PROBLEMS for Great Theorems

Chapter 1. Hippocrates' Quadrature of the Lune

VOLUME OF A TRUNCATED SQUARE PYRAMID

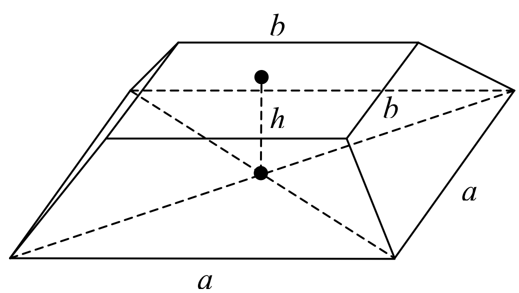
1. Suppose we know that the volume of a pyramid with square base is a third of the product of the height and the area of the base. We want to derive a formula for the volume of a truncated square pyramid having lower base of side *a*, upper base of side *b*, and height *h*.

   (a) If we put the top back onto the truncated pyramid so that the restored pyramid has height *H*, use similar triangles to show that $H = \dfrac{ah}{a-b}$.

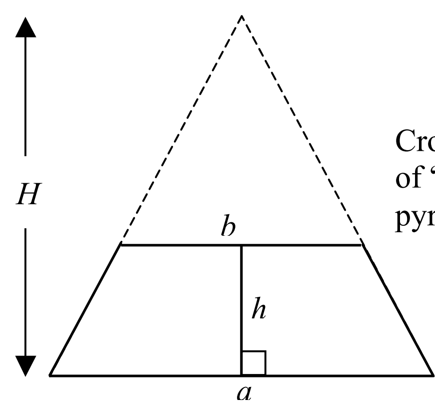
   (b) Now subtract the small upper pyramid from the large one to derive the formula

   $$V = \frac{1}{3}h\left(a^2 + ab + b^2\right)$$

   for the volume of the truncated pyramid.



Truncated pyramid



Cross section of "restored" pyramid

   (c) Recall the Egyptian recipe:

   "If you are told: A truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top, you are to square this 4, result 16; you are to double 4, result 8; you are to square 2, result 4; you are to add the 16, the 8, the 4, result 28; you are to take a third of 6, result 2; you are to take 28 twice, result 56. See, it is 56. You will find the formula right!"

   Show that the formula in (b) yields the correct volume for this problem.

2. Derive the volume of the truncated pyramid in Problem #1(b) using the calculus technique known as "volumes by slicing."

IRRATIONALITY OF $\sqrt{2}$

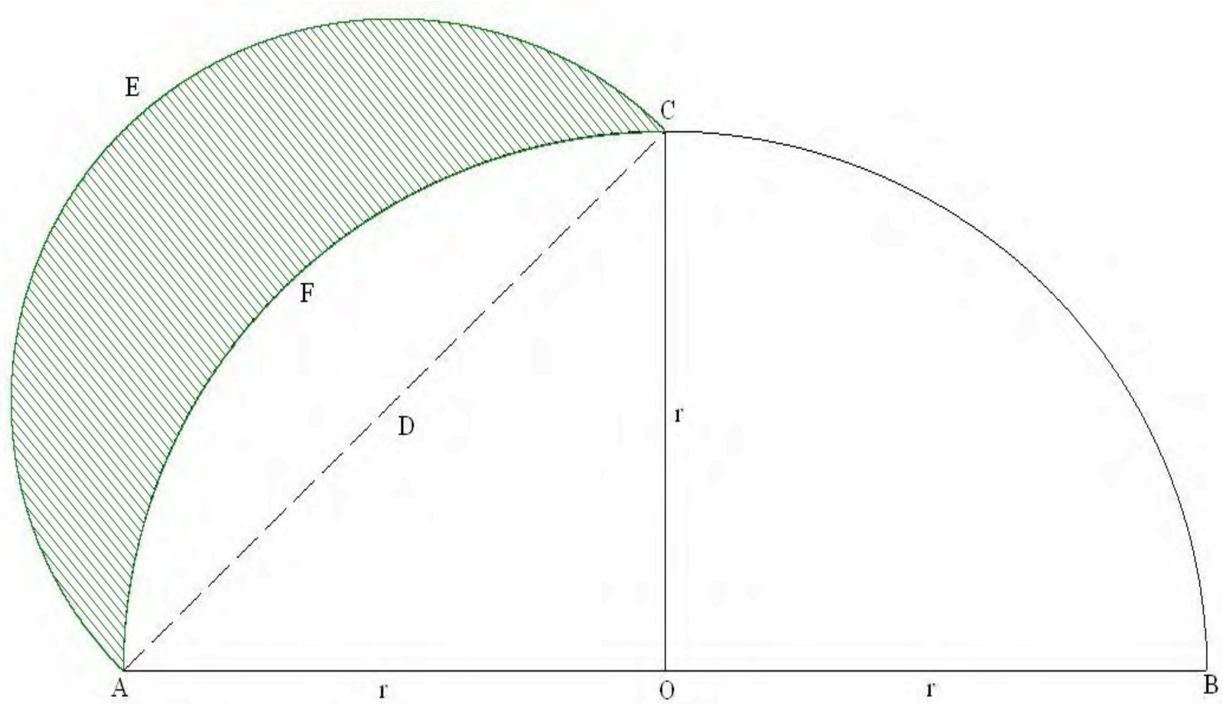
One of the great discoveries of the School of Pythagoras was that the side and diagonal of a square are incommensurable magnitudes – which, in numerical terms, says that $\sqrt{2}$ cannot be written as the ratio of two integers. There follows a nice proof of the irrationality of $\sqrt{2}$.

3. (a) If m is a positive integer, explain why each prime in the factorization of m^2 must occur an even number of times.

   (b) Now suppose $\sqrt{2}$ is rational. That is, $\sqrt{2} = \frac{a}{b}$ for integers a and b. Square and cross-multiply to conclude $a^2 = 2b^2$. From this, determine the number of times the prime "2" occurs on each side of this equation and thereby derive a contradiction to the unique factorization theorem.

QUADRATURE OF THE LUNE

4. Below is the figure Hippocrates used in his geometric proof of the quadrature of the lune. In what follows, we re-prove his result algebraically (which is of course totally alien to the thrust of Greek mathematics).



   (a) Letting r be the radius of semicircle ACB, find Area (ΔAOC) in terms of r.

(b) Now determine the area of segment AFCD in terms of r. (Use the post-Hippocratean fact that the area of a circle is πr^2.)

(c) Next, find the area of semicircle AECD in terms of r.

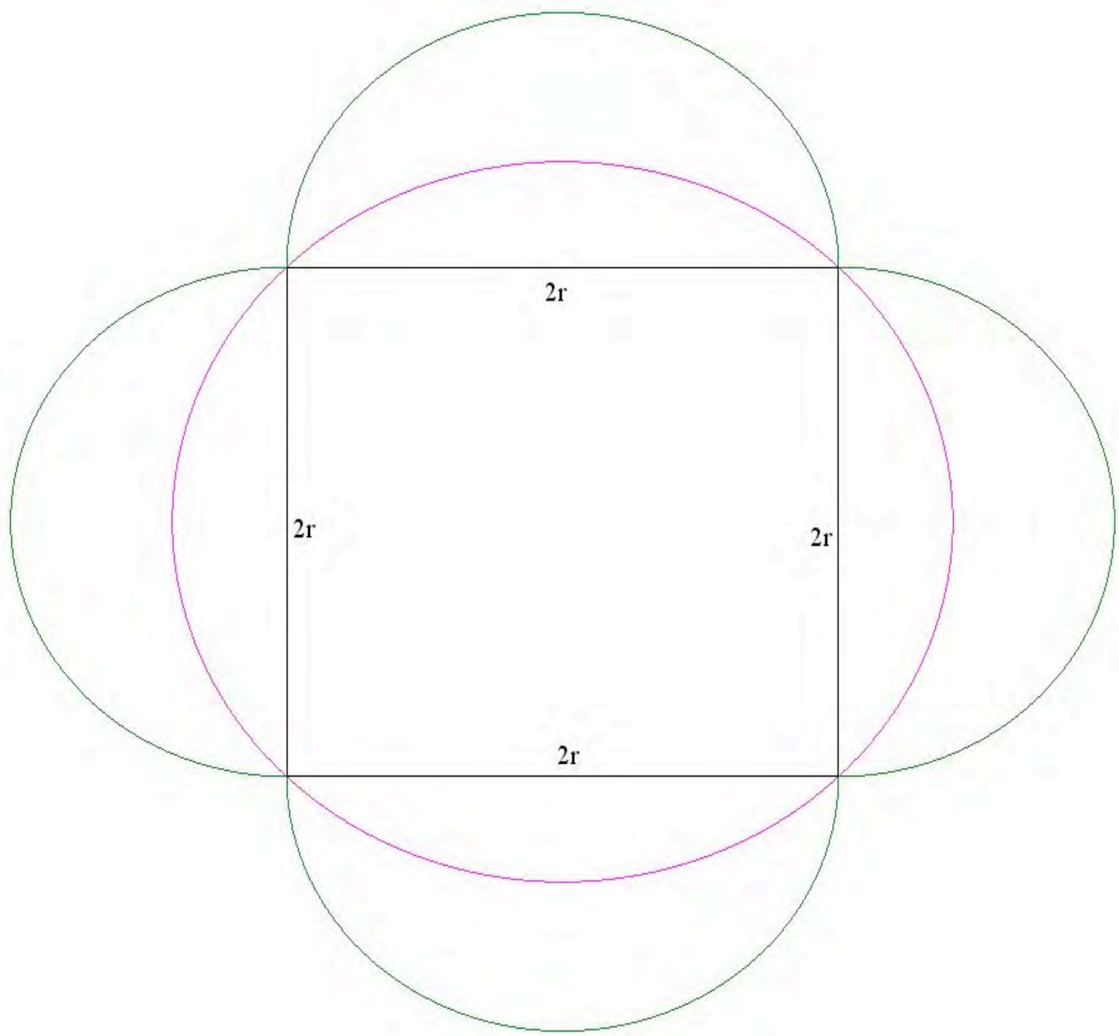
(d) Using (b) and (c), find the area of lune AECF in terms of r.

(e) Finally, put all of this together to explain why Hippocrates' lune is quadrable.

LEONARDO'S "DYNAMIC DISSECTION" QUADRATURE

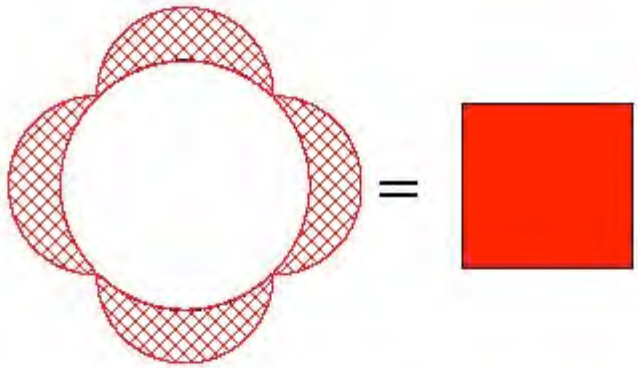
5. Here is an algebraic version of Leonardo's quadrature:

(a) Begin with a square of side $2r$ and upon each side construct a semicircle. Simultaneously circumscribe a circle about the square (as shown). Find the combined area of the four semicircles in terms of r.



(b) Now find the area of the circumscribed circle in terms of r. Notice anything?

(c) According to (a) and (b), we get the same area if we remove the circumscribed circle from the figure above as we get if we instead remove the four semicircles from the figure above.



Use this to explain why a single lune from the picture is quadrable.

SQUARING THE CIRCLE?

6. Trace through the (fallacious) argument attributed to Hippocrates to the effect that the quadrature of the lune implies the quadrature of the circle (*Journey Through Genius*, pp. 20-22). Be sure you can identify where the error occurs. Does anyone seriously believe this would have deceived the great Hippocrates?

ALGEBRAIC/TRANSCENDENTAL NUMBERS – PART ONE (MORE LATER)

7. (a) Show that $\sqrt[3]{3}+\sqrt{2}$ is algebraic by determining a specific polynomial with integer coefficients which this number satisfies. (HINT: Let $x=\sqrt[3]{3}+\sqrt{2}$; then $x-\sqrt{2}=\sqrt[3]{3}$ and cube both sides, etc.)

   (b) Likewise, find a polynomial that guarantees that $\dfrac{1}{\sqrt[3]{3}+\sqrt{2}}$ is algebraic.

8. Prove that, if c is algebraic, so is $\sqrt[k]{c}$, for $k = 2, 3, 4, \ldots$. To get started, note that c being algebraic implies that there exists a polynomial

$$p(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0,$$

where the coefficients are integers, such that $p(c) = 0$. Then consider the polynomial

$$q(x)=a_nx^{kn}+a_{n-1}x^{kn-k}+\ldots+a_2x^{2k}+a_1x^k+a_0.$$

9. Prove that, if c is algebraic, so is $\frac{1}{2}c$. Of course, you'll need to find a polynomial $r(x)$ with integer coefficients such that $r\left(\frac{1}{2}c\right) = 0$.

10. Prove that if $c \neq 0$ is algebraic, so is $1/c$. (Possible HINT: Problem #7(b))

11. Prove that if c is algebraic, so is $1 + c$.

12. As we noted, in 1882 Ferdinand Lindemann proved that π was transcendental. Use the previous problems to explain why π^7, $\pi + \pi$, $\pi - 1$, and $1/\pi$ are transcendental as well.

13. Prove or disprove the following conjectures:

    (a) The difference of two transcendentals is transcendental.

    (b) The product of two transcendentals is transcendental.

Chapter 2. Euclid's Proof of the Pythagorean Theorem

THE "TRIANGLE INEQUALITY" IN *THE ELEMENTS*

14. (a) Do the proof of Proposition I.18. Note that it hinges on the Exterior Angle Theorem.

    (b) Do the proof of Proposition I.19. This is a "double *reductio ad absurdum*" argument – that is, it shows that of the three possibilities AC > AB, AC = AB, and AC < AB, two lead to contradictions.

    (c) Now prove Proposition I.20: "In any triangle, two sides taken together in any manner are greater than the remaining one" – the Triangle Inequality the Epicureans thought was patently obvious even to an ass.

THE LOGICAL ROLE OF THE PARALLEL POSTULATE

15. None of the propositions in Book I prior to I.29 uses the Parallel Postulate in its proof, whereas all of the later results in Book I depend on the Parallel Postulate, with a single exception. Find it. Speculate as to why Euclid didn't put it before I.29.

SOME MISSING EUCLIDEAN PROPOSITIONS

One thing that some people find surprising about Book I of the *Elements* is not the familiar propositions that appear but the familiar ones that are omitted. Everyone may have his or her own "favorite" that appears nowhere in Euclid, but the following are mine:

16. Prove a result sometimes called "Playfair's Postulate":

    Through a point not on a line there can be drawn one and only one parallel to the given line.

    Assume that you are inserting this in the *Elements* as Proposition I.31½.

17. Prove the following result as if it were to be included as Proposition I.16½:

    From a point not on a line there can be drawn one and only one perpendicular to the given line.

    Note that proving the uniqueness of parallels (Problem #16) requires the Parallel Postulate, but proving the uniqueness of perpendiculars (Problem #17) does not.

18. Prove that in a right triangle the midpoint of the hypotenuse is equidistant from the triangle's three vertices. Give a proof that could be inserted immediately after Proposition I.32. Be careful – this one's a bit tricky.

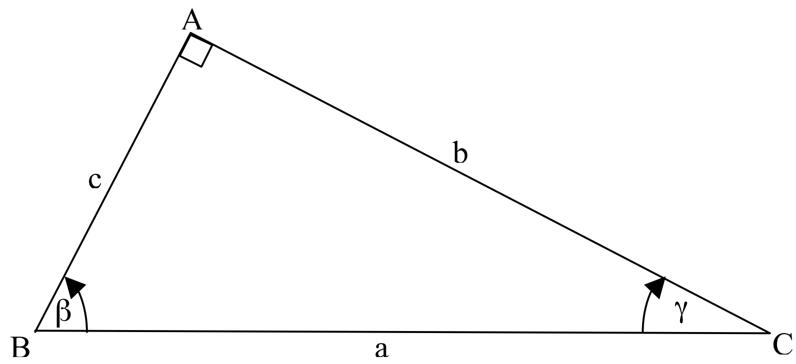
19. Now suppose Euclid wanted to follow up Problem #18 with a proof of the Side-Side congruence scheme for right triangles – that is, if two right triangles have two sides of one respectively congruent to two sides of the other, then the triangles are themselves congruent. This is easy to do if we already have proved the Pythagorean Theorem, since that result transforms Side-Side into Side-Side-Side. But suppose Euclid wanted to prove it prior to Proposition I.47. Provide the proof. Note that there are really two cases – when the equal sides are both legs (easy as π!) and when the equal sides are a leg and a hypotenuse.

PYTHAGOREAN THEOREM, OVER AND OVER AND OVER AND …

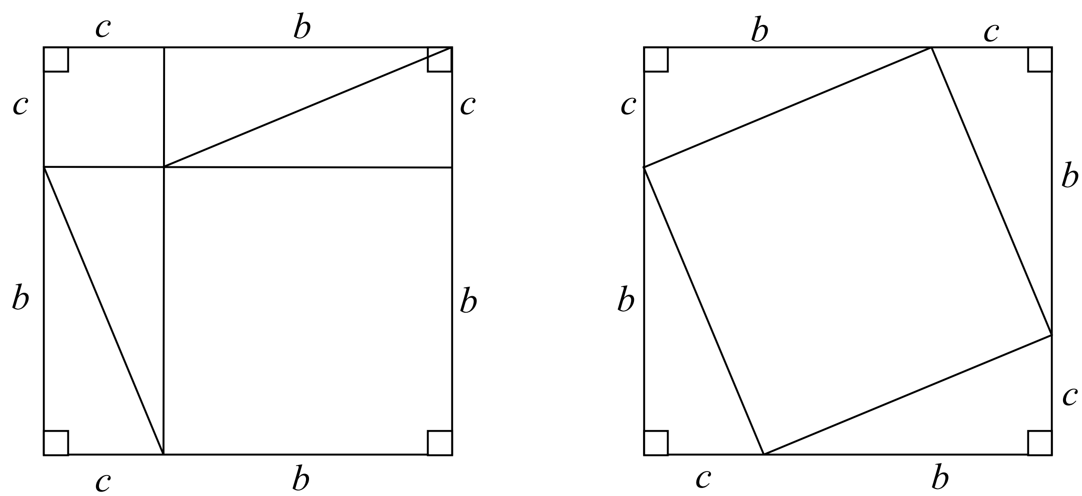
Certainly no mathematical proposition boasts a greater number of different proofs than the theorem of Pythagoras, for which well over 400 different arguments can be found in E.S. Loomis' *The Pythagorean Proposition* (NCTM, 1968). These, along with Euclid's proof from the *Elements,* may seem to approach mathematical overkill; on the other hand, they should satisfy even the most hard-boiled skeptic.

For each proof outlined below, we begin with the right triangle BAC as shown below.

As you work through the details of the proofs in Problems #20–26, ask yourself where in each one we use the assumption that we have a **right** angle.



20. The proof that many attribute to Pythagoras himself (i.e., the "ox-killer proof"):



Begin with identical squares of side $b + c$, decomposed as shown.

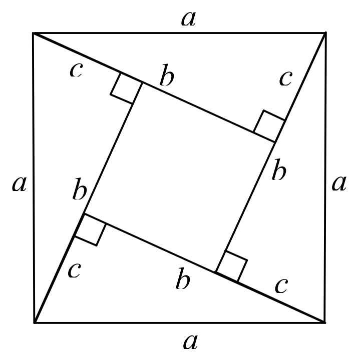
(a) Show the area of the left-hand square is $2bc + b^2 + c^2$.

(b) Prove that the inner figure on the right is itself a square.

(c) Show the area of the right-hand square is $2bc + a^2$.

(d) Now prove the Pythagorean Theorem from these preliminaries.

21. This proof is due to the 12th century Hindu mathematician Bhaskara:



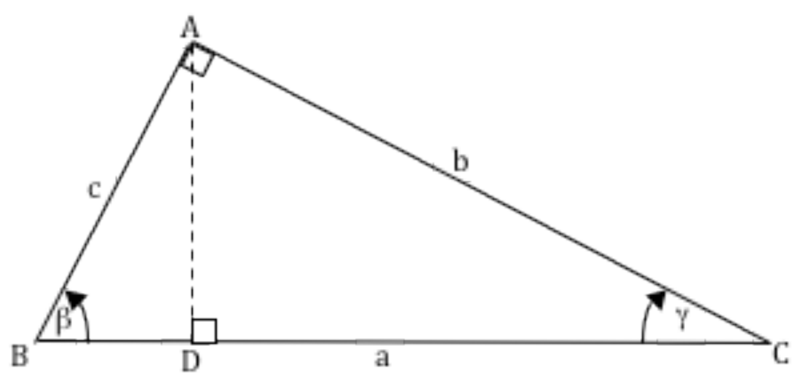
Assemble four copies of the original right triangle as shown.

(a) Prove that the large quadrilateral is a square.

(b) Prove that the inner quadrilateral is a square.

(c) Equate the area of the large square with the total areas of the small square and the four triangles to prove the Pythagorean Theorem.

22. This proof of the Pythagorean Theorem is usually credited to the 17th century British mathematician John Wallis, although it surely had been discovered prior to him. It is regarded as the shortest proof of all.

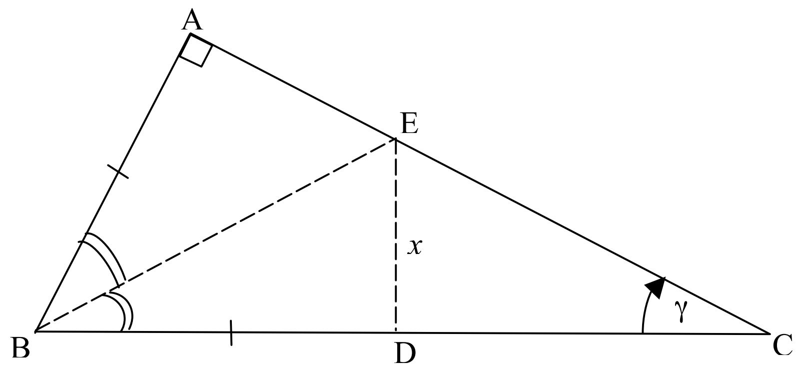


(a) From A draw altitude AD to the hypotenuse and prove ΔACD ~ ΔBAC ~ ΔBDA.

(b) Conclude that $\frac{b}{CD} = \frac{a}{b}$ and $\frac{c}{DB} = \frac{a}{c}$. From this, complete the proof.

23. Here is another similarity proof:

In right triangle BAC, mark off BD = BA, then bisect ∠ABC by line BE, where E is on side AC. (By the way, could you give a convincing proof – independent of the diagram – guaranteeing that the line bisecting ∠ABC MUST in fact meet AC?) Also, draw ED and call its length x.



(a) Prove ΔBAE ≅ ΔBDE.

(b) Use (a) to show ΔEDC ~ ΔBAC.

(c) Set up the resulting proportions from (c), use these to eliminate x and thereby derive the Pythagorean Theorem.

24. Here is yet another similarity proof of the Pythagorean Theorem.

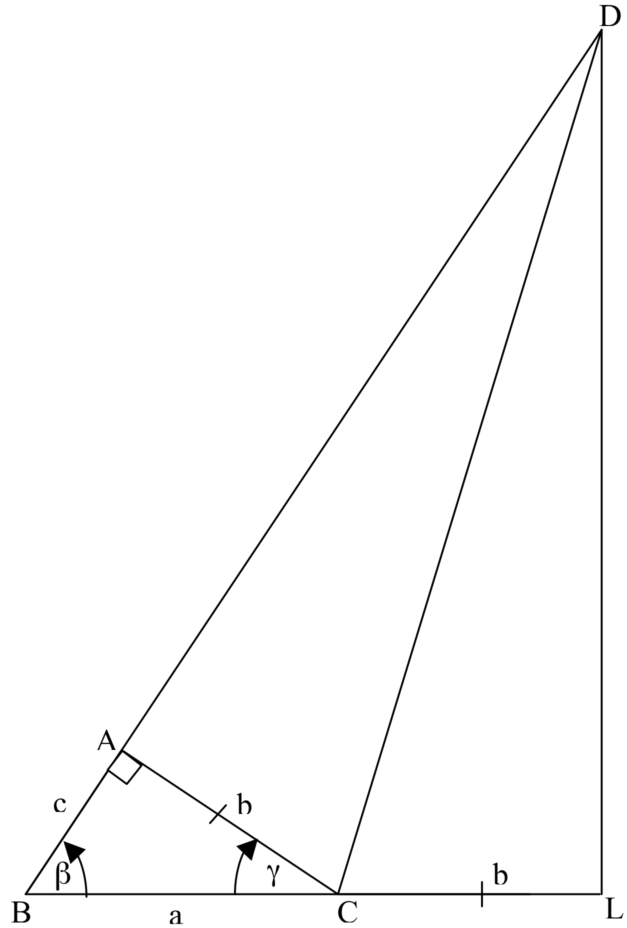
Begin with right ΔBAC and extend BC to L so that AC = LC. Construct CD bisecting ∠ACL, where D is on segment BA extended (as shown). Draw DL.

(a) Prove ΔBAC ≅ ΔDLC.

(b) Use (a) to show ΔBAC ~ ΔBLD.

(c) Deduce that DL = $\dfrac{(a+b)b}{c}$.

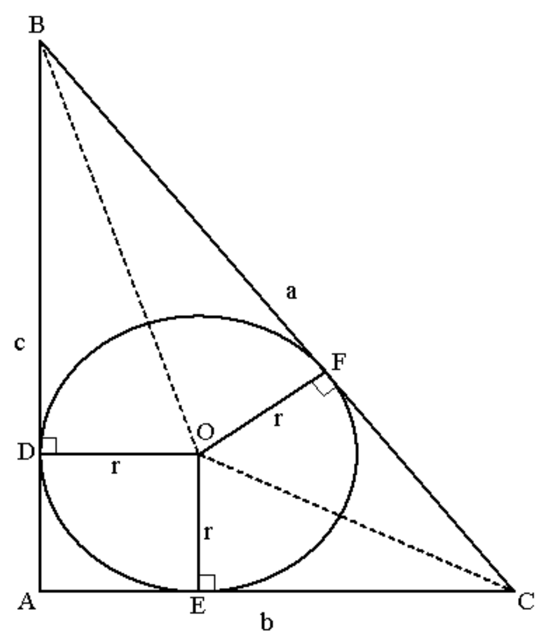
(d) Now attack the area of large ΔBLD in two ways: First, treat it as one large triangle; second regard it as the amalgamation of the three sub-triangles. Equate the areas of these figures, do a bit of algebra, and presto!



25. Here is a clever "inscribed circle" proof of the Pythagorean Theorem.

Again begin with ΔBAC. Inscribe within it a circle having center O and radius *r*. Draw OD, OE, and OF as shown. You may use one key fact from Euclid about inscribed circles:

Proposition IV.4 guarantees that the point O – i.e., the center of the inscribed circle – is the intersection of the three bisectors of the angles of the triangle.



(a) Carefully prove ΔBOD ≅ ΔBOF and ΔCOE ≅ ΔCOF.

(b) Prove the quadrilateral AEOD is a square.

(c) Explain why $b + c - 2r = a$ and solve this for r.

(d) Why is Area (ΔBAC) $= \dfrac{bc}{2}$?

(e) Decomposing ΔBAC into the square and the two pairs of congruent triangles, prove that

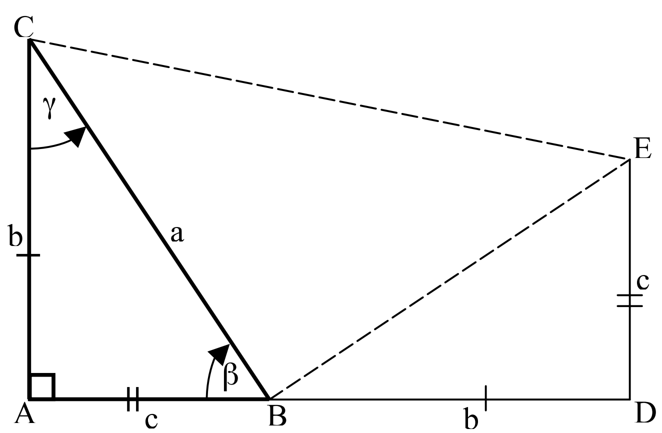
$$\text{Area }(\Delta BAC) = r\,(b + c - r).$$

(f) Finally, equate the two expressions for Area (ΔBAC) from (d) and (e), substitute for r from (b), and do some algebra to derive the Pythagorean Theorem.

26. The last proof here is due to Congressman (later President) James A. Garfield of Ohio, who published it in the *New England Journal of Education* in 1876.

(a) As a preliminary, prove that the area of a trapezoid is half the product of the height and the sum of the bases.

(b) Now consider right ΔBAC. Extend AB to D so that BD = b and construct DE ⊥ AD at D, with DE = c. Draw BE and CE. With this out of the way, prove that ΔBAC ≅ ΔEDB.



(c) Explain why CB = BE and why ∠CBE is right.

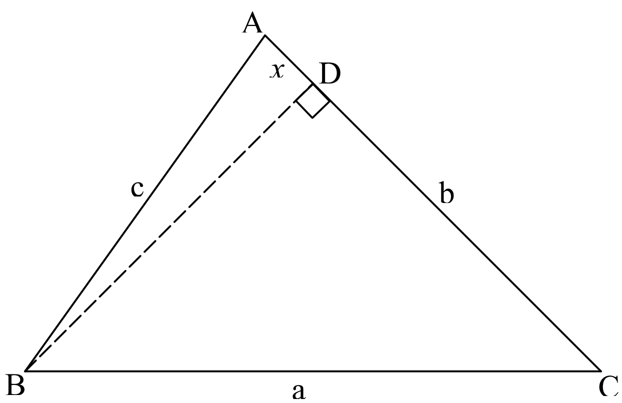
(d) Explain why quadrilateral CADE is a trapezoid.

(e) Find the area of the trapezoid in two different ways: first by using the formula from part (a) and second by assembling the trapezoidal area as the sum of its three component triangles. From this, prove the Pythagorean Theorem.

27. The previous seven proofs give an array of possible ways of establishing the theorem of Pythagoras, but it should be noted that Euclid used none of them in the *Elements*. Which of these could he have used as Proposition I.47 and which could he not have used? In this light, do you give him high marks for the proof he actually devised for the *Elements*, or did he miss an easy one?

CONVERSE OF THE PYTHAGOREAN THEORM

28. Begin with ΔBAC and assume that $a^2 = b^2 + c^2$. From B construct BC ⊥ AC and let AD have length x.



(a) Explain why $c^2 - x^2 = a^2 - (b - x)^2$.

(b) Now solve the equation from part (a) for x and go on to explain why Δ BAC must be right. Where in here did you use the hypothesis?

29. The previous argument, like Euclid's Proposition I.48, proves the converse of the Pythagorean theorem by using the Pythagorean Theorem itself. But of course we do not always use a theorem in the proof of its converse (e.g., Propositions I.27 and I.29). I'd be curious (1) if anyone could find a proof of the converse of the Pythagorean Theorem that does not require the Pythagorean Theorem itself. I'd be more curious (2) if someone could find a proof of the converse that would fit perfectly as Proposition I.46½. And, I'd be most curious (3) if someone, having proved the converse as Proposition I.46½, could then derive the Pythagorean Theorem from the converse instead of vice versa. In short, could Euclid have reversed the order of proof of these two great results while still keeping them at the end of Book I?

The next problem – Problem 30 – is a proof of the converse independent of Proposition I.47 that satisfies (1), but it requires results about circles and similar triangles, which don't come until Books III and IV of the *Elements*. As to queries (2) and (3), I haven't a clue.

30. The following proof of the converse uses circles and similarity.

Begin with ΔBAC where we assume that $a^2 = b^2 + c^2$.

With center B and radius BC, construct a circle (Postulate 3). Extend BA in both directions, forming a diameter that meets the circle in D and E (Postulate 2). Next, extend CA to F on the circle (Postulate 2) and draw CE, DF, and BF.

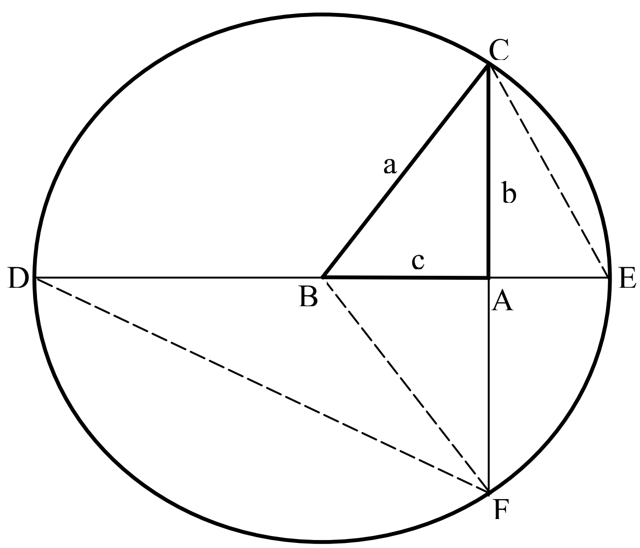
(a) Prove that $\Delta CAE \sim \Delta DAF$.

(b) Conclude that $\overline{DA}/\overline{AF} = \overline{CA}/\overline{AE}$ and then deduce $\overline{AF} = b$.

(c) Prove that $\Delta BAC \cong \Delta BAF$.

(d) Finish up by showing $\angle BAC$ is right.

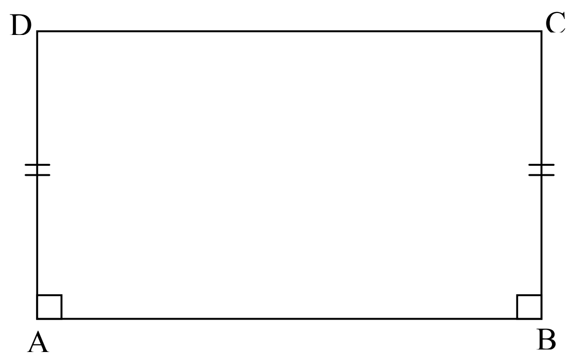
(e) Where in your argument did you use the hypothesis? (that $a^2 = b^2 + c^2$)



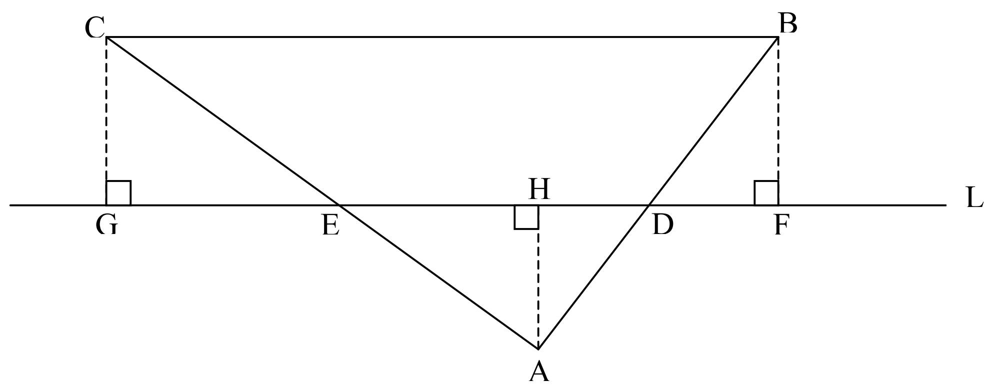
A QUICK TOUR THROUGH HYPERBOLIC GEOMETRY

For the next few problems, suppose we are operating under Saccheri's Hypothesis of the Acute Angle (HAA). That is, we suppose that the congruent summit angles of any Saccheri Quadrilateral are acute. The thrust of these problems is to prove (1) that under HAA, triangles have fewer than 180° and (2) that under HAA the Pythagorean relationship between hypotenuse and legs of a right triangle cannot hold. This development is taken from Richard Trudeau's *The Non-Euclidean Revolution*.

31. Under HAA, prove that if ABCD is a Saccheri Quadrilateral as shown, then $\overline{AB} \neq \overline{CD}$.



32. Again, we are working under HAA. Let ΔABC be given. Bisect AB at D and AC at E. Draw line *L* between D and E and construct BF ⊥ *L*, CG ⊥ *L*, and AH ⊥ *L*.



(a) Prove ΔAHD ≅ ΔBFD and ΔAHE ≅ ΔCGE.

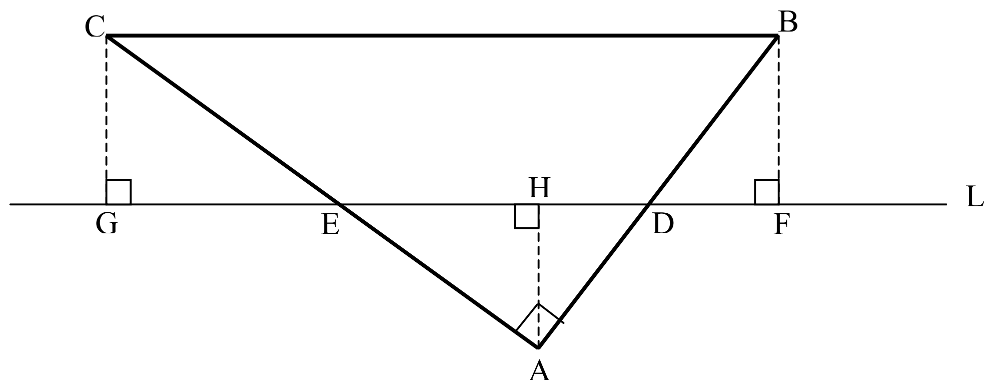
(b) Conclude that BCGF is a Saccheri Quadrilateral.

(c) Prove that $\overline{DE} = \frac{1}{2}\overline{FG}$

(d) Finally, prove that the sum of the measures of the three angles in ΔABC is just ∠FBC + ∠GCB.

33. Now prove that under HAA, the angle sum of a triangle must be less than 180°.

34. Finally, we'll prove that under HAA, the Pythagorean Theorem is not valid. Of course, we could immediately reject the Pythagorean Theorem since "squares" do not exist under HAA (that is, under HAA, all quadrilaterals contain fewer than 360° and thus there is no quadrilateral possessing four right angles). But we could instead interpret the Pythagorean Theorem as a relationship about (arithmetic) squares; while the "square of AB" might be meaningless under HAA, "a^2" is still a numerical value. So, the goal is to show that under HAA it is false that $a^2 = b^2 + c^2$ for right triangles.



Assume, for the purpose of contradiction, that the Pythagorean Theorem holds. Begin with right triangle BAC as shown above and repeat the constructions of Problem #32.

(a) Prove that $\overline{BC} = 2\,\overline{DE}$.

(b) Now derive a contradiction by looking back at Problem #31. Conclude that the Pythagorean Theorem fails in hyperbolic geometry.

35. We showed that a valid congruence scheme in hyperbolic geometry is AAA. What about Angle-Angle? That is, if two triangles have two angles of one congruent to two corresponding angles of another, must the triangles be congruent? Prove your result.

Chapter 3. Euclid and the Infinitude of Primes

A SMATTERING OF RESULTS FROM *THE ELEMENTS*, BOOKS II - IV

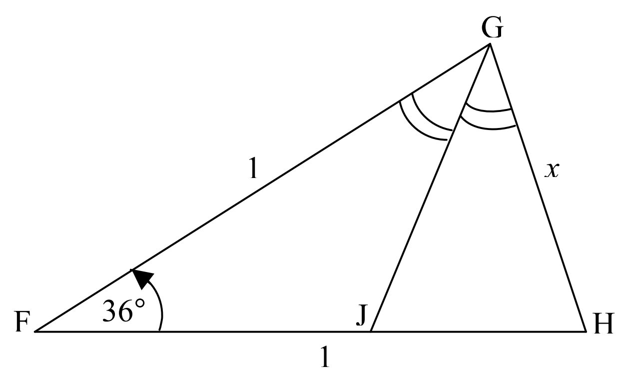
36. Run through the easy proof of Proposition II.1. Translate this into a simple algebraic identity.

37. Prove Proposition II.13. Do you recognize a famous result from trigonometry here?

38. Prove Proposition III.1, the construction and proof for finding the center of a given circle.

39. Euclid gave his clever construction of the regular pentagon in Proposition IV.11, but following is the more familiar and more algebraic one:

   (a) For future reference, consider isosceles ΔFGH where $\overline{FG} = \overline{FH} = 1$ and $\overline{GH} = x$. Suppose also that ∠GFH = 36°. Construct GJ bisecting ∠FGH. Use similar triangles to prove that $x = \dfrac{\sqrt{5}-1}{2}$.

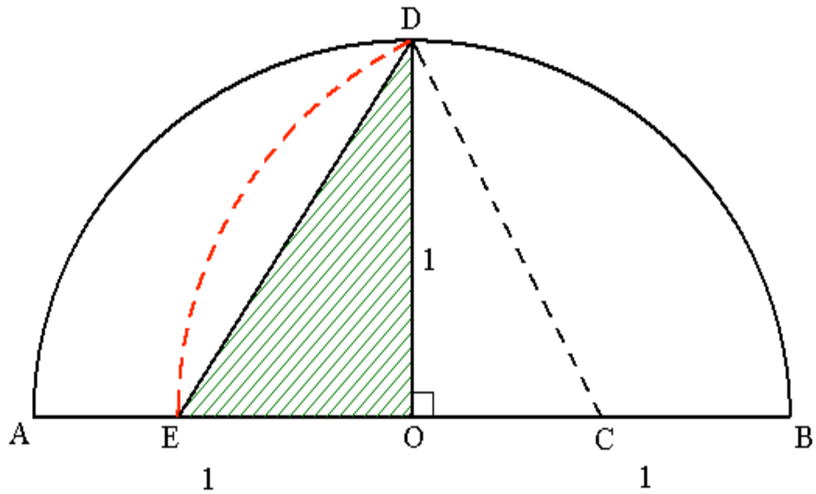


   (b) Now, for the regular pentagon. Begin with a circle of unit radius with center O and diameter AB as shown below. Bisect OB at C and construct radius OD ⊥ AB. With center C and radius $\overline{CD}$, draw an arc cutting AB at E. This yields the remarkable ΔEDO. Prove the following three properties about this triangle:

$\overline{OE}$ is the side of a regular inscribed decagon.

$\overline{OD}$ is the side of a regular inscribed hexagon.

$\overline{DE}$ is the side of a regular inscribed pentagon.



40. If you haven't done so for a while, get out a (big) piece of paper and construct a regular pentagon with the technique from Problem #39.

41. If you haven't done so for a while, use your construction in Problem #40 to construct a regular pentadecagon (i.e., 15-gon) based on Euclid's argument from Proposition IV.16.

THE DISTRIBUTION OF PRIMES

42. Use the table of primes to count the number of primes in each batch of 250 consecutive integers – i.e., 1 – 250, 251 – 500, ... up to (say) 2001 – 2250. Can you see why one might suppose we'd eventually run out?

43. A famous old problem is to find a polynomial function $f(x)$ so that, for each positive integer $n = 1, 2, 3, \ldots$, the result $f(n)$ is a prime. Among other things, this would give an alternate proof of the infinitude of primes, since there must be infinitely many *different* outputs among the numbers $f(1)$, $f(2)$, $f(3)$, … .

    (a) Prove this last statement. That is, assume an n^{th} degree polynomial only generated finitely many different values among $f(1)$, $f(2)$, … and derive a contradiction.

    (b) A good candidate for $f(x)$ is $f(x) = x^2 + x + 41$. Find and factor the first three integers which, when substituted into this polynomial, yield composites.

44. Explain how you could find 100 consecutive numbers, none of which is prime. How about a billion consecutive non-primes? (HINT: Factorials!)

THE GOLDBACH CONJECTURE

This is one of the most troubling unsolved problems in mathematics, for it is so ridiculously simple to state: any even number greater than or equal to 4 can be written as the sum of two primes. Alas, no one from Goldbach to Erdös has proved it.

45. (a) Verify the Goldbach Conjecture for 38, 538, and 1988.

(b) Is the conjecture true if we replace "even" with "odd"?

(c) Is the conjecture true if we replace "sum" with "product"?

46. Suppose that tomorrow someone proved the Goldbach Conjecture. Show that it would then follow that any integer (even or odd) greater than 5 can be written as the sum of three primes.

DE POLIGNAC'S CONJECTURE

47. In 1848, De Polignac asserted: "Every odd number can be expressed as the sum of a power of 2 and a prime."

(a) Verify this for odd numbers 3, 5, 7, 9, 11, 13, 15, 17, and 19.

(b) Verify this for 61 and 119.

(c) Find a counterexample to De Polignac's assertion.

PRIME TWINS AND PRIME TRIPLETS

48. Recall that if p and $p + 2$ are both prime, we call them "twin primes."

(a) Find the first ten pairs of twin primes.

(b) How many pairs of twin primess are there between 100 and 199? Between 1500 and 1599?

49. Clearly, any positive integer must assume one of the six forms

$6k$, $6k + 1$, $6k + 2$, $6k + 3$, $6k + 4$, or $6k + 5$

for some non-negative integer k.

(a) Use this to show that any prime greater than 3 is either one more or five more than a multiple of 6.

(b) Now suppose we say that if p, $p + 2$, and $p + 4$ are all prime, we'll call them "triplet primes." While it is unknown whether there are finitely or infinitely many pairs of prime twins, we can determine whether the following conjecture is true or false:

"There are infinitely many prime triplets."

Do so, using (a).

PRIMES OF THE FORM $p = 4n + 3$

From Euclid's Proposition IX.20, we know we never run out of primes. In addition, since all primes but 2 are odd, we can divide the odd primes into two categories – those that are one more than a multiple of 4 (e.g., 5, 13, 17, 29, …) and those that are three more than a multiple of 4 (e.g., 3, 7, 11, 19, …). Obviously, at least one of these two categories of primes must be infinite. In what follows, we'll modify Euclid's proof of Proposition IX.20 to show that there are infinitely many primes of the form $p = 4n + 3$.

50. (a) Prove that the product of two numbers, each of which is one more than a multiple of 4, is itself one more than a multiple of 4. In other words, if $M = 4m + 1$ and $N = 4n + 1$, then MN also has this form. (Of course, all numbers in sight here are positive integers.)

(b) Now suppose that $\{a, b, c, d, \ldots, e\}$ is a finite collection of primes (in other words, an "assigned multitude"), each having the form $4n + 3$. We introduce the new number $f = 4(abcd\ldots e) - 1$. Mimic Euclid's two cases from Proposition IX.20 to show that there must be a prime of the form $4n + 3$ not among the original multitude. Conclude that there are infinitely many primes of the form $4n + 3$. (HINT: Unique Factorization Theorem and part (a))

51. Are there finitely or infinitely many composites of the form $4n + 3$? Explain.

52. One of the most remarkable theorems from number theory states that a prime of the form $4n + 1$ can be written as the sum of the two perfect squares in one and only one way, whereas a prime of the form $4n + 3$ cannot be written as the sum of two perfect squares in any fashion whatsoever. The first half of this theorem (originally stated by Fermat and first proved by Euler) is difficult. The second half (i.e., the "$4n + 3$" case) isn't too bad. Prove it. (HINT: If $p = 4n + 3 = a^2 + b^2$, consider the three cases: a, b both even; a, b both odd; and one of a, b even and the other odd.)

THE REGULAR SOLIDS REVISITED

53. Show the Euler-Descartes formula holds for the Great Pyramid of Cheops, the World Trade Center, and the Pentagon (which has a central courtyard and is thus not convex).

54. Now suppose we have a regular solid with V vertices, F faces, and E edges, where each face of the solid is a regular n-gon. Suppose further that the number of regular n-gons meeting at each vertex is k. Note immediately that $n \geq 3$ and $k \geq 3$.

(a) Initially, one might conclude that, since each of the F faces is a regular polygon with n edges, there should be nF edges in the regular solid. Explain why the correct result is E = nF/2.

(b) Explain why V = nF/k.

(c) Now substitute (a) and (b) into the Euler-Descartes formula to get

$\frac{F}{4k}(2n + 2k - kn) = 1$, and conclude that $2n + 2k - kn > 0$.

(d) The inequality in part (c) provides the key to the analysis of the regular solids. Begin with the minimal case $n = 3$. Show that this implies that $3 \leq k < 6$ and thus yields three regular solids.

(e) If $n = 4$, show the inequality yields only one regular solid.

(f) What happens if $n = 5$?, $n = 6$?, $n = 7$?, etc.

55. Complete the summary table below for the regular solids, combining the results of Euclid with those from the previous page. You might look for interesting symmetries hidden among the entries on the chart – you can be sure the Greeks would have.

Name of Solid	No. of Faces (F)	No. of Vertices (V)	No. of Edges (E)	Type of regular n-gon (polygon) at each face (n)	No. of faces at each vertex (k)	No. of degrees in each face angle	No. of deg. in ea. polyhedral angle
Tetrahedron							
Cube							
Octahedron							
Dodecahedron							
Icosahedron							

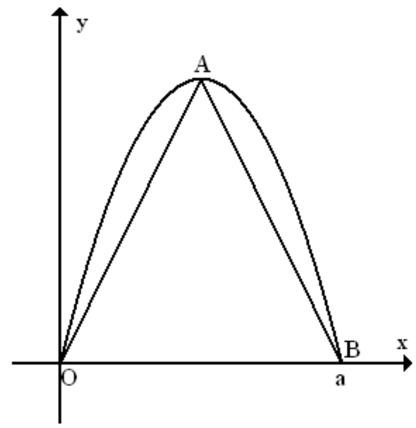
The Regular Solids[1]

Tetrahedron Cube Octrahedron Dodecahedron Icosahedron

[1] Source: http://en.wikipedia.org/wiki/Platonic_solid

Chapter 4. Archimedes' Determination of Circular Area

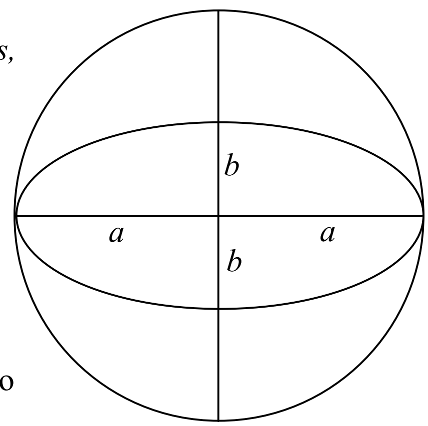
ARCHIMEDES' QUADRATURE OF THE PARABOLA



56. (a) Use calculus to verify that the area under the segment of the parabola $y = ax - x^2$ from $x = 0$ to $x = a$ is 4/3 of the area of triangle OAB.

    (b) Suppose someone gave Archimedes the parabolic segment at right. Describe how he could then have constructed, with compass and straightedge, a square of equal area. In other words, explain how his result successfully completed the *quadrature* of a given parabola.

ARCHIMEDES' AREA OF THE ELLIPSE

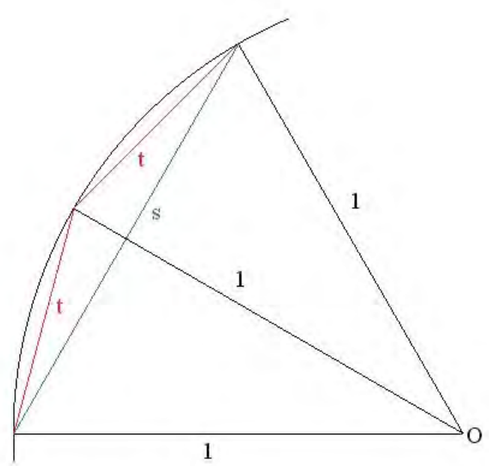


57. In Proposition 4 of his work *On Conoids and Spheroids,* Archimedes proved that "The area of any ellipse is to that of the auxiliary circle as the minor axis is to the major axis." Here, the ellipse's major axis is the longest diameter ($2a$ in the figure to the right), its minor axis is its shortest ($2b$), and the auxiliary circle is the circumscribed circle – i.e. that having diameter equal to the major axis.

    (a) Show that Archimedes' wordy statement amounts to saying that the area of the ellipse is πab, where a and b are as shown.

    (b) Now prove the area of an ellipse really is πab using calculus.

ARCHIMEDES' APPROXIMATION OF π



58. Prove that, if s is the side of a regular inscribed n-gon and t is the side of a regular inscribed $2n$-gon, then

$$t = \sqrt{2 - \sqrt{4 - s^2}}$$

(Assume the n-gon and $2n$-gon are inscribed in the same unit circle.)

59. Now begin with a regular hexagon inscribed in a unit circle. The hexagon's perimeter is 6, a rough approximation for the circle's circumference 2π, and so $\pi \approx 3.00$. Now use Problem #58 through seven doublings, until you have the perimeter of a regular inscribed 768-gon. What is the corresponding approximation of π based on these inscribed figures?

60. In the midst of his approximation, Archimedes needed a value for $\sqrt{3}$ and he used

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}.$$ How good is this as a decimal?

61. An interesting question raised above is how one would get such a sharp approximation for square roots without benefit of calculator. A very nice algorithm is the following:

To approximate $\sqrt{A}$, begin with an initial approximation (obtained by "eyeballing it") of x_0.

Then, let the next approximation be $x_1 = \dfrac{x_0^2 + A}{2x_0}$, then $x_2 = \dfrac{x_1^2 + A}{2x_1}$ and generally, use the

recursive definition $x_{n+1} = \dfrac{x_n^2 + A}{2x_n}$. (*)

(a) Assuming that the sequence of successive approximations converges to a limit *L,* show that $L = \sqrt{A}$. (HINT: Take limits of both sides of (*).)

(b) Now suppose we want to approximate $\sqrt{3}$ and we start with the rational number $x_0 = 5/3$ (this is reasonable since $(5/3)^2 = 25/9 \approx 27/9 = 3$). Apply the recursion formula twice to approximate this square root.

(c) Notice anything? Do you think Archimedes was onto something?

ARCHIMEDEAN "FORMULAS" IN GEOMETRIC GARB

62. In his masterpiece *On the Sphere and the Cylinder,* Archimedes stated his results about spherical volume and area by computing his figures with such better-understood figures as cylinders and cones. Assuming we know the modern formulas for the key properties of cones and cylinders, translate the following Archimedean statements into familiar, modern-day formulas:

(a) "Any sphere is equal (by volume) to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere."

(b) "Every cylinder whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere is half again as large as the sphere."

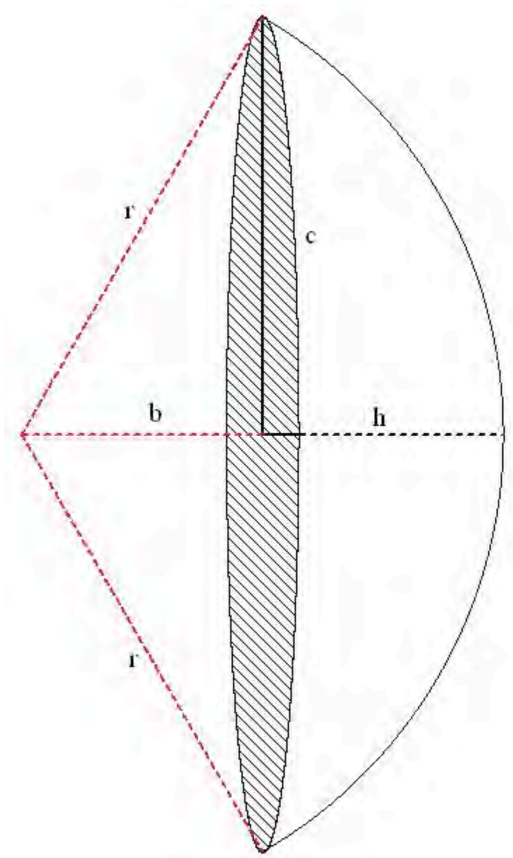
(c) "Every cylinder whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere has surface (together with its bases) that is half again as large as the surface of the sphere."

63. Suppose we know that $\pi = C/D$ is the constant ratio of a circle's circumference to its diameter. As noted, Proposition XII.2 of Euclid implies that, for circular area, $A = k_2D^2$ for some constant k_2, and Proposition XII.18 of Euclid yields that, for spherical volume, $V = k_3D^3$ for some constant k_2. But Euclid, of course, gave no clue as to the relationships among the one-dimensional constant π, the two-dimensional constant k_2, and the three-dimensional constant k_3. Use Archimedes' insights from *Measurement of a Circle* and *On the Sphere and the Cylinder* to express k_2 and k_3 in terms of π.

ARCHIMEDES AND SPHERICAL SEGMENTS

64. Recall that a spherical **segment** is that part of a sphere cut off by a plane. Archimedes, in Proposition 42 of Book I of *On the Sphere and the Cylinder*, determined the surface area of a segment (as we've seen). But he wasn't finished. In Proposition 2 of the second book of *On the Sphere and the Cylinder*, he polished off the volume of a segment as well. He proceeded as follows:

From a sphere with radius r, slice off a segment with a plane that is b units from the sphere's center. Let $h = r - b$ be the height of the segment, and let c be the radius of the circle that forms the base of the segment. Archimedes then proved that "… the volume of the segment is equal to that of a CONE whose base is the same as that of the segment…" and whose height is x, where x satisfies the (bizarre) proportion:



$$\frac{x}{r-b} = \frac{2r+b}{r+b}$$

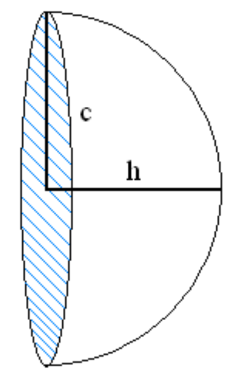
(a) Solve the proportion for x and show that Archimedes' result amounts to

$$\text{Vol}_{\text{seg}} = \frac{1}{3}\pi(r-b)^2(2r+b).$$

(b) Now find the volume of the segment using calculus and verify that your result is in agreement with Archimedes' result. (NOTE: If one of you is wrong, you can guess who it is.)

(c) The only trouble with the formula from (a) is that it expressed the volume of the segment in terms of r and b, which are parameters of the original sphere. However, if one were presented with an actual segment, one would know only the dimensions h and c of the segment itself, since the rest of the sphere would have been discarded. Thus, it makes sense to adjust the formula of (a) to introduce h and c at the expense of r and b. Show that such an adjustment yields:

$$\mathrm{Vol}_{\mathrm{seg}} = \frac{1}{6}\pi h\left(3c^2 + h^2\right).$$



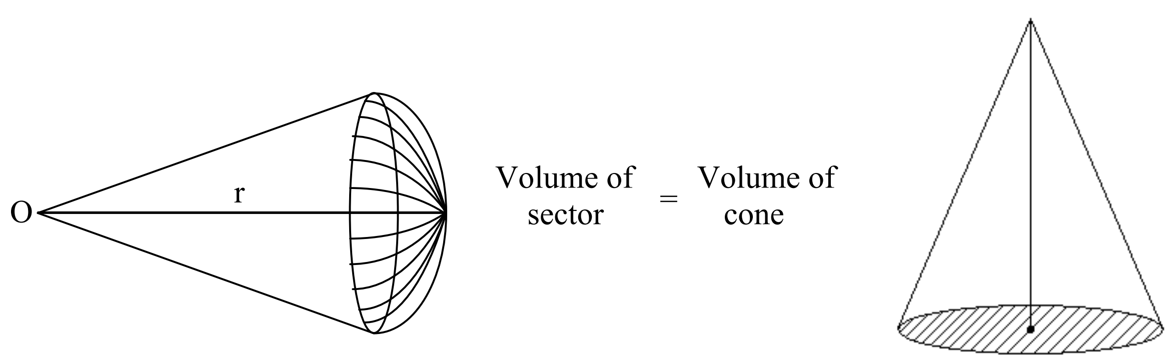
(d) Finally, return to part (c) and determine what happens to the formula ALGEBRAICALLY if $h = r$, while also determining what happens to the segment GEOMETRICALLY if $h = r$. Put these together to re-derive the formula for spherical volume.

ARCHIMEDES AND SPHERICAL SECTORS

65. A sector of a sphere is a region formed by intersecting the sphere with a cone having its vertex at the sphere's center. Thus, a sector looks sort of like an ice cream cone. In Proposition 44 of Book I of *On the Sphere and the Cylinder*, Archimedes proved:

"The volume of any sector of a sphere is equal to a CONE whose base is equal to the surface of the segment of the sphere included in the sector, and whose height is equal to the radius of the sphere."

In other words, he took the shaded area on the left – i.e., the area of the "surface of the segment," squashed it flat into a circle of equal area, and then upon the base erected a cone with height equal to the sphere's radius, as shown:



Note the fascinating mixing of the segment's surface – a surface in three dimensions – and the equal plane area of the cone's base.

(a) By adding the volume of the cone and the segment atop (from the previous problem), show

$$\text{Vol}_{\text{sec}} = \frac{\pi}{6}\left(2rc^2 + hc^2 + h^3\right) \quad \text{(same notation as in Problem \#64)}$$

(b) Eliminate r from the previous formula to show

$$\text{Vol}_{\text{sec}} = \frac{\pi\left(c^2 + h^2\right)^2}{6h}.$$

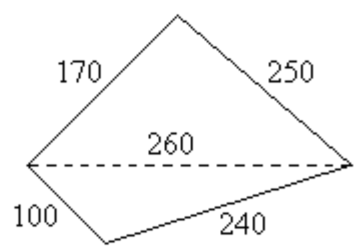
sr

(c) Finally, show that Archimedes' peculiar statement with the squashed circle yields precisely this same volume for a spherical sector.

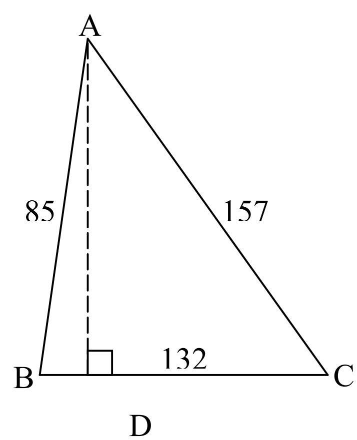
Chapter 5. Heron's Formula for Triangular Area

SOME EXAMPLES WITH HERON'S FORMULA

66. We own a four-sided piece of land, whose area we must determine for tax purposes. Pacing off the sides, we get lengths of 100 yards, 170 yards, 250 yards, and 240 yards (as shown), and then we march along the diagonal (dotted) and find its length to be 260 yards. How many square yards of property do we own?



67. Consider the triangle shown at the right.



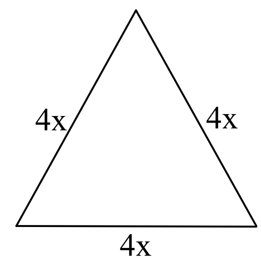
(a) Find its area via Heron's formula.

(b) Find its altitude, $\overline{AD}$.

(c) What do you notice about ΔABC from (b)?

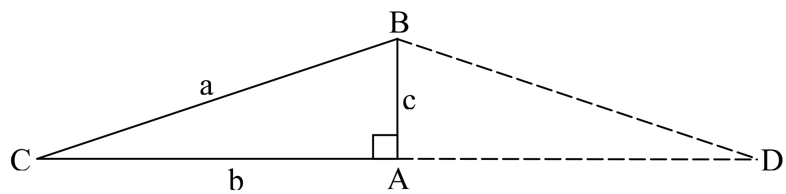
(d) Determine the area that is left if we delete the inscribed circle from ΔABC.

68. An equilateral triangle has each side $4x$ units long, as shown at right. Find its area both by the standard formula, $A = \frac{1}{2}bh$, and by Heron's formula, and verify that the results agree.



WHAT IF HERON AND PYTHAGORAS "SQUARE OFF"?

69. Suppose we have right triangle BAC with right angle BAC and sides of length *a, b,* and *c,* as shown. We extend CA to D so that AD = AC = *b* and then we draw BD.



(a) Show that $\Delta BAC \cong \Delta BAD$.

(b) Explain why the semi-perimeter of ΔDBC is $s = a + b$.

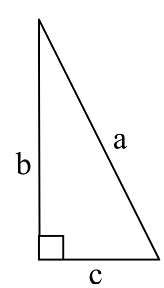
(c) Apply Heron's formula to ΔDBC to deduce Area (Δ DBC) $= \sqrt{b^2\left(a^2 - b^2\right)}$.

(d) Apply the simple formula for triangular area, Area $= \frac{1}{2}$(base)(height), to prove that the area of ΔDBC is bc.

(e) Finally, equate the area expressions from (c) and (d), simplify algebraically, and deduce the theorem of Pythagoras.

70. (a) Beginning with a right triangle of sides a, b, and c, determine its area in two ways – by the standard $A = \frac{1}{2}$(base)(height) – and by Heron's formula. Equate these and manipulate the resulting, dreadful equation until you have derived the theorem of Pythagoras.

(b) Which proof of Heron $\Rightarrow$ Pythagoras do you like better, the one in Problem #69 or the one in this problem?

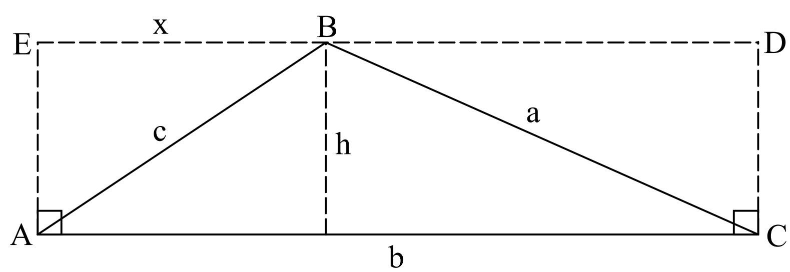


71. Do the arguments of Problem #69 or Problem #70 provide valid proofs of the Pythagorean Theorem? That is, do they contain circular reasoning?

HERON'S FORMULA OVER AND OVER AND OVER AND …

72. While I am not a fan of algebraic proofs of Heron's Formula, the following is at least a little better than most. It was sent to me by John C. Torrez of Chicago:

Begin with ΔABC having sides of lengths *a, b,* and *c,* as shown, where AC is at least as long as the other two sides. Draw perpendiculars to AC at A and C and draw the line through B parallel to AC and meeting the perpendiculars at E and D, respectively. Draw the altitude from B to AC and call its length *h*. Also, let $x = \overline{EB}$.



(a) Using ΔAEB and ΔCDB, show $x = \dfrac{b^2 + c^2 - a^2}{2b}$.

(b) Show Area (ΔABC) = $K = \dfrac{1}{2} b\sqrt{c^2 - x^2}$.

(c) Using (a) and (b), derive the relatively simple expression:

$$K = \frac{1}{4} b\sqrt{4b^2c^2 - \left(b^2 + c^2 - a^2\right)^2}.$$

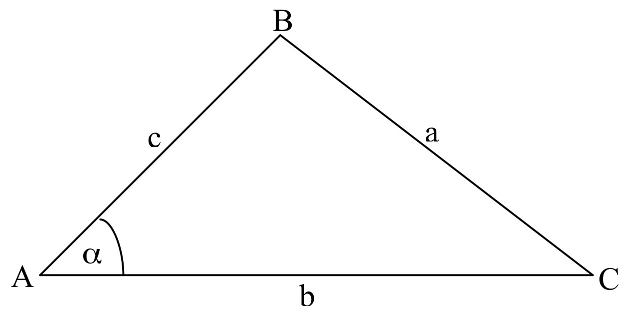
(Note that this is a difference of squares)

(d) Continue with (c) to get

$$K = \frac{1}{4}\sqrt{\left((b+c)^2 - a^2\right)\left(a^2 - (b-c)^2\right)}$$

(e) Now factor the expression beneath the radical and derive Heron's formula.

73. Here is a derivation of Heron's formula, from the Law of Cosines. It has found its way into a number of texts.



(a) Prove Area (ΔBAC) $= \frac{1}{2} bc \sin\alpha$.

(b) Now begin with the Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos\alpha$. Add and subtract $2bc$ from the right side and deduce that

$$(b + c - a)(b + c + a) = 2bc(1 + \cos\alpha).$$

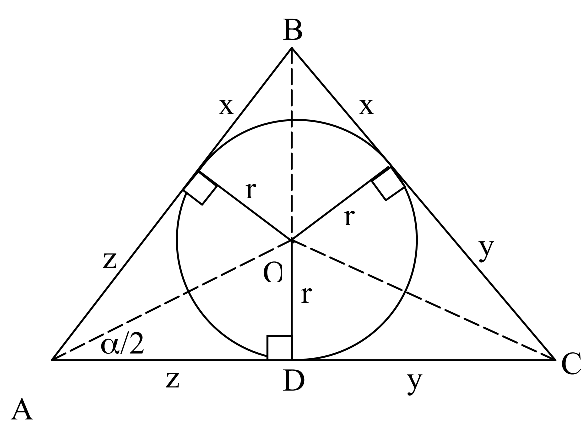
(c) Again begin with the Law of Cosines, but this time alter your argument a bit to prove

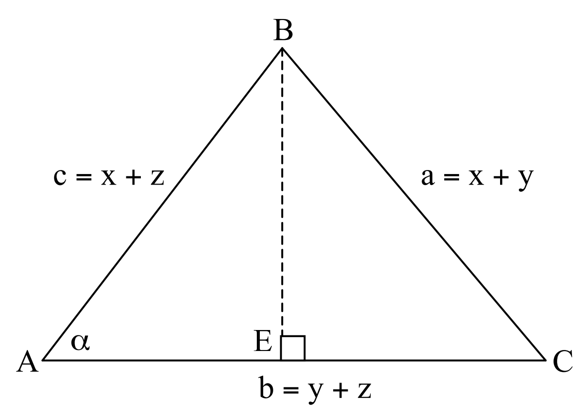
$$(a - b + c)(a + b - c) = 2bc(1 - \cos\alpha).$$

(d) Finally, multiply the results of (b) and (c), mix in a pinch of (a), and end up with a proof of Heron's formula.

74. This is what I called an "Ancient/Modern" proof of Heron's formula when I contributed it to *Mathematics Teacher* in 1986.

Begin with two copies of ΔBAC. For the one on the left, mimic Heron's argument by inscribing a circle, generating $x = s - b$, $y = s - c$, and $z = s - a$, as shown. For the one on the right, just draw the altitude from B to E, having length h. Also, let $\alpha = \angle BAC$.





(a) As Heron did, quickly show that K = Area (ΔBAC) = sr .

(b) For the right-hand triangle, show: $\sin\alpha = \dfrac{h}{x+z}$.

(c) For the left-hand triangle, why is ∠OAD = α/2 ?

(d) Returning to the left-hand triangle, show that

$$\sin(\alpha/2) = \frac{r}{\sqrt{r^2+z^2}} \text{ and } \cos(\alpha/2) = \frac{z}{\sqrt{r^2+z^2}}.$$

(e) Using the famous identity that $\sin(2\theta) = 2\sin\theta\cos\theta$, with θ = α/2, apply (b), (c), and (d) to prove

$$\frac{h}{x+z} = \frac{2rz}{r^2+z^2}.$$

(f) Cross multiply and simplify (f) to show: $s(r^2 + z^2) = z(xy + sz)$.

(g) Cross multiply and simplify (f) to show: $s(r^2 + z^2) = z(xy + sz)$.

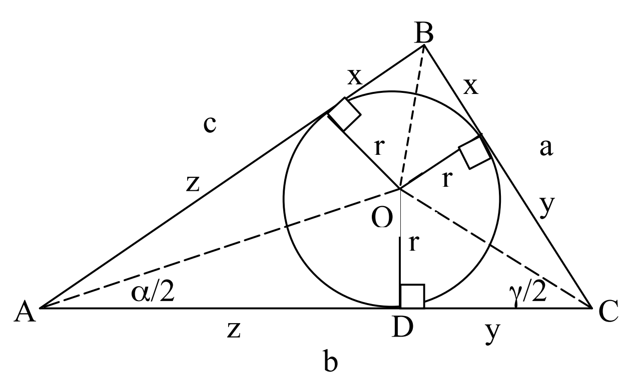
(h) Conclude from (g) that $sr^2 = xyz$.

(i) Finally, combine (a) and (h) to show $K^2 = s(xyz)$ and from this deduce Heron's formula.

75. Here is a wonderful trigonometric proof of Heron's formula, sent to me by Barney Oliver, who works at NASA at Moffett Field, California (don't ask why he was working on Heron's formula instead of space flight). This one uses the trigonometric identities:

$\sin(\alpha \pm \theta) = \sin\alpha\ \cos\theta \pm \cos\alpha\ \sin\theta$ $\qquad$ $\cos(\alpha \pm \theta) = \cos\alpha\ \cos\theta \mp \sin\alpha\ \sin\theta$

It begins, as did Heron, with the inscribed circle.



(a) Prove that, for any angles δ and θ, that $\tan(\delta+\theta)=\dfrac{\tan\delta+\tan\theta}{1-\tan\delta\tan\theta}$.

(HINT: $\tan(\delta+\theta)=\dfrac{\sin(\delta+\theta)}{\cos(\delta+\theta)}$)

(b) Prove that if $\delta + \theta$, = 90°, then $\tan\delta\tan\theta = 1$.

(c) Show that if α, β, and γ are three angles of our triangle, then $\dfrac{\alpha}{2}+\dfrac{\beta}{2}+\dfrac{\gamma}{2}=$ 90°.

(d) For α, β, and γ in our triangle, explain why $\tan\dfrac{\alpha}{2}\tan\left(\dfrac{\beta}{2}+\dfrac{\gamma}{2}\right)=1$.

(e) Now use (d) to prove that $1=\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}+\tan\dfrac{\alpha}{2}\tan\dfrac{\gamma}{2}+\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}$.

(HINT: Somewhere in here you want to apply (a) to $\tan\left(\dfrac{\beta}{2}+\dfrac{\gamma}{2}\right)$)

(f) Next, use (e) to prove $1=\dfrac{r^2}{xz}+\dfrac{r^2}{yz}+\dfrac{r^2}{xy}$.

(g) Get a common denominator and simplify (f) to conclude that $r^2=\dfrac{xyz}{s}$.

(h) Now use (g) to derive Heron's formula.

Chapter 6. Cardano and the Solution of the Cubic

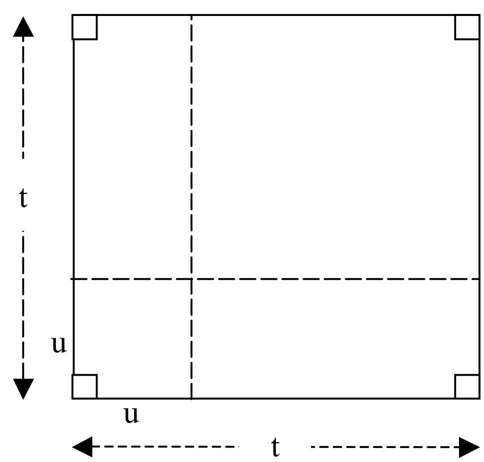
A "CARDANESQUE" APPROACH TO QUADRATICS

76. Here we adapt the geometric/algebraic ideas of Cardano's *Ars Magna* to an old friend – the quadratic equation $ax^2 + bx + c = 0$.

(a) Subdivide a square of side t to prove **geometrically** that

$(t-u)^2 + 2u\,(t-u) = t^2 - u^2$.

(b) Now convert the original quadratic equation into $x^2 + \dfrac{b}{a}x = -\dfrac{c}{a}$ and introduce the auxiliary variables t and u, where $x = t - u$. Then, using (a), find t and u (and ultimately x) in terms of a, b, and c. Does your solution for x ring a bell?



DEPRESSING POLYNOMIALS

77. (a) Prove by induction that if $n \geq 1$, then the first two terms in the expansion of $(y - r)^n$ are $y^n - nry^{n-1}$.

(b) Suppose we begin with the nth degree polynomial $ax^n + bx^{n-1} + cx^{n-2} + \ldots + dx + e$. Use (a) to verify that the substitution $x = y - \dfrac{b}{na}$ converts the nth degree polynomial in x into an nth degree polynomial in y that lacks its $(n-1)$st degree term – i.e., one that is depressed.

CARDANO'S FORMULA IN ACTION

78. Use Cardano's formula to get one real solution of $x^3 + 63x = 316$.

79. Use Cardano's formula to show that one real solution of the cubic $x^3 - 24x + 96 = 0$ is

$$x = 2\sqrt[3]{2\sqrt{7} - 6} - 2\sqrt[3]{2\sqrt{7} + 6}$$

80. Use Cardano's formula to get one real solution of $9x^3 - 9x = 4$ and retain all roots in order to express your solution in terms of "radicals" (rather than using a decimal approximation). Simplify your solution by radicals as far as you can.

81. (a) Find a "solution by radicals" of the depressed cubic $x^3 - 3x = \dfrac{5}{2}$.

(b) Use Cardano's formula to find ALL real solutions to $x^3 - 6x^2 + 9x - 4 = 0$.

82. Use Cardano's formula to get one real solution of the simple cubic $ax^3 + b = 0$. Your result, of course, should agree with what you can easily see to be the solution.

83. Cardano's original example of his method in the *Ars Magna* was to solve the depressed cubic $x^3 + 6x = 20$.

(a) Show, as he did that the solution is $x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$.

(b) Cardano somehow recognized his solution as being simply "2." Of course, he didn't do this on a calculator. Find an **algebraic** confirmation that $x = 2$.
(HINT: $10 + \sqrt{108} = 10 + 6\sqrt{3} = 1 + 3\sqrt{3} + 9 + 3\sqrt{3} = (\ \ ?\ \ )^3$.)

VIETE'S SOLUTION OF THE CUBIC

84. In the late 1500s, Francois Viete devised an alternate approach to solving the cubic. Beginning with the depressed cubic $x^3 + mx = n,$ he made the substitution $x = \dfrac{m}{3y} - y$.

Show that this reduces the original cubic to a quadratic in y^3 (which can thus be solved for y^3, then for y, then for x).

85. (a) Re-do Problem #78 with Viete's method (your answers, of course, should agree).

(b) Re-do Problem #80 with Viete's method (ditto).

(c) Which technique do you prefer – Cardano's or Viete's?

THE IRREDUCIBLE CASE OF THE CUBIC

It was Rafael Bombelli whose 1572 *Algebra* took the first successful steps toward resolving the troubling "irreducible case of the cubic." This case arose from the depressed cubic $x^3 + mx = n$ in the situation where $\dfrac{n^2}{4} + \dfrac{m^3}{27} < 0$. Here Cardano's formula introduces the square root of a negative number. Bombelli realized that, in such a case, the real solutions of a real cubic require us to detour into the realm of the complex numbers. That is, while we begin and end in the comfortable domain of the reals, we seem to have to journey into the strange, uncharted world of the imaginaries to complete the trip. To the sixteenth century mind, this seemed mighty strange. (Come to think of it, it still does.)

86. (a) Recall that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. Use this to verify that $(5 + i)^3 = 110 + 74i$. Conclude that $\sqrt[3]{110 + 74i} = 5 + i$.

(b) Similarly show $\sqrt[3]{110 - 74i} = 5 - i$.

(c) Use these cube roots and Cardano's formula to find one real solution of $x^3 - 78x = 220$.

87. Another irreducible cubic:

(a) Verify that $\sqrt[3]{-9 + 46i} = 3 - 2i$.

(b) Now find **all** real solutions for the real cubic $x^3 - 3x^2 - 36x + 56 = 0$ by first depressing and then solving the resulting equation via Cardano. How many *real* solutions did you get?

88. A final irreducible case, for good measure:

(a) Verify that $\sqrt[3]{259+286i} = 7+2i$.

(b) Now find **all** real solutions for the real cubic $x^3 + 3x^2 - 156x - 676 = 0$. How many *real* solutions did you get for this cubic?

NOTE: The previous three examples are artificial, since the cube roots of the pertinent complex numbers seem to appear out of nowhere. There is a technique discovered by Euler in the eighteenth century whereby the cube roots of any complex number can be found, although it involves introducing the trig functions (!).

VIETE'S ATTACK ON THE IRREDUCIBLE CASE

Not to be outdone, Viete likewise made an assault on the irreducible case. Interestingly, his approach avoided the need for complex numbers; alas, it moved beyond the "algebraic" by introducing the trigonometry of the reals. We present Viete's idea in the following and then try a pair of examples.

89. (a) Prove the trig identity $\cos^3\alpha - \frac{3}{4}\cos\alpha = \frac{1}{4}\cos 3\alpha$.

(HINT: $\cos(\alpha+\beta) = \cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta$.)

(b) Now, to solve the depressed cubic $x^3 + mx = n$, Viete made the substitution $x = b\cos\alpha$, where α and b are unknowns to be determined. Use (a) to show

$$\alpha = \frac{1}{3}\cos^{-1}\left(\frac{n/2}{\sqrt{-m^3/27}}\right) \qquad \text{and} \qquad b = \sqrt{\frac{-4m}{3}}.$$

90. Apply Viete's technique to get one real solution of $x^3 - 78x = 220$. You may resort to decimals and calculators here. Your result should, needless to say, agree with that obtained in Problem #86(c).

91. Apply Viete's technique to get one real solution of $x^3 - 6x = 4$. This time, express your answer as a "solution by radicals" if you can – no decimal approximation and no trig functions in your final answer!

(HINT: Along the way, you may want to derive the not-so-famous fact that

$$\cos\frac{\pi}{12} = \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}}\ )$$

Chapter 7. A Gem from Isaac Newton

SOME FUN WITH NEWTON'S GENERALIZED BINOMIAL THEORM

92. Write out the first five terms (i.e., through x^4) in the binomial expansion of $\sqrt[3]{1+x}$.

93. Use the previous problem to get a quick numerical estimate of $\sqrt[3]{70}$.

    Use the previous problem to get a quick numerical estimate of $\int_0^{1/2} \sqrt[3]{1+x}\, dx$,

    and compare it with the exact result from calculus.

94. Now CUBE the expansion you found in Problem #92, carrying the result at least through the x^3 term. What do you notice (besides hand cramps)?

AN ANALYTIC APPROXIMATION OF π

95. The following "post-Newtonian" approximation of π uses the "Gregory Series":

    Here is an informal derivation of the series: $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

    (a) Use the substitution $t = \tan \alpha$ to show $\int_0^x \frac{1}{1+t^2}\, dt = \tan^{-1} x$.

    (b) Now expand $\frac{1}{1+t^2} = \left(1+t^2\right)^{-1}$ by the binomial series and integrate the series term-wise from 0 to x, and thereby derive the Gregory series.

96. Using trig identities (NO CALCULATORS ALLOWED!!), prove the peculiar formula of John Machin (1706):

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

(HINT: Introduce $\alpha = \tan^{-1}\left(\frac{1}{5}\right)$ and $\beta = \tan^{-1}\left(\frac{1}{239}\right)$ and then attack $\beta = \tan\left(4\alpha - \beta\right)$ with your trig identities.)

97. Now use the first four terms of the Gregory series to estimate both $\tan^{-1}\left(\frac{1}{5}\right)$ and $\tan^{-1}\left(\frac{1}{239}\right)$ and then, from Machin's identity, estimate π.

98. Finally, compare the efficiencies of the classical approach of Archimedes (see Problems #58–59), of Newton, and of Machin.

(a) Which estimate gives the best accuracy for the buck?

(b) How many root extractions does the classical method need? Newton? Machin?

Chapter 8. The Bernoullis and the Harmonic Series

EXPLORING THE HARMONIC SERIES

99. (a) Use your calculator or a computer to determine exactly how many terms we must sum in the harmonic series before it totals 4.50 or more.

(b) A famous constant in mathematics is the so-called "Euler-Mascheroni constant," denoted by γ (gamma) and defined by the equation $\gamma = \lim\left[\left(\sum_{k=1}^{n}\frac{1}{k}\right) - \ln n\right]$.

This constant is approximately equal to 0.577215. (It remains one of the great unsolved problems of mathematics to determine whether the Euler-Mascheroni constant is or is not rational.) In any event, use this value of γ to estimate the number of terms it takes the harmonic series to grow to 10.00 or more.

(c) How long does it take the harmonic series to reach 15.00 ?

LEIBNIZ AND HIS SERIES SUMMATIONS

100. Prove by induction that $\sum_{k=1}^{n}\frac{1}{k(k+1)} = \frac{n}{n+1}$ and use this to give a modern proof (i.e., one using partial sums) that $\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = 1$.

101. We've seen Leibniz cleverly sum the series of reciprocals of triangular numbers:

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\ldots+\frac{1}{k(k+1)/2}+\ldots = 2.$$

But he didn't stop there, for he next summed the reciprocals of the so-called "pyramidal numbers":

$$P = 1 + \frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \frac{1}{35} + \ldots + \frac{1}{k(k+1)(k+2)/6} + \ldots .$$

Here is his approach:

(a) By considering $\frac{2}{3}P = \frac{2}{3} + \frac{2}{12} + \frac{2}{30} + \frac{2}{60} + \frac{2}{105} + \ldots$ and cleverly decomposing the terms into a telescoping series, follow Leibniz' reasoning to sum *P*.

(b) Now re-do the sum by the modern technique of partial sums. That is, first prove by induction that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)/6} = \frac{3n(n+3)}{2(n+1)(n+2)} .$$

and then take limits.

102. Here we follow a similar line of reasoning to sum the infinite series:

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \frac{1}{99} + \frac{1}{143} + \ldots + \frac{1}{4k^2 - 1} + \ldots$$

where the denominators are $3 = 1 \times 3$; $15 = 3 \times 5$; $35 = 5 \times 7$; $63 = 7 \times 9$; etc.

(a) If we call this series S, determine the series 2S.

(b) As Leibniz did with Huygens' challenge problem, decompose the series 2S into cancelling "+" and "–" terms, and thereby evaluate 2S and thus S.

(HINT: $\frac{2}{3} = 1 - \frac{1}{3}; \frac{2}{15} = \frac{1}{3} - ?$; etc.)

THE GEOMETRIC SERIES IN ACTION

103. (a) Find a specific geometric series which sums to 30.

(b) Can you find a specific geometric series which sums to -30?

(c) Can you find a specific geometric series which sums to $-\frac{1}{3}$?

(d) Determine precisely which real numbers can be sums of geometric series.

JAKOB'S DIVERGENCE PROOF (1689)

104. Perhaps to show up his brother Johann, Jakob Bernoulli devised his own clever proof for the divergence of the harmonic series in 1689 and included it in his *Tractatus de Seriebus Infinitis*. His key idea was to prove that, starting at **any** point $\frac{1}{a}$ of the harmonic series, the sum $\frac{1}{a}+\frac{1}{a+1}+\ldots$ will, after a FINITE number of terms, exceed 1. In Jakob's words, "No matter where we start in the harmonic series, the sum of a finite number of consecutive terms will exceed 1." Below appears a (streamlined) proof:

(a) For any $a \geq 1$, look at that portion of the harmonic series:

$$\frac{1}{a+1}+\frac{1}{a+2}+\ldots+\frac{1}{a^2}.$$

How *many* terms are in this portion of the harmonic series?

(b) Explain why $\frac{1}{a+1}+\frac{1}{a+2}+\ldots+\frac{1}{a^2} \geq \frac{1}{a^2}+\frac{1}{a^2}+\frac{1}{a^2}+\ldots+\frac{1}{a^2}$

$$=\left[a^2-a\right]\frac{1}{a^2}.$$

(c) Now use (b) to show that, as Jakob claimed, the portion of the harmonic series

$$\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a^2} \geq 1.$$

(d) Just to be sure, check numerically that

$$\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{9} \geq 1 \quad \text{and} \quad \frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{16} \geq 1.$$

(e) Finally, explain why Jacob's conclusion in (c) proves the harmonic series diverges to $+\infty$. What is so important about the **finite** number of terms?

AN EARLY COMPARISON TEST FOR SERIES

105. We noted that Jakob Bernoulli knew $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converged to something less than 2 by comparing this series with a term-wise greater series:

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\ldots=2\left[\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\ldots\right]=2(1)=2.$$

Here we'll come up with a somewhat sharper upper bound:

(a) First use a clever decomposition of terms to show

$$\frac{2}{3}+\frac{2}{15}+\frac{2}{35}+\frac{2}{63}+\frac{2}{99}+\ldots=1.$$

(b) Now if we let $S_1=\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{1}{64}+\ldots$, it is clear that $S_1<\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\ldots$. Use this and part (a) to get the bound $S_1<\frac{1}{2}$.

(c) Similarly, if $S_2=1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\ldots$, show $S_2<\frac{5}{4}$.

(d) Finally, explain why $\sum_{k=1}^{\infty}\frac{1}{k^2}<1.75$

Chapter 9. The Extraordinary Sums of Leonhard Euler

L'HOSPITAL'S RULE MAKES ITS FIRST HOUSE CALL

106. Consider the example l'Hospital gave as the first (!) illustration of his rule in his 1696 *Analyse de Infiniment Petit*:

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}.$$

(a) Verify that if $x = a$, both numerator and denominator are zero.

(b) Now use l'Hospital's Rule to determine the limit as x approaches a.

MORE OF EULER'S SERIES

107. Begin with the Taylor Series $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ and mimic Euler's work (i.e., equate this infinite sum with an infinite product) to derive the sum of the reciprocals of squares of the odd integers:

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \ldots = \frac{\pi^2}{8}.$$

108. Now use the previous result to get an independent derivation of our great theorem:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \ldots = \frac{\pi^2}{6}.$$

109. (a) Evaluate $\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \ldots$

(b) Now evaluate the alternating series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \ldots$

110. Finally, use the product expansion for cos x from Problem #107 and a clever choice for x to show

$$\sqrt{2} = \frac{2 \cdot 2 \cdot 6 \cdot 6 \cdot 10 \cdot 10 \cdot 14 \cdot 14 \cdot \ldots}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot \ldots}$$

EULER'S EVALUATION OF $\sum_{k=1}^{\infty} \frac{1}{k^6}$

111. Push Euler's argument one step further to get a formula relating $\sum r_k$, $\sum r_k^2$, and $\sum r_k^3$ with coefficients A, B, and C in the factorization of

$$1 - \mathrm{A}y + \mathrm{B}y^2 - \mathrm{C}y^3 + \mathrm{D}y^4 - \cdots = (1 - r_1 y)(1 - r_2 y)(1 - r_3 y)\cdots$$

112. Now use the previous result to find the sum

$$1 + \frac{1}{64} + \frac{1}{729} + \frac{1}{4096} + \ldots + \frac{1}{k^6} + \ldots$$

Chapter 10. A Sampler of Euler's Number Theory

A BIT OF NUMBER THEORY

113. Numerically check the Little Fermat Theorem for ...

(a) $a = 12$ and $p = 7$.

(b) $a = 3$ and $p = 17$.

114. (a) Is the Little Fermat Theorem still true if we drop the restriction that p not be a factor of a (keeping everything else the same)?

(b) Is the Little Fermat Theorem still true if we drop the restriction that p be prime (keeping everything else the same)?

115. Suppose that 5 is not a factor of a, $a + 1$, or $a^2 + 1$. Explain why 5 must be a factor of $a - 1$.

116. CONJECTURE: "If a is any positive integer, then 7 must divide evenly into one of the following: a, $a + 1$, $a + 6$, or $a^4 + a^2 + 1$." Either prove this conjecture or find a counterexample. HINT: The Little Fermat that could!)

117. Euler showed that a prime p divides evenly into $a^4 + 1$ (a even) only if $p = 8k + 1$. Use this insight to factor 234,257 into the product of primes. In so doing, keep track of how many divisions you actually must do before hitting on a prime factor.

118. (a) Verify: $1 + a^4(1 + ab - b^4) = (1 + ab)\,[a^4 + (1 - ab)\,(1 + a^2b^2)\,]$

(b) Now substitute $a = 128$ and $b = 5$. What astonishing fact emerges?

119. Recall we mentioned in Problem #52 that Euler proved Fermat's assertion that a prime of the form $p = 4k + 1$ can be written as the sum of two squares in one and only one way.

(a) Verify this fact numerically for the prime $p = 4(22) + 1 = 89$.

(b) Show that $2^{32} + 1$ does have the form $4k + 1$.

(c) Now use Euler's result to explain why $2^{32} + 1$ must be composite.
(HINT: $2^{32} + 1 = 418161601 + ?$)

REPRESENTATION OF INTEGERS

120. Check by listing all possibilities that the number of ways of writing 12 as the sum of distinct integers is equal to the number of ways of writing 12 as the sum of (not necessarily distinct) odd integers.

THE FUNDAMENTAL THEOREM OF ALGEBRA

121. We've seen that, in the eighteenth century, the Fundamental Theorem of Algebra was framed in terms of factoring real polynomials into the product of real linear and/or real quadratic factors. Recall that Euler managed to factor Nicolaus Bernoulli's proposed counterexample $x^4 - 4x^3 + 2x^2 + 4x + 4$ into the product of

$$x^2 - \left(2+\sqrt{4+2\sqrt{7}}\right)x + \left(1+\sqrt{4+2\sqrt{7}}+\sqrt{7}\right) \text{ and}$$
$$x^2 - \left(2-\sqrt{4+2\sqrt{7}}\right)x + \left(1-\sqrt{4+2\sqrt{7}}+\sqrt{7}\right).$$

Figure out how Euler might have done this.

122. Factor the real depressed quartic $x^4 + x^2 + 4$ into the product of two real quadratic factors AND check your result. (HINT: Since this is depressed, if one factor looks like $x^2 + ux + a$, the other must look like $x^2 - ux + a$.)

EULER'S OBSERVATION ABOUT CONJUGATE PAIRS

123. Suppose $z = x + iy$ and $w = u + iv$.

(a) Prove that $\overline{z + w} = \overline{z} + \overline{w}$.

(b) Prove that Re $(z + w)$ = Re (z) + Re (w)

(c) Is it true that Re $(z \cdot w)$ = Re $(z) \cdot$ Re (w) ?

124. Prove that z is REAL if and only if $z = \overline{z}$.

125. Now suppose that $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_2z^2 + a_1z + a_0$ is an nth degree polynomial for which the coefficients $a_n, a_{n-1}, \ldots a_1, a_0$ are all REAL numbers. Suppose also that $z = z_0$ is a complex solution of the equation $P(z) = 0$. Prove that $z = \overline{z_0}$ is also a solution of $P(z) = 0$. In other words, as Euler observed in 1742, solutions to polynomial equations with real coefficients come in conjugate pairs. (HINT: You might want to use the two previous problems here.)

126. (a) Verify that $z = 2 + 4i$ is a solution of $z^2 + 6z - 40i = 0$.

(b) Is $\overline{2 + 4i} = 2 - 4i$ also a solution of $z^2 + 6z - 40i = 0$?
Does this contradict the observation of Problem #116? Why or why not?

DE MOIVRE'S (EULER'S) THEOREM AND APPLICATIONS

127. Prove DeMoivre's Theorem by induction on *n.*

128. Now prove $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$. (HINT: Use $-\theta$ in Problem #127.)

129. Use DeMoivre's Theorem to get easy evaluations of $\left(\sqrt{3}+i\right)^{15}$ and of $\left(3+4\,i\right)^{8}$.

130. We've seen that DeMoivre's Theorem provides the vehicle for determining roots of any complex numbers. Use this technique to find all three cube roots of $110 + 74i$. Compare this with the result of Problem #86. Notice how this technique allows us to overcome the "irreducible case" of the cubic that so badly troubled Cardano.

131. We know that, for positive real numbers, square roots come in "± pairs" – e.g., the two square roots of 9 are +3 and −3. Does this same principle hold for general complex numbers? That is, if we apply our technique to find the two square roots of a complex number *z,* and if one of these square roots is *w,* must the other be $-w$? Explain, using the square root formula.

132. (a) Determine $\sqrt{48+14\,i}$ by the usual (DeMoivre) technique.

(b) Now determine $\sqrt{48+14\,i}$ by this alternative:

Set $\sqrt{48+14\,i} = a + bi$, so that $(a + bi)^2 = 48 + 14i$.

Then expand the binomial, equate the real and imaginary parts, and solve the resulting pair of equations for *a* and *b*. (BEWARE: *a* and *b* are real numbers here.)

133. The algebraic technique of the previous problem seems, at first glance, to be a way to make an end-run around DeMoivre when taking roots of complex numbers, and thus a purely algebraic way to overcome the irreducible case of the cubic. However, before getting too elated about all this, try to adapt this technique to find $\sqrt{65+142\,i}$. That is, let $\sqrt{65+142\,i} = a + bi$, so that $(a+bi)^3 = 65+142i$. Then expand, generate a pair of equations in *a* and *b,* and boogie. (WARNING!! – This technique may be hazardous to your health since it doesn't work. Don't spend too much time on this; just convince yourself that some of life's promising things somehow just don't pan out.)

THE ROOTS OF UNITY

134. (a) Find all six 6th roots of unity and write them in rectangular form.

(b) Use (a) to check numerically that $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = -1$.

135. Suppose that $\alpha = a + bi$ is an arbitrary complex number. Prove that $(x - \alpha)(x - \overline{\alpha})$ is a quadratic polynomial with REAL coefficients.

136. Factor the real cyclotomic polynomial $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ into the product of real linear and/or real quadratic factors.

137. Factor the real cyclotomic polynomial $x^{23} + x^{22} + \ldots + x^3 + x^2 + x + 1$ into the product of real linear and/or real quadratic factors. Stop at any point when you want to wring my neck. (HINT: $\pi/12 = \pi/3 - \pi/4$.)

138. Prove the peculiar trigonometric fact that for any positive integer n, we have:

$$1+\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cos\frac{6\pi}{n}+\ldots+\cos\frac{(2n-2)\pi}{n}=0\,.$$

(HINT: Take the real parts of both sides of $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = -1$.)

139. In the following, we attack the trig identity:

$$\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\ldots+\sin\frac{(n-1)\pi}{n}=\cot\frac{\pi}{2n}\,.$$

(a) First, derive the finite geometric series formula for any complex number $z \neq 1$:

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{1-z^n}{1-z}\,.$$

(b) Now let $z = \cos\frac{\pi}{n}+i\sin\frac{\pi}{n}$ on both sides of the equation in (a); use DeMoivre, conjugates, trig, and whatever else to prove the identity. (Pretty neat, huh?)

THE REGULAR HEPTADECAGON

140. Be sure you can show that, if a and b are constructible magnitudes, then so are the magnitudes $a + b$, $a - b$, ab, a/b, and $\sqrt{a}$ (provided these are all real numbers).

141. For the 17-gon construction, explicitly show that $a_1 a_2 = -4$. (UGH!!)

142. For the 17-gon construction, find the explicit formula for b_1 in terms of square roots; in other words, show that b_1 is indeed a surd. Of course, you'll first have to determine a_1 as a surd. Good luck.

143. List all the values of n less than 200 for which a regular n-gon can be constructed with compass and straightedge. (HINT: There are more than 30 of them.)

COROLLARIES OF THE TRISECTION PROBLEM

144. Pierre Wantzel's key result was:

If a cubic equation with integer coefficients $x^3 + ax^2 + bx + c = 0$ has no rational solutions, then it has no constructible solutions.

Recall the proof by contradiction went as follows: It began by assuming that, although this cubic had no rational solution, it nonetheless did have a constructible solution of the form $r = p + q\sqrt{\beta}$, where this solution was "minimal" in the sense that it used as few or fewer square roots in its formula than any other constructible solutions. He next showed that $s = p - q\sqrt{\beta}$ will also be a root of the cubic above.

(a) Explain why this implies that $a_1 a_2 = -4$ is also a root of the cubic.

(b) What did this contradict and why?

(c) Now, carefully indicate exactly where in this argument we used the fact that the cubic had no rational roots. That is, show precisely how the desired contradiction falls apart if rational roots are permitted.

145. With the key result behind him, Wantzel proved that the 60° angle could not be trisected with compass and straightedge. Use this fact to prove:

(a) It is impossible to construct a regular nonagon (9-gon) with Euclidean tools.

(b) It is impossible to construct a regular 27-gon with Euclidean tools.

(c) It is impossible to construct an 85° angle with Euclidean tools.

Chapter 11. The Non-Denumerability of the Continuum

ONE-TO-ONE CORRESPONDENCES

146. Convince yourself that $f: \mathbf{N} \to \mathbf{Z}$ defined by

$$f(n) = \frac{\sin^2 \dfrac{\pi n}{2} + n\cos \pi n}{2}$$

is our one-to-one correspondence between the set of natural numbers and the set of (positive, negative, and zero) integers.

147. Find an explicit function that generates the same correspondence as that of Problem #147 but which has only algebraic – as opposed to trigonometric – components.

148. Write down an explicit function $g: (0, 1) \to [0, 1]$ that is both one-to-one and onto. Note that this establishes $\overline{[0,1]} = \overline{(0,1)} = \mathbf{c}$ directly from the definition.

149. By finding an explicit one-to-one correspondence, prove that for any real number a, the cardinality of the ray $(a,+\infty)$ is $\mathbf{c}$.

150. What is the cardinality of the set of all complex numbers? Why?

CANTOR'S NESTED SET THEOREM

151. In his first (1874) proof of the non-denumerability of the continuum, Cantor needed the theorem from analysis which says that any nested sequence of closed intervals has at least one point belonging to all the intervals. Today this is often called "Cantor's Nested Set Theorem." Here we want to show why the nested intervals must be *closed*.

Consider the nested sequence of open subintervals of (0,1):

$$(0,1) \supset \left(0,\frac{1}{2}\right) \supset \left(0,\frac{1}{3}\right) \supset \ldots \supset \left(0,\frac{1}{n}\right) \supset \ldots$$

And let **A** be the set of points that belong to all of these intervals. Explain why there can be nothing in **A**.

(Possible HINT: Assume x is in **A**, and do a triple *reductio ad absurdum* on the cases $x \le 0$, $0 < x < 1$, and $1 \le x$.)

CANTOR'S DIAGONALIZATION PROCESS

152. Let **B** be the set of all real numbers in the interval (0,1) whose decimal expansions contain only the digits "0" and "5." For instance, **B** contains the numbers 0.05, ½, and 0.05005000500005…, whereas 0.050050504555… is not a member of **B**. Use the diagonalization process to explain why **B** is a non-denumerable subset of (0,1).

153. (a) Explain why ½ belongs to set **B** in Problem #152, and why $\frac{1}{20}$ also belongs to **B**. More generally, list an infinite number of rationals belonging to **B**.

(b) Does set **B** from Problem #152 contain finitely or infinitely many irrationals? Use a set theoretic argument to answer this question.

ALGEBRAIC/TRANSCENDENTAL NUMBERS – PART TWO

154. The height h of a polynomial with integer coefficients is the largest absolute value of any coefficient of the polynomial. List all polynomials of height $h = 4$ and solve them to determine which new algebraic numbers they yield – i.e., which new algebraic numbers appear here that did not appear for heights $h = 1, 2$, or 3. (HINT: There are a dozen new ones.)

155. Prove that $\cos\frac{\pi}{7}$ is algebraic by exhibiting a specific polynomial with integer coefficients for which $\cos\frac{\pi}{7}$ is a zero. (HINT: $7\alpha = 3\alpha + 4\alpha$.)

156. (a) Show that $P(x) = x^2 - 3$ and $Q(x) = 4x^3 - 8x - 1$ are two *different* polynomials with integer coefficients that take the same value for $x = \sqrt{2}$.

(b) Using Lindemann's 1882 result that π is transcendental, explain why there do NOT exist two different polynomials $P(x)$ and $Q(x)$, each having integer coefficients, for which $P(\pi) = Q(\pi)$. Put another way, this says that, if for some polynomial $P(x)$ with integer coefficients, you find $P(\pi) = a$, then there is no other such polynomial $Q(x)$ taking the value a when $x = \pi$.

(c) Does any polynomial with integer coefficients have x-intercept at $x = \pi$? Does any have y-intercept at $y = \pi$? (Which of these two parts is very elementary and which is very deep?)

Chapter 12. Cantor and the Transfinite Realm

SCHRÖDER-BERNSTEIN IN ACTION

157. (a) Use Schröder-Bernstein to prove $[0,1] = \mathbf{c}$ (compare with Problem #148).

(b) Use Schröder-Bernstein to prove $(a, +\infty) = \mathbf{c}$ (compare with Problem #149).

158. (a) Look over the proof in *Journey Through Genius* (pp. 270-271) that uses Schröder-Bernstein to prove that the cardinality of the irrationals is **c**.

(b) Can you cook up a similar argument to establish that the cardinality of the set of transcendentals is **c**? (I can't – I need the continuum hypothesis for this, and that's quite a different thing.)

159. Determine the cardinality of the set of all constructible numbers – i.e., the set of real numbers we called the "surds." (HINT: This is still the section on Schröder-Bernstein, after all.)

160. Let $\mathbf{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers and $\mathbf{O} = \{1, 3, 5, \dots\}$ be the set of odd natural numbers. Introduce the mappings

$$f : \mathbf{N} \to \mathbf{O} \text{ defined by } f(n) = 4n - 1 \text{ and}$$

$$g : \mathbf{O} \to \mathbf{N} \text{ defined by } g(n) = \frac{n+5}{2}.$$

(a) Show that f and g are both one-to-one but NOT onto.

(b) Use these specific functions to determine the sets $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ from the Schröder-Bernstein proof, and thereby write down the first few members of the set

$$\mathbf{E} = \mathbf{E}_0 \cup \mathbf{E}_1 \cup \mathbf{E}_2 \cup \dots$$

(c) Next, construct the specific formula for the one-to-one and onto function $h : \mathbf{N} \to \mathbf{O}$ and give your answer in "piecewise" form:

$$h(n) = \begin{cases} ? & \text{if } n \text{ is in } \mathbf{E} \\ ? & \text{if } n \text{ is in } \mathbf{N} \setminus \mathbf{E}. \end{cases}$$

(d) Finally, "check" your function h in (c) by making a chart showing $h(1)$, $h(2)$, $h(3)$, …, $h(15)$ and noting whether h really does seem to be one-to-one and onto, as Schröder-Bernstein promised.

CANTOR'S THEOREM

161. Prove by induction on n: If a set **A** has n elements, its power set has 2^n elements. (This, of course, gives the finite version of Cantor's Theorem.)

162. In his 1891 proof that the cardinality of the power set exceeds the cardinality of the set itself, Cantor began with any matching between elements of the set **A** and elements of its power set and had to come up with an element of the power set that could not be on the proposed list. The fact that he figured out how to find such a set *in general* is a clear testimony to his remarkable genius. Just to see how his insight works in a specific case or two, work on the examples below:

(a) Suppose we match all of $\mathbf{A} = \{a, b, c, d, e, f, g\}$ with some of the elements of its power set P(**A**) as follows:

$$
\begin{array}{lll}
a & \to & \{b, c, g\} \\
b & \to & \{b, c, e, g\} \\
c & \to & \{b, c\} \\
d & \to & \{d\} \\
e & \to & \mathbf{A} \\
f & \to & \{a, b, c\} \\
g & \to & \{\ \}\ \text{(empty set)}
\end{array}
$$

Find the set **B** from Cantor's proof. Is it an element of P(**A**)? Is it matched with anything in the matching above?

(b) Do the same thing for the matching between **N** and its power set P(**N**) given by

$$
\begin{array}{llll}
n & \to & \{n-1, n, n+1\} & \text{if } n \text{ is even} \\
n & \to & \{n^2, n^2+2, n^2+4, \ldots\} & \text{if } n \text{ is odd.}
\end{array}
$$

For this matching, explicitly find the subset **B** of **N** and explain why it is or isn't matched with any element n under this matching.