A Curious Rule for Computing Cube Roots from Medieval Iraq
Student Explorations

The goal of these exercises is to explore a curious rule for finding the "side," or cube root, of a perfect cube. The rule was inserted by a later author as a kind of appendix to some manuscript copies of the 10th-century *Epistles* of the Brethren of Purity (Ikhwān al-Ṣafāʾ). The *Epistles*, which were intended as a compendium of all scientific and philosophical knowledge of the time, were written by an anonymous, secretive group of scholars centered in Basra, in southern Iraq.

Below is a translation of the appendix. Before the advent of modern algebraic notation, rules like these were routinely expressed verbally. Because the verbal explanation may be difficult to decipher, we follow it with two worked-out examples.

> Abū Ṭālib Aḥmad bin Jaʿfar bin Ḥammād recounted that he found a characteristic property of the side of the cubic number that is being mentioned, which is that when asked for the side of a cubic number, the way [to find] it is that you take a sixth of the number being investigated. You see if the number [being investigated] is even, [in which case] you add up the squares of the odd numbers in a series until [you reach] what resulted from the sixth, and you multiply the remainder of the sum of the squares by six, to get the sought-after side. If the number being sought is odd, you take the squares of the even numbers in a series and you add them up. Where the sum ends, you look at the remainder and you multiply six by it, which gives the side. If the number that is investigated is odd, take the even squares in a series and add them up to whereof the addition ends, then look into the remainder and multiply it by six, wherein the side comes out.[1]

We illustrate the rule by examples for both the even case and the odd case.

Example 1 *Suppose we wish to compute the cube root of the even number* 2744.

- *Dividing* 2744 *by* 6*, we get* $457\frac{1}{3}$.
- *Then we add up the squares of the odd integers, starting with* 1^2 *and stopping just before the sum exceeds the number we computed in the first step:*

$$1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 = 455.$$

- *The difference* $457\frac{1}{3} - 455$ *is* $2\frac{1}{3}$*, and multiplying this by* 6*, we find the cube root of* 2744*:*

$$6 \times 2\frac{1}{3} = 14.$$

- *To check that this is correct, we simply cube the result:* $14^3 = 2744$.

Example 2 *Suppose we wish to compute the cube root of the odd number* 4913*. Dividing* 4913 *by* 6 *gives* $818\frac{5}{6}$*. Adding the squares of the even integers as long as we stay below that value gives* $2^2 + 4^2 + \cdots + 16^2 = 816$*. The difference is* $818\frac{5}{6} - 816 = 2\frac{5}{6}$*, and multiplying this by* 6*, we find the cube root of* 4913*:* $6 \times 2\frac{5}{6} = 17$*. Checking this by cubing gives:* $17^3 = 4913$.

[1] This translation of the primary source excerpt is based on El Bizri's translation of *Epistles of the Brethren of Purity: On Composition and the Arts, Epistles 6–8*, Oxford University Press, 2018, p. 68, with some adjustments by the authors of these exercises. This Abū Ṭālib is otherwise unknown.

Exercises

1. In this exercise you will compute the cube root of a perfect cube using Abū Ṭālib's method. You may use a standard calculator to add, subtract, multiply and divide numbers, but not for any other operation.

   (a) Fill in the following tables with the squares, and the sums of the squares.
   The first table is for odd numbers and the second for even numbers.
   The first two entries of each table are already filled.

n	1	3	5	7	9	11	13	15	17
n^2	1	9							
$1^2 + 3^2 + \cdots + n^2$	1	10							

n	2	4	6	8	10	12	14	16	18
n^2	4	16							
$2^2 + 4^2 + \cdots + n^2$	4	20							

   (b) All the numbers in the table below are perfect cubes. Choose the one that corresponds to today's day of the week. We will call this number n^3, and we need to find n.

Mon	Tue	Wed	Thu	Fri	Sat	Sun
3375	1728	2197	9261	4096	6859	5832

   (c) Divide the number you have chosen by 6. So it will now be $n^3/6$. Enter both n^3 and $n^3/6$ in the table below. But **do not** use decimals. Use mixed numbers instead, as in Example 1, where $4913/6 = 818\frac{5}{6}$. Also enter in the table the largest sum of squares from the appropriate table that is not larger than $n^3/6$.

n^3	$n^3/6$	Sum of squares

   (d) Find the difference $n^3/6 -$ (Sum of squares) from the last table, and multiply it by 6. Your answer will be the cube root of n^3, or n. Check by cubing your answer to see if you get the number you originally chose.

2. The numbers used in Exercise 1 are all perfect cubes, and Abū Ṭālib's method works fine in that case—even though it is not a very practical means of finding the cube root of a perfect cube! In this exercise, we will show that Abū Ṭālib's method does not even give a good approximation of the cube root if the initial number is not itself a perfect cube.

   (a) Choose a number in the following table according to the last digit of your birthday. So if your birthday is August 17 you would choose the number for 7.

0	1	2	3	4	5	6	7	8	9
5132	4501	1955	3222	4997	2538	6047	7001	2988	4001

   (b) Go through the same steps as before, and cube the final answer. Check that what you get is not even close to the original number.

3. This exercise is for more advanced students.

A brute force method for finding the cube root of a number n is to compute successively all the cubes $1^3, 2^3, 3^3, \ldots$ until we find some k such that $k^3 = n$. This method can be stated as follows:

$$\sqrt[3]{n} = \max\{k : n - k^3 \geq 0\}$$

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Show that Abū Ṭālib's method can be re-stated as the following variation of the brute force method:

If n is a perfect cube, then

$$\sqrt[3]{n} = \begin{cases} 1 + 2\max\{m : n - 2m(2m+1)(2m+2) \geq 0\} & \text{if } n \text{ is odd} \\ 2\max\{m : n - (2m-1)(2m)(2m+1) \geq 0\} & \text{if } n \text{ is even} \end{cases}$$

In other words, instead of using the perfect cubes k^3 to determine the required maximum value, we use the products of three consecutive integers $k-1, k, k+1$.

Hint: Medieval Islamic mathematicians were very familiar with summation formulas, which they also stated in verbal form. Here are modern symbolic translations of their verbal descriptions for two such formulas that will be useful for this exercise:

$$\text{If } n \text{ is even:} \quad 1^2 + 3^2 + 5^2 + \ldots + (n-1)^2 = \frac{(n-1)n(n+1)}{6}$$

$$\text{If } n \text{ is odd:} \quad 2^2 + 4^2 + 6^2 + \ldots + (n-1)^2 = \frac{(n-1)n(n+1)}{6}$$