendre's error term, the difference between the approximation and actual amount of $\pi(x)$, was smaller than Gauss', the logarithmic integral's error term grew more slowly than Legendre's. Thus, Gauss suspected that the logarithmic integral was more accurate for higher numbers.

The next major step was Bernhard Riemann connection of the study of numbers to the complex plane as he developed what is now called the Riemann Zeta function. For complex s,

$$\zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s}. \tag{5}$$

The connection of (5) to the prime numbers comes from what is called Riemann's explicit formula

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}) \tag{6}$$

where ρ traverses the non-trivial zeros of (5) and

$$\psi(x) = \sum_{\substack{p^m \leq x \\ p \text{ prime} \\ m \in \mathbb{N}}} \log p. \tag{7}$$

The mathematics leading to this formula, while monumental, is not too important for this discussion. However, what is important is that Riemann's explicit formula intimately links the zeros of $\zeta(s)$ with prime numbers. This is important because it allows for results in prime numbers to be found by studying the Riemann's zeta function.

One such result is the Prime Number Theorem. Poussin and Hadamard proved independently in 1896 that $\zeta(1+it) \neq 0$ for all t. By proving this region of the Riemann Zeta function was free of zeros, the are necessary implications on the distribution of primes. In particular, they proved

Theorem 1.3 (Prime Number Theorem)**.**

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1. \tag{8}$$

This means that the number of primes less than x and the fraction $x/\log x$ get closer and closer together as $x \to \infty$. The prime number theorem can be stated in another way.

$$\pi(x) = \frac{x}{\log x} + o(\frac{x}{\log x}) \tag{9}$$

where $o(\frac{x}{\log x})$ is a function $f(x)$ satisfying the relationship

$$\lim_{x\to\infty} \frac{f(x)}{x/\log x} = 0 \tag{10}$$

so that $o(\frac{x}{\log x})$ is outpaced in growth compared to $\frac{x}{\log x}$ and therefore becomes insignificant at large x. In this way we can see how results concerning the zeros of the Riemann zeta function can lead to results about the distribution of primes.

Bernhard Riemann made many conjectures about the zeta function when he first studied it. Today, almost all of these conjectures have been answered. There is still one though which has seen no answer. It concerns the zeros of the zeta function, and thus also has profound implications for the distribution of primes.

Conjecture 1.4 (Riemann Hypothesis). *All non-trivial zeros of $\zeta(s)$ have real part 1/2.*

In essence, all non-trivial zeros lie on the line in the complex plane that has real part of 1/2. This allows for very sharp approximation on $\pi(x)$.

$$\textit{content...} \tag{11}$$

By extending the Riemann hypothesis to include the entire class of functions of which the zeta function is just one, we find the Generalized Riemann Hypothesis, or GRH. Sharp bounds can also be introduced for number of primes in arithmetic progression, $\pi_{q,a}$, by supposing the GRH.

$$\pi_{q,a}(x) = \frac{\mathrm{Li}}{\phi(q)} + O(x^{1/2}\log x). \tag{12}$$

2 The Hardy-Littlewood Circle Method

One of the most fundamental methods in the study of Additive Prime Number Theory is the Circle Method. The Circle Method was first conceptualized in *Asymptotic formulae in combinatory analysis* [5] by G.H. Hardy and S. Ramanujan in 1918, approximately 20 years after the development of the prime number theorem. However, the method was not formalized for general use until a series of papers by Hardy and J. E. Littlewood in 1919 in a paper titled *A new solution to Waring's problem*. Within this series of papers Hardy and Littlewood also addressed the use of the circle method in solutions of Goldbach's conjectures — see [3] and [4]. In this section we will overview the circle method and its application to the weak Goldbach conjecture. In particular, we will see that all applications of the circle method follow the same structure of 3 main parts:

1. Construction of the generating function and corresponding integral

2. Splitting of the integral into major and minor arcs
3. Estimation of the major and minor arcs

We will then examine a timeline of results which utilize the circle method. We will also discuss differences between Hardy and Littlewood's approach and a refinement on the process that was innovated by I. M. Vinogradov.

2.1 Hardy-Littlewood's Approach

Here we introduce the basic ideas behind the circle method and its application to a Goldbach's weak conjecture. Consider the set of prime numbers

$$A = \{2, 3, 5, 7, \ldots\}. \tag{13}$$

Then consider the function $r(n)$ which returns the number of representations of n as the sum of 3 primes, or 3 elements of A. We wish to derive a generating function for $r(n)$. Let

$$F(x) = \sum_{a \in A} x^a. \tag{14}$$

Take then the 3^{rd} power of $F(x)$

$$\begin{aligned} F(x)^3 &= \left(\sum_{a \in A} x^a \right)^3 \\ &= \underbrace{x^{a_1} \cdot x^{a_2} \cdot x^{a_3}}_{\text{First term of the expansion}} + \underbrace{x^{a_1} \cdot x^{a_2} \cdot x^{a_4} \cdot x^{a_{k+1}}}_{\text{Second term of the expansion}} + \ldots \\ &= x^{a_1+a_2+a_3} + x^{a_1+a_2+a_4} + \ldots \\ &= \sum_{j=1}^{\infty} r(j) x^j. \end{aligned} \tag{15}$$

As $r(n)$ is the coefficient to the x^n term in $F(x)^3$, it is the generating function of $r(n)$. In order to recover the coefficients of the generating function, Hardy and Littlewood set up an integral in the complex plane over a circle $\mathcal{C}$ with

$$\mathcal{C} = \{z \in \mathbb{C} \ : \ |z| = r, 0 < r < 1\}. \tag{16}$$

The integral they set up is

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{F(x)^3}{x^{n+1}} dx \tag{17}$$

Utilizing a basic result from complex analysis which essentially states

$$\frac{1}{2\pi i} \int_{\mathcal{C}} x^m dx = \begin{cases} 1 \text{ if } m = -1, \\ 0 \text{ otherwise} \end{cases} \tag{18}$$

they can evaluate the integral

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{F(x)^3}{x^{n+1}}dx &= \frac{1}{2\pi i}\int_{\mathcal{C}}\sum_{j=1}^{\infty}\frac{r(j)x^j}{x^{n+1}}dx \\ &= \sum_{j=1}^{\infty}\frac{r(j)}{2\pi i}\int_{\mathcal{C}}x^{j-n-1}dx \\ &= r(n)\end{aligned} \tag{19}$$

and recover $r(n)$. Of course, setting up such a recovery is functionally pointless and is merely used to illustrate the equivalence of the circle integral and $r(n)$. What they can, and do, accomplish instead is to establish bounds for the circle integral which depend on n and show that there is a constant N such that, for all $n > N$, the integral is greater than or equal to one. This is done by splitting the integral into what are called major and minor arcs. Hardy and Littlewood's bounds in their work are however dependent on the Generalized Riemann Hypothesis. Before expounding on the usage of these major and minor arcs, we examine a technical refinement by Vinogradov who works around the necessity of the GRH.

2.2 Vinogradov's Approach

In I. M. Vinogradov's 1937 work *Some theorems concerning the theory of primes* [9], he reformulates the approach used by Hardy and Littlewood so that the bounds are not conditional on the assumption of the Generalized Riemann Hypothesis. For notational simplicity, set

$$e(x) = e^{2\pi i x}. \tag{20}$$

Usually, x is taken in Vinogradov's formulation when $x \in \mathbb{R}/\mathbb{Z}$ where $\mathbb{R}/\mathbb{Z}$ denotes the quotient group under addition which is cyclic over some real interval of length 1. Note that $e(x)$ taken by $x \in [0, 1]$ yields the unit circle.

The first step in Vingadov's reformulation is a new choice of generating function. Rather than use a power series sum like $F(x)$ in equation (15), Vinogradov uses a trigonometric sum

$$f(x) = \sum_{a\in A} e(a). \tag{21}$$

By processes similar to (15) and (19), he develops the trigonometric generating function

$$f(x)^3 = \sum_{j=1}^{\infty} r(j)e(jx) \tag{22}$$

and the integral

$$\int_0^1 \frac{f(x)^3}{e(nx)}dx = r(n). \tag{23}$$

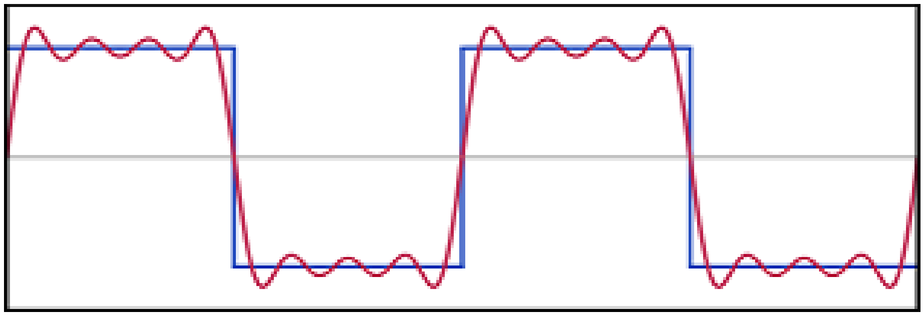

Figure 1: Example of a fourier series.—en.wikipedia.org

However, this integral is subject to issues of convergence which are solved in Hardy-Littlewood's formulation by the choice of the contour $\mathcal{C}$. A key insight of Vinogradov is that it is sufficient to consider

$$f_P(x)^3 = \sum_{j=1}^{P} r'_3(j)e(jx) \tag{24}$$

where $r'(n)$ works similarly to $r_(n)$ but on an correspondingly truncated and finite set. With an appropriately chosen P, dependent on n, $r'(n) = r(n)$ and, for all $P' \geq P$, $f_{P'}(x) = f(x)$. With this consideration,

$$\int_0^1 \frac{f_P(x)^3}{e(nx)} dx \tag{25}$$

converges well and appropriate bounds dependent on n can be found.

2.3 Major and Minor Arcs

A key insight to estimating the integral in (25) lies in the observation that it is a Fourier integral and, as a result, its behavior is nicely described by examining two different sections of the interval of integration: the so-called major and minor arcs. While in the major arcs, we expect for there to be certain asymptotic behavior, and while in the minor arcs, we expect for there to be different asymptotic behavior. To illustrate this concept with an unrelated Fourier series, refer to Figure 1. Those areas where the function is relatively consistent are analogous to the major arcs, just as the areas of transition are analogous to the minor arcs.

Informally, analysis of the integral in (25) reveals that when x is near a fraction with a "small" denominator, $f_P(x)$ is expected to be "large", hence the major arcs, and otherwise, when x is in a minor arc, it is transitioning across the unit circle, and the contribution of $f_P(x)$ is then expected to be relatively small.

Formally, we find a constant Q which is determined by P and denote a major arc, for relatively prime s and t, $(s,t)=1$, and $1 \leq s \leq t \leq Q$,

$$\mathfrak{M}_{s,t} = \left[\frac{s}{t} - \frac{Q}{tn}, \frac{s}{t} + \frac{Q}{tn}\right] \tag{26}$$

so that the set of all major arcs is

$$\mathfrak{M} = \bigcup_{t \leq P} \bigcup_{\substack{1 \leq s \leq t \\ (s,t)=1}} \mathfrak{M}_{s,t} \tag{27}$$

and the set of all minor arcs is correspondingly

$$\mathfrak{m} = [0,1] \backslash \mathfrak{M}. \tag{28}$$

The integral to find lower bounds on $r(n)$ can then be split up accordingly.

$$\int_{\mathfrak{M}} \frac{f_P(x)^3}{e(nx)} dx + \int_{\mathfrak{m}} \frac{f_P(x)^3}{e(nx)} dx \tag{29}$$

Bounding of the integral then rests on finding bounds for the behavior over the major arcs and then showing the any contribution by the minor arcs is subsumed by the asymptotic behavior of the main term of the estimation of the major arcs. Unfortunately the path to bounding Vinogradov's integral below, and by implication bounding $r(n)$, involves mathematics well outside the assumption of knowledge of this paper. We can mention the main points of Vinogradov's estimation. Bounding of the major arcs typically resulted in the product of the singular series, $\mathfrak{S}(n)$ and the singular integral, $J(n)$ where

$$\mathfrak{S}(n) = \prod_{p|n} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \nmid n} \left(1 + \frac{1}{(p-1)^3}\right) \tag{30}$$

and

$$J(n) = \int_{-\infty}^{\infty} \frac{\left(\int_2^n \frac{e(\beta u)}{\log u} du\right)^3}{e(\beta n)} d\beta. \tag{31}$$

Thus, improvement of the bounds of the major arcs typically relied on improvement on the bounds of the singular integral, $J(n)$. On the other hand, study of the minor arcs centers around the study of an expression S with

$$S = \sum_{k=1}^{n} \sum_{u < l \leq \frac{n}{k}} \sum_{\substack{d|k \\ d \leq u}} \mu(d) \Lambda(l) e(akl) \tag{32}$$

where the choice of u depends on n. Following, we examine the development of results in improvements to proofs Goldbach's weak conjecture.

2.4 Results utilizing the Circle Method

In this section we will give an overview of results in the endeavor towards proof of Goldbach's weak conjecture. In 1920 Hardy and Littlewood introduced the circle method as a tool to address Goldbach's conjectures [3]. Under the assumption of the Generalized Riemann Hypothesis, which gave strong bounds for primes in arithmetic progression, they found

$$r(n) \sim \frac{n^2}{2(\log n)^3}\mathfrak{S}(n) \tag{33}$$

so that

$$\lim_{n\to\infty} \frac{r(n)}{\frac{n^2}{2(\log n)^3}\mathfrak{S}(n)} = 1. \tag{34}$$

In particular using these bounds confirmed Goldbach's weak conjecture for all $n \geq 10^{50}$.

Vinogradov improved upon Hardy and Littlewood's efforts by overcoming the reliance on the Generalized Riemann Hypothesis [9]. By developing a new method for bounding sums over primes without using the Generalized Riemann Hypothesis, he was able to remove the reliance. He found then

$$r(n) = \frac{n^2}{2(logn)^3}\mathfrak{S}(n) + O(n^2(\log n)^{-4}). \tag{35}$$

This result is similar to that of Hardy and Littlewood with just the added asymptotic function. This is the result of using Siegel-Walfisz's theorem providing bounds for primes in arithmetic progression. While this theorem allowed for Vinogradov's result, it was deficient in that he was left unable to evaluate the right side of the above equation, and was thus unable to compute the constant after which all odd numbers would satisfy the weak Goldbach conjecture. It was not until 1939 that Borodzin was able to overcome the deficiency of the Siegel-Walfisz theorem and compute a numerical constant [6]. He found that the weak conjecture was satisfied using Vinogradov's method for all $n \geq 3^{3^{15}}$.

This numerical constant was marginally improved over time but failed to be lowered to the point such that all remaining cases less than the constant could be exhaustively verified computationally [8]. Ground was broken though in 1997 when Deshouillers, Effinger, Riele, and Zinoviev reimplemented the use of the Generalized Riemann Hypothesis to improve the numerical constant. They lowered it to the point that all $n \geq 10^{20}$ satisfy the weak conjecture and were able to exhaustively verify all numbers below that constant.

Gold was proverbially struck in 2012 when H. A. Helfgott published a series of papers in which he definitively proved Goldbach's weak conjecture by unconditionally proving all $n \geq 10^{30}$ satisfied the conjecture and completed exhaustive verification below that point. There were several key points of Helgott's proof which resulted in his success, one of which was a verification in a large region of the complex plane that the Generalized Riemann Hypothesis. This allowed for sharp enough bounds for primes in arithmetic progression.

3 Conclusions

That the results brought about by the utilization of the Circle Method have come so far is nothing short of amazing. Such results are indicative of the persistence of mathematicians through the ages. The ingredients of the successive development suggests a correlative relationship with developments on the bounds of primes in arithmetic progression. Other improvements were technical methods developed in order to better facilitate the use of such bounds. Efforts in providing the machinery to utilize the bounds for the strong conjecture have yet been unsuccessful, but, considering the success in the case of the weak conjecture, they certainly warrant further study.

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