

# Gaussian Guesswork: Elliptic Integrals and Integration by Substitution

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Just prior to his 19th birthday, the mathematical genius Carl Friedrich Gauss (1777–1855) began a “mathematical diary” in which he recorded his mathematical discoveries for nearly 20 years.<sup>1</sup> Among these discoveries is the existence of a beautiful relationship between three particular numbers:

- the ratio of the circumference of a circle to its diameter, or  $\pi$ ;
- a specific value of a certain (elliptic) integral, which Gauss denoted by  $\varpi = 2 \int_0^1 \frac{dx}{\sqrt{1-x^4}}$ ;
- a number called “the arithmetic-geometric mean” of  $\sqrt{2}$  and 1, which he denoted by  $M(\sqrt{2}, 1)$ .

Like many of his discoveries, Gauss uncovered this particular relationship through a combination of the use of analogy and the examination of computational data, a practice that historian Adrian Rice calls “Gaussian Guesswork” in his *Math Horizons*<sup>2</sup> article subtitled “Why 1.19814023473559220744 . . . is such a beautiful number” [Rice, November 2009].

This project is one of a set of three projects that looks at the power of Gaussian guesswork via the story of his discovery of this beautiful relationship through excerpts from his mathematical diary<sup>3</sup> and related manuscripts. In this project, we focus especially on how Gauss’ discovery led him to the creation of a powerful new technique for evaluating certain types of integrals, such as the one that defines the number  $\varpi$ .

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<sup>1</sup>Gauss’ extraordinary mathematical talent was recognized at a young age and nurtured by his teachers at St. Katherine’s Public School in the family’s home region of Brunswick, Germany. The first in his family to obtain a higher education (his father worked at various trades and his mother was a housemaid before marriage), he completed his degree in mathematics by age 18 and his doctorate by age 22. His academic studies and early research were made possible by the patronage of Charles William Ferdinand, Duke of Brunswick (1735–1806), who paid Gauss’ educational expenses and continued to provide him with a stipend until his own death in 1806. Gauss acknowledged the generosity of the Duke—whom he addressed as “Most Serene Prince”—in the dedication of his famous groundbreaking book, *Disquisitiones Arithmetica (Number Theory Investigations)*, published in 1801. In recognition of his own generous contributions to the field, Gauss has become known as the “Prince of Mathematics.” During his lifetime, he made important discoveries in every area of mathematics that was then known, as well as a few new ones that he helped to create, including elliptic functions. He also applied his mathematical insights to problems in physics, geodesy, magnetism, optics and astronomy. For more about Gauss’ life and works, see the references [Dunnington, 2004], [O’Connor and Robertson, 1996] and [Famous Scientists, 2016].

<sup>2</sup>*Math Horizons* is the undergraduate magazine of the Mathematical Association of America (MAA). It publishes expository articles about mathematics and the culture of mathematics, including mathematical people, institutions, humor, games and book reviews. For more information, visit <https://www.maa.org/press/periodicals/math-horizons>.

<sup>3</sup>Gauss’ diary remained in the possession of his family until 1898 and was first published by Felix Klein (1849–1925) in [Klein, 1903]. An English translation with commentary on its mathematical contents by historian of mathematics Jeremy Gray appears in [Gray, 1984], and was later reprinted in [Dunnington, 2004, 469–496]. A facsimile of the original diary can also be found in Volume X.1 of Gauss’ *Werke (Collected Works)*.



**Task 1** In this task, we consider the integral  $\int \frac{1}{\sqrt{1-x^4}} dx$ .

- Consider the substitution  $x^2 = \sin \theta$ . Explain why this is a natural substitution to try here.
- Now verify that the substitution from part (a) gives the integral result that Gauss noted in his diary entry of January 7, 1797. What strategies might we use in order to evaluate the resulting  $d\theta$  integral? What prevents these strategies from going through?

Interestingly, there is a slight variation on the substitution from Task 1 which allowed Gauss to eventually succeed in evaluating the lemniscate arc length integral. In the next task, we will use this other substitution to transform this integral into a new form—a form that may not seem promising at first glance. In the subsequent sections of this project, we will then work through Gauss’ treatment of the transformed integral to see how he was eventually able to evaluate it.

**Task 2** (a) Let  $x = \sin \theta$ . Show  $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\sin^2 \theta}}$ .

(b) Now re-write  $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\sin^2 \theta}}$  as an integral of the form  $\int_0^{\pi/2} \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}}$ .

State the values of  $m$  and  $n$  clearly.

## 2 The Arithmetic-Geometric Mean

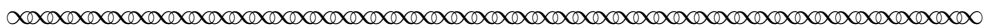
In the next section of this project, we will see how Gauss finally managed to evaluate integrals of the form  $\int_0^{\pi/2} \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}}$ . In this section, we first take a look at the mathematical tool that would eventually allow him to tackle this tricky integral. This tool—a type of average called the *arithmetic-geometric mean*—was discovered by Gauss independently of his work on the lemniscate, but also quite early in his mathematical studies. Although his *Nachlass* (a German term meaning roughly “mathematical legacy”) contains extensive notes about the arithmetic-geometric mean and its properties, it was mentioned only once in his published works, in an important astronomical paper on the gravitational attraction of planets [Gauss, 1818].<sup>6</sup>

Here is what Gauss had to say about the arithmetic-geometric mean in his 1818 paper:<sup>7</sup>

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<sup>6</sup>The rather long (but descriptive!) title of Gauss’ 1818 paper is “Determinatio Attractionis, quam in punctum quodvis positionis datae exerceret planeta, si eius massa per totam orbitam ratione temporis, quo singulae partes describuntur, uniformiter esset dispersita,” or “Determination of the Attraction, which a planet exerts on any point, if its mass is distributed uniformly through the time of the orbit.” All excerpts from Gauss’ 1818 paper used in this project are taken from pages 352–353 of Volume III of his *Werke*; their translations were prepared by George W. Heine III, Math and Maps, 2018.

<sup>7</sup>Additional details of Gauss’ work on the arithmetic-geometric mean can be found in the author’s primary source project *Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean*, available at [https://digitalcommons.ursinus.edu/triumphs\\_calculus/2/](https://digitalcommons.ursinus.edu/triumphs_calculus/2/).



Let  $m, n$  be two positive quantities, and set<sup>8</sup>

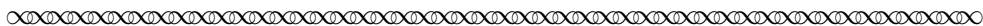
$$m_1 = \frac{1}{2}(m + n), \quad n_1 = \sqrt{mn}$$

so that  $m_1, n_1$  represent the arithmetic mean and the geometric mean, respectively, of  $m$  and  $n$ . The geometric mean will always be taken to be positive.

Similarly set

$$\begin{aligned} m_2 &= \frac{1}{2}(m_1 + n_1), & n_2 &= \sqrt{m_1 n_1} \\ m_3 &= \frac{1}{2}(m_2 + n_2), & n_3 &= \sqrt{m_2 n_2} \end{aligned}$$

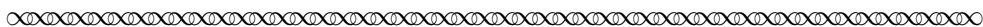
and so on, by which manner [are obtained] the sequences  $m, m_1, m_2, m_3$ , etc., and  $n, n_1, n_2, n_3$ , etc., converging rapidly to a *common limit*, which we denote  $\mu$ , and call simply the *arithmetic-geometric mean* between  $m$  and  $n$ .



**Task 3**

Write (at least) two comments and (at least) two questions about what Gauss' description of the *arithmetic-geometric mean* in the preceding excerpt.

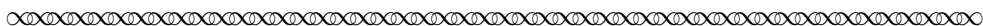
You probably noticed that the value of the arithmetic-geometric mean  $\mu$  depends on the starting values that are used for  $m$  and  $n$ ; we will use the function notation  $\mu(m, n)$  to remind us of this fact.<sup>9</sup> Although Gauss did not give any examples of how to compute arithmetic-geometric mean in his 1818 astronomy paper, here is one of the examples<sup>10</sup> that he worked out much earlier in one of his unpublished *Nachlass* papers, [Gauss, 1799].



*Example 1:  $m = 1, n = 0.2$*

$m = 1.00000\ 00000\ 00000\ 00000\ 0$	$n = 0.20000\ 00000\ 00000\ 00000\ 0$
$m_1 = 0.60000\ 00000\ 00000\ 00000\ 0$	$n_1 = 0.44721\ 35954\ 99957\ 93928\ 2$
$m_2 = 0.52360\ 67977\ 49978\ 99964\ 1$	$n_2 = 0.51800\ 40128\ 22268\ 36005\ 0$
$m_3 = 0.52080\ 54052\ 86123\ 95414\ 3$	$n_3 = 0.52080\ 78709\ 39876\ 24344\ 0$
$m_4 = 0.52080\ 16381\ 06187$	$n_4 = 0.52080\ 16381\ 06187$

Here  $m_5, n_5$  differ in the 23<sup>rd</sup> decimal place.



<sup>8</sup>Gauss himself used prime notation (i.e.,  $m', m'', m'''$ ) to denote the terms of the sequence. In this project, we instead use indexed notation (i.e.,  $m_1, m_2, m_3$ ) in keeping with current notational conventions. To fully adapt Gauss' notation to that used today, we could also write  $m_0 = m$  and  $n_0 = n$ .

<sup>9</sup>Gauss used this same type of function notation in his unpublished writing about the arithmetic-geometric mean. Today, the arithmetic-geometric mean is also sometimes denoted as  $M(m, n)$ .

<sup>10</sup>We have slightly altered Gauss' notation in this example in order to be consistent with the notation used in [Gauss, 1818].

**Task 4**

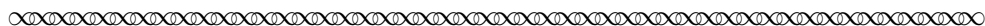
This task examines examples from Gauss' 1799 paper on the arithmetic-geometric mean.

- (a) Verify that the values given by Gauss in Example 1 above are correct. Are you able to use your calculator to obtain the same degree of accuracy (21 decimal places!) that Gauss obtained by hand calculations?
- (b) In Example 1 above, notice how quickly the two sequences converge to the same limiting value, thereby allowing us to assert that  $\mu(1, 0.2) \approx 0.520801638106187$ . Compile some additional numerical evidence concerning Gauss' claim that 'the sequences  $m, m_1, m_2, m_3$ , etc., and  $n, n_1, n_2, n_3$ , etc. converge rapidly to a common limit' by computing the first few terms of the sequences  $(m_k), (n_k)$  in the following two examples from Gauss' 1799 paper. For each, compute a sufficient number of terms to approximate the value of the arithmetic-geometric mean  $\mu(m, n)$  to at least 10 decimal places.
  - (i)  $m = 1, n = 0.8$
  - (ii)  $m = \sqrt{2}, n = 1$
- (c) Based on these three examples, how convincing do you find Gauss' assertion concerning the rapidity of the convergence of the two sequences to the value of  $\mu(m, n)$ ?

In the next section, we will look at how Gauss used the arithmetic-geometric mean to evaluate integrals.

### 3 Gauss' evaluation of the integral $\int \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}}$

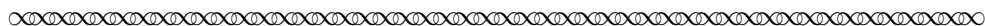
We continue now with our reading of Gauss' paper on the gravitational attraction of planets [Gauss, 1818]. Just after his definition of  $\mu$  as the arithmetic-geometric mean of the numbers  $m$  and  $n$ , Gauss asserted:<sup>11</sup>



Now we shall demonstrate,  $\frac{1}{\mu}$  to be the value of the integral

$$\int \frac{d\theta}{2\pi \sqrt{mm \cos^2 \theta + nn \sin^2 \theta}}$$

from  $\theta = 0$  extended to  $\theta = 2\pi$ .



Let's pause before reading Gauss' proof of this theorem to take a look at how it can be applied, and especially how it relates to the lemniscate's arc length.

**Task 5**

Recall from Task 4 of the previous section that Gauss approximated the arithmetic-geometric mean  $\mu$  for the values of  $m = 1$  and  $n = 0.2$ , finding that  $\mu(1, 0.2) \approx 0.520801638106187$ .

- (a) Write down the integral corresponding to these values of  $m$  and  $n$  that Gauss' theorem tells us how to evaluate.
- (b) Use Gauss' approximated value for  $\mu(1, 0.2)$  and his theorem to give an approximation for the integral from part (a).

<sup>11</sup>In this and subsequent excerpts from Gauss' paper, we have stated all limits of integration in radians; Gauss himself used degrees for this purpose.

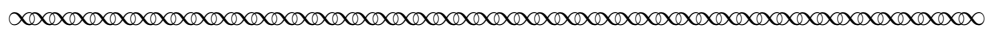
**Task 6**

Recall from Task 2 that  $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2 \theta + 2 \sin^2 \theta}}$ , where this latter integral has the form  $\int \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}}$  with  $m = 1$  and  $n = \sqrt{2}$ . In Task 4, part (b-ii), you also approximated the value of the arithmetic-geometric mean  $\mu(\sqrt{2}, 1)$  by computing the first few terms of the associated arithmetic mean sequence  $(m_k)$  and the associated geometric mean sequence  $(n_k)$ .

Use your approximate value for  $\mu(\sqrt{2}, 1)$  to approximate the arc length of the full lemniscate curve.

Now that we have an idea of the power of Gauss' theorem for evaluating  $\int_0^{2\pi} \frac{d\theta}{2\pi \sqrt{mm \cos^2 \theta + nn \sin^2 \theta}}$ ,

let's tackle his proof! **Read through this proof at least twice**, making note of any questions or comments you may have. The tasks that follow this excerpt will then guide us through the full details.



Now we shall demonstrate,  $\frac{1}{\mu}$  to be the value of the integral

$$\int \frac{d\theta}{2\pi \sqrt{mm \cos^2 \theta + nn \sin^2 \theta}}$$

from  $\theta = 0$  extended to  $\theta = 2\pi$ .

PROOF. We suppose the variable  $\theta$  is expressed by another variable  $\theta_1$ , so that

$$\sin \theta = \frac{2m \sin \theta_1}{(m+n) \cos^2 \theta_1 + 2m \sin^2 \theta_1}$$

[where] it is easily observed that while  $\theta_1$  is increased from 0 to  $\frac{\pi}{2}$ ,  $\pi$ ,  $\frac{3\pi}{2}$ ,  $2\pi$ ,  $\theta$  also (although by different intervals) increases from 0 to  $\frac{\pi}{2}$ ,  $\pi$ ,  $\frac{3\pi}{2}$ ,  $2\pi$ . The expansion duly performed, it is found to be that

$$\frac{d\theta}{\sqrt{mm \cos^2 \theta + nn \sin^2 \theta}} = \frac{d\theta_1}{\sqrt{m_1 m_1 \cos^2 \theta_1 + n_1 n_1 \sin^2 \theta_1}}$$

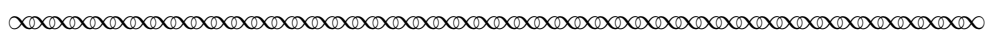
and indeed the values of the integrals

$$\int \frac{d\theta}{2\pi \sqrt{mm \cos^2 \theta + nn \sin^2 \theta}}, \quad \int \frac{d\theta_1}{2\pi \sqrt{m_1 m_1 \cos^2 \theta_1 + n_1 n_1 \sin^2 \theta_1}}$$

if each of the variables is extended continuously from the value 0 to the value  $2\pi$ , equal to each other. And if this [process] is permitted to continue further, clearly these values also are equal to the integral value

$$\int \frac{d\theta}{2\pi \sqrt{\mu\mu \cos^2 \theta + \mu\mu \sin^2 \theta}}$$

from  $\theta = 0$  to  $\theta = 2\pi$ , which evidently becomes  $= \frac{1}{\mu}$ .



Gauss' idea of setting up the substitution  $\sin \theta = \frac{2m \sin \theta_1}{(m+n) \cos^2 \theta_1 + 2m \sin^2 \theta_1}$  certainly gives us some insight into why he is considered a mathematical genius! Before we look at the full details of this particular substitution, let's pause to look at Gauss' overall argument.

The critical idea here is that Gauss' (clever!) substitution produces a sequence of definite integrals that all have the same (constant!) value:

$$\int_0^{2\pi} \frac{d\theta}{2\pi \sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}} = \int_0^{2\pi} \frac{d\theta_1}{2\pi \sqrt{m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1}} = \int_0^{2\pi} \frac{d\theta_2}{2\pi \sqrt{m_2^2 \cos^2 \theta_2 + n_2^2 \sin^2 \theta_2}} = \dots$$

Notice also how the arithmetic mean sequence  $(m_k)$  and geometric mean sequence  $(n_k)$  associated with the starting values of  $m$  and  $n$  come into this integral sequence. As we know from Section 2, the sequences  $(m_k)$  and  $(n_k)$  both converge to the arithmetic-geometric mean  $\mu(m, n)$ . Connecting this fact to the constant sequence of definite integrals above gives us the following surprising, but quite powerful, equality:

$$\int_0^{2\pi} \frac{d\theta}{2\pi \sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}} = \int_0^{2\pi} \frac{d\theta}{2\pi \sqrt{\mu^2 \cos^2 \theta + \mu^2 \sin^2 \theta}}.$$

In the next task, you will give the details that show exactly why this "limit integral" is especially nice. The final task in this section will then guide us through the rather complicated algebraic details of the substitution that Gauss simply told us to 'duly perform'.

**Task 7**

Evaluate the integral  $\int_0^{2\pi} \frac{d\theta}{2\pi \sqrt{\mu^2 \cos^2 \theta + \mu^2 \sin^2 \theta}}$ .

Then explain how this relates back to Gauss' claim that  $\int_0^{2\pi} \frac{d\theta}{2\pi \sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}} = \frac{1}{\mu}$ .

**Task 8**

In this task, we examine the details of Gauss' substitution, which we restate here for ease of reference:

$$\mathbf{GIVEN:} \quad \sin \theta = \frac{2m \sin \theta_1}{(m+n) \cos^2 \theta_1 + 2m \sin^2 \theta_1}$$

We will do this through a series of steps<sup>12</sup> to see how the "expansion duly performed" gives the following:

$$\mathbf{GOAL:} \quad \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}} = \frac{d\theta_1}{\sqrt{m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1}}$$

The key to setting up the new integrand in  $\theta_1$  will be to find two different (but equal) expressions in  $\theta_1$  for the differential  $d(\sin \theta)$ . We first do this using the (original) constants  $m, n$  from the  $d\theta$  integral

<sup>12</sup>If you enjoy algebraic challenges, you could first try your hand at carrying the substitution through on your own first. But even if you decide not to take the full plunge, get ready for a bit of a workout! The particular outline given in this task is based on details provided by Carl Gustav Jacob Jacobi (1804–1851) in his own celebrated book on elliptic functions [Jacobi, 1829]. Gauss himself left no record of his reasoning.

throughout; the end result of this work will come from Differential Equality #2 in part (e) below. Part (f) will then look at how to bring the (new) constants  $m_1, n_1$  into the  $d\theta_1$  integrand.

**In each part of this task, you should complete the details that appear in bold font.**

- (a) We start by proving a fact that will be useful in re-writing the given substitution (in the first box above) so that only one trigonometric function ( $\sin \theta_1$ ) appears on its right-hand side; this will make certain steps (e.g., the derivative of the right-hand side of the given substitution) easier to carry out. (Eventually, we will need to bring the cosine back into the equation.)

**Use the Pythagorean identity  $\cos^2 \theta_1 = 1 - \sin^2 \theta_1$  to show that the following holds:**

<b>Part (a) End Result</b>	$(m + n) \cos^2 \theta_1 + 2m \sin^2 \theta_1 = (m + n) + (m - n) \sin^2 \theta_1$
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- (b) Using the end result of part (a), we can now re-write the given substitution as follows:

<b>GIVEN (re-written):</b>	$\sin \theta = \frac{2m \sin \theta_1}{(m + n) + (m - n) \sin^2 \theta_1}$
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**Use the quotient rule to differentiate the right-hand side of this expression, then simplify in order to show that the following holds:**

<b>Differential Equality #1:</b>	$\cos \theta d\theta = \frac{2m \cos \theta_1 [(m + n) - (m - n) \sin^2 \theta_1]}{[(m + n) + (m - n) \sin^2 \theta_1]^2} d\theta_1$
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- (c) Our next step will be to re-write  $\cos \theta$  in terms of the variable  $\theta_1$ . (Eventually, we will substitute this into the left-hand side of Differential Equality #1.)
- (i) We start by again using the Pythagorean Identity  $\cos^2 \theta = 1 - \sin^2 \theta$  and the re-written form of the given substitution from part (b) as follows. *Be sure you see why each step holds!*

$$\begin{aligned} \cos^2 \theta &= 1 - \sin^2 \theta = && 1 - \left[ \frac{2m \sin \theta_1}{(m + n) + (m - n) \sin^2 \theta_1} \right]^2 \\ &= && \frac{[(m + n) + (m - n) \sin^2 \theta_1]^2 - 4m^2 \sin^2 \theta_1}{[(m + n) + (m - n) \sin^2 \theta_1]^2} \\ &= && \frac{\overbrace{(m + n)^2 + 2(m + n)(m - n) \sin^2 \theta_1 + (m - n)^2 \sin^4 \theta_1 - 4m^2 \sin^2 \theta_1}}{\text{Set this equal to } V. \\ &&& [(m + n) + (m - n) \sin^2 \theta_1]^2} \end{aligned}$$

**Simplify the numerator  $V$  of the right-hand side of this last expression to show that the following holds:**

$$V = (m + n)^2 - 2(m^2 + n^2) \sin^2 \theta_1 + (m - n)^2 \sin^4 \theta_1 \quad (\diamond)$$



Next show that the expression for  $V$  given in  $(\diamond)$  can be re-written to get:<sup>13</sup>

$$V = \cos^2 \theta_1 [(m+n)^2 - (m-n)^2 \sin^2 \theta_1] \quad (\heartsuit)$$

Finally, substitute the expression given for  $V$  in  $(\heartsuit)$  back into the numerator of the expression we obtained above for  $\cos^2 \theta$  in order to get the following:

<b>Part (c-i) End Result</b>	$\cos^2 \theta = \frac{\cos^2 \theta_1 [(m+n)^2 - (m-n)^2 \sin^2 \theta_1]}{[(m+n) + (m-n) \sin^2 \theta_1]^2}$
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(ii) Now use the result of part (i) to write an expression for  $\cos \theta$ .

*(There will be a square root in this expression!)*

- (d) Before we go back to the differential expression, it will be helpful to also re-write the expression  $\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}$  in terms of  $\sin \theta_1$ . *(Do you remember why we are interested in the expression  $\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}$ ? If not, look back at Task 7!)*

Dropping the square root for now, let's start by substituting the expressions for  $\sin \theta$  and  $\cos^2 \theta$  from parts (b) and (c) respectively into this expression, simplifying a bit, and using the Pythagorean Identity once more. *Again, be sure you see why each step holds!*

$$\begin{aligned} m^2 \cos^2 \theta + n^2 \sin^2 \theta &= m^2 \left[ \frac{\cos^2 \theta_1 [(m+n)^2 - (m-n)^2 \sin^2 \theta_1]}{[(m+n) + (m-n) \sin^2 \theta_1]^2} \right] + n^2 \left[ \frac{2m \sin \theta_1}{(m+n) + (m-n) \sin^2 \theta_1} \right]^2 \\ &= \frac{m^2 \cos^2 \theta_1 [(m+n)^2 - (m-n)^2 \sin^2 \theta_1] + 4m^2 n^2 \sin^2 \theta_1}{[(m+n) + (m-n) \sin^2 \theta_1]^2} \\ &= m^2 \frac{\overbrace{(1 - \sin^2 \theta_1) [(m+n)^2 - (m-n)^2 \sin^2 \theta_1] + 4n^2 \sin^2 \theta_1}}{\text{Set this equal to } W}{[(m+n) + (m-n) \sin^2 \theta_1]^2} \end{aligned}$$

Expand, then simplify, the expression  $W$ ; then show that the following holds:

$$W = [(m+n) - (m-n) \sin^2 \theta_1]^2$$

Conclude that the following holds:

$$m^2 \cos^2 \theta + n^2 \sin^2 \theta = \left[ m \frac{(m+n) - (m-n) \sin^2 \theta_1}{(m+n) + (m-n) \sin^2 \theta_1} \right]^2,$$

or equivalently,

<b>Part (d) End Result</b>	$\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta} = m \frac{(m+n) - (m-n) \sin^2 \theta_1}{(m+n) + (m-n) \sin^2 \theta_1},$
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<sup>13</sup>A sophisticated way to do this is to use the equality  $2(m^2 + n^2) = (m+n)^2 + (m-n)^2$  to rewrite the middle term of  $(\diamond)$ , then factor  $(\diamond)$  by grouping. Another option is to start with the right-hand side of  $(\heartsuit)$  and work backwards to get  $(\diamond)$ . Either way, the Pythagorean Identity will be useful.

(e) We're now ready to go back to the differential equality that we found in part (b):

$$\text{Differential Equality \#1: } \cos \theta d\theta = \frac{2m \cos \theta_1 [(m+n) - (m-n) \sin^2 \theta_1]}{[(m+n) + (m-n) \sin^2 \theta_1]^2} d\theta_1$$

Substituting the expression for  $\cos \theta$  from part (c-ii) into Differential Equality #1, we get:

$$\text{Differential Equality \#2: } \frac{\cos \theta_1 \sqrt{(m+n)^2 - (m-n)^2 \sin^2 \theta_1}}{(m+n) + (m-n) \sin^2 \theta_1} d\theta = \frac{2m \cos \theta_1 [(m+n) - (m-n) \sin^2 \theta_1]}{[(m+n) + (m-n) \sin^2 \theta_1]^2} d\theta_1$$

Use the end result of part (d) and some simplification to show that we can re-write Differential Equality #2 as follows:

$$\text{Part (e) End Result } \sqrt{(m+n)^2 - (m-n)^2 \sin^2 \theta_1} d\theta = 2\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta} d\theta_1$$

(f) Rearranging the result of part (e) gives us the following, which is very close to what we want:

$$\text{Differential Equality \#3: } \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}} = \frac{2d\theta_1}{\sqrt{(m+n)^2 - (m-n)^2 \sin^2 \theta_1}}$$

All that remains is to see how the expression involving the constants  $m, n$  on the right-hand side relates to one involving the constants  $m_1, n_1$ . The key to doing this is to recall that  $m_1 = \frac{m+n}{2}$  and  $n_1 = \sqrt{mn}$ . Use these facts to first show the following:

$$\text{Part (f) Result \#1 } 4(m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1) = (m+n)^2 - (m-n)^2 \sin^2 \theta_1$$

*Hint: The Pythagorean Identity will be useful once more!*

Then use this Part (f) Result #1 to show that the right-hand side of Differential Equation #3 can be re-written as follows:

$$\text{Part (f) Result \#2 } \frac{2d\theta_1}{\sqrt{(m+n)^2 - (m-n)^2 \sin^2 \theta_1}} = \frac{d\theta_1}{\sqrt{m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1}}$$

(g) Substituting Result #2 of Part (f) into Differential Equation #3 leaves us with just one last piece to explain about Gauss' final conclusion that

$$\int_0^{2\pi} \frac{d\theta}{2\pi \sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}} = \int_0^{2\pi} \frac{d\theta_1}{2\pi \sqrt{m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1}}$$

Complete the proof by explaining why the limits of integration do not change under Gauss' substitution. (Gauss commented on these limits in the first sentence of his proof.)

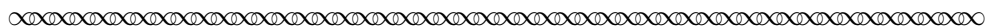
## 4 Why *is* 1.19814023473559220744... such a beautiful number?

In the introduction to this project, we mentioned that historian Adrian Rice’s article “Gaussian Guesswork” was cleverly subtitled “Why 1.19814023473559220744... is such a beautiful number” [Rice, November 2009]. Since then, we’ve met this very same number in Task 6, where we found that 1.19814023473559220744... is a good approximation for the arithmetic-geometric mean  $\mu(1, \sqrt{2})$ . Task 6 also highlights one of the reasons why the specific arithmetic-geometric mean  $\mu(1, \sqrt{2})$  was of special interest to Gauss; namely

$$\frac{1}{\mu(\sqrt{2}), 1} = \int_0^{2\pi} \frac{d\theta}{2\pi\sqrt{\cos^2 \theta + 2\sin^2 \theta}}$$

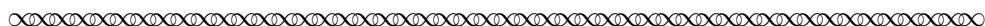
Now recall that  $\int_0^{2\pi} \frac{d\theta}{\sqrt{\cos^2 \theta + 2\sin^2 \theta}} = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$ , where  $\varpi = 2 \int_0^1 \frac{1}{\sqrt{1-t^4}} dt$ . Rearranging things as needed (*do this!*), this lets us conclude (along with Gauss) that  $\mu(\sqrt{2}, 1) = \frac{\pi}{\varpi}$  ... a rather surprising connection between three apparently unrelated numbers!!

Here is Gauss’ recording of his initial *discovery* of this relationship:



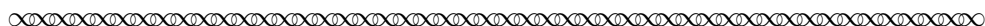
We have established that the arithmetic-geometric mean between 1 and  $\sqrt{2}$  is  $\pi/\varpi$  to 11 places; the proof of this fact will certainly open up a new field of analysis.

May 30, 1799



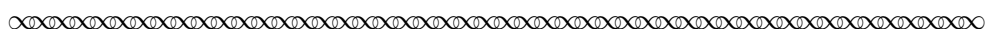
Interestingly, this discovery was largely based on numerical guesswork! In this project, we have only been able to hint at aspects of that guesswork. Gauss’ work with the arithmetic-geometric mean illustrates this somewhat: he first calculated the value of  $\mu(\sqrt{2}, 1) = 1.19814023473559220744...$  as one of his examples in [Gauss, 1799], well before he saw its connection to the elliptic integral  $\int_0^1 \frac{1}{\sqrt{1-t^4}} dt$ . Gauss also used power series techniques on this integral to numerically estimate the value of  $\varpi$ , finding that  $\varpi = 2.662057055429211981046...$ . And, of course, he knew the value  $\pi = 3.14159265258979323846...$ . Although it is straightforward to check that the relationship  $\mu(\sqrt{2}, 1) = \frac{\pi}{\varpi}$  is reasonable once these three estimates are put in front of us in this way—you should do this!—we will likely never know exactly what stroke of genius led Gauss to think about their possible connection in the first place.

Following his discovery that  $\mu(\sqrt{2}, 1) = \frac{\pi}{\varpi}$ , it took Gauss nearly another full year to prove his guesswork was correct. But, as he predicted, this discovery went well beyond just this one numerical relationship. We have seen, for instance, how Gauss’ theorem allows us to numerically estimate *any* integral of the form  $\int_a^b \frac{d\theta}{2\pi\sqrt{m\cos^2 \theta + n\sin^2 \theta}}$  by simply making use of the very rapid convergence of the arithmetic-geometric mean. The ‘new field of analysis’ that opened up in connection with this proof led him well beyond the study of elliptic functions of a single real-valued variable, and into the realm of functions of several complex-valued variables. Today, a special class of such functions known as the ‘theta functions’ provide a powerful tool that is used in a wide range of applications throughout mathematics—providing yet one more piece of evidence of Gauss’ extraordinary ability as a mathematician and a guesswork genius!



The arithmetic-geometric mean itself is an integral quantity. Proved.

December 23, 1799



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## Notes to Instructors

### PSP Content: Topics and Goals

This mini-Primary Source Project (mini-PSP) is one of a set of three mini-PSPs designed to consolidate student proficiency of the following traditional topics from a first-year Calculus course:

- Gaussian Guesswork: Elliptic Integrals and Integration by Substitution
- Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve
- Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean

Each of these mini-PSPs can be used either alone or in conjunction with any of the other three. All three are based on excerpts from Gauss's mathematical diary and related primary texts that will introduce students to the power of numerical experimentation via the story of his discovery of a relationship between three particular numbers: the ratio of the circumference of a circle to its diameter ( $\pi$ ); a specific value ( $\varpi$ ) of the elliptic integral  $u = \int_0^x \frac{dt}{\sqrt{1-t^4}}$ ; and the Arithmetic-Geometric Mean of 1 and  $\sqrt{2}$ . Like many of his discoveries, Gauss uncovered this particular relationship through a combination of the use of analogy and the examination of computational data, a practice referred to as "Gaussian Guesswork" by historian Adrian Rice in his *Math Horizons* article subtitled "Why 1.19814023473559220744... is such a beautiful number" [Rice, November 2009].

The primary content goal of this particular mini-PSP is to consolidate students' understanding of integration by substitution. The project's core content appears in Task 8, in which students work through the details of a sophisticated substitution used by Gauss to evaluate integrals of the form  $\int_0^{2\pi} \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}}$ . Although this particular substitution is not itself part of the standard Calculus II curriculum, working through its details provides an excellent opportunity for Calculus II students to apply and consolidate core concepts and techniques, and to witness their interplay within the context of some amazingly beautiful, and important, mathematics!

### Student Prerequisites

Completion of this project requires an understanding of basic integration concepts, including familiarity with integration by substitution. Some familiarity with the technique of trigonometric substitution is also assumed, but only for the completion of Task 1. Especially for Section 3, basic knowledge of trigonometric identities and derivatives (for the sine and cosine function only) is essential, as is an understanding of the principles of algebraic manipulation. Although infinite sequences appear in the project, they do so in a fairly straightforward way that requires little more than the ability to numerically compute arithmetic and geometric means. The project could thus easily be used in the course prior to the formal study of infinite sequences.

### PSP Design and Task Commentary

Following a brief introduction to set the historical stage, Section 1 sets the mathematical stage by prompting students to consider the difficulties involved in evaluating the definite integral  $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ . Task 2 in that same section then has students use substitution and the Pythagorean Identity in order to transform that integral  $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$  to an integral of the form  $\int_0^{\pi/2} \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}}$ . Section 2 next provides a brief introduction to the concept of the arithmetic-geometric mean which is sufficient for understanding the idea behind Gauss'

proof of the theorem which states  $\int_0^{2\pi} \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}} = \mu(m, n)$ , where  $\mu(m, n)$  denotes the arithmetic-geometric mean of  $m$  and  $n$ . That theorem and its proof are then presented in Section 3, starting with several tasks that explore the meaning (and power!) of the theorem statement, as well as the general idea behind Gauss’ proof plan. As noted above, Task 8 in that section then guides students through the details of the substitution used in the proof. Although that task is heavily scaffolded, verifying the algebraic details outlined therein does require considerable focus and effort on the part of the students. A short concluding section then ties the mathematical details of this particular project to the larger story of Gauss’ amazing discovery.

## Suggestions for Classroom Implementation

Classroom implementation of this and other mini-PSPs in the collection may be accomplished through individually assigned work, small-group work and/or whole-class discussion; a combination of these instructional strategies is recommended in order to take advantage of the variety of questions included in the project.

To reap the full mathematical benefits offered by the PSP approach, students should be required to read assigned sections in advance of in-class work, and to work through primary source excerpts together in small groups in class. The author’s method of ensuring that advance reading takes place is to require student completion of “Reading Guides” (or “Entrance Tickets”). These Reading Guides typically include “Classroom Preparation” exercises (drawn from the PSP Tasks) for students to complete prior to arriving in class; they may also include “Discussion Questions” that ask students only to read a given task and jot down some notes in preparation for class work. On occasion, tasks are also assigned as follow-up to a prior class discussion. In addition to supporting students’ advance preparation efforts, these guides provide helpful feedback to the instructor about individual and whole-class understanding of the material. The author’s students receive credit for completion of each Reading Guide (with no penalty for errors in solutions). A sample guide (based on the Day 1 Advanced Preparation Work suggested in the Sample Implementation Schedule given below) is appended to the end of these Notes.

## Sample Implementation Schedule (based on a 75-minute class period)

To complete this particular mini-PSP in its entirety, the following implementation schedule is recommended:

- **Advance Preparation Work for Day 1** (to be completed before class): Read through the end of Section 2, completing Tasks 1–4 for class discussion along the way, per the sample Reading Guide in Appendix I.
- **Class Work for Day 1**
  - Brief whole-class or small-group comparison of answers to Tasks 1–4.
  - Small-group work on Tasks 5–8 (supplemented by whole-class discussion as deemed appropriate by the instructor). Note that Task 8 forms the core of the material in this mini-PSP, but will require careful attention to algebraic details on the part of the students. Small-group work interspersed with whole-class discussion is recommended in order to help students keep track of the overall direction and goal of this task. As needed (or desired), continued work on Task 8 could be assigned as follow-up to the work completed in class. **An in-class worksheet that guides students through the derivations in Task 8 and offers helpful suggestions at key junctures is included at the end of these Notes to Instructors.**

- **Homework (optional):** A complete formal write-up of student work on some or all of Tasks 2–7 could be assigned, to be due at a later date (e.g., one week after completion of the in-class work).
- **Advance Preparation Work for Day 2** (to be completed before class): Read the project conclusion in Section 4. As needed (or desired), continued “advance work” on Task 8 could also be assigned in preparation for small-group or whole-class discussion on Day 2.
- **Class Work for Day 2** (based on a 75-minute class period)  
*The amount of time required to complete the following on Day 2 will naturally vary depending on the instructor’s goals and students’ backgrounds, but is not expected to require an entire class period.*
  - As appropriate, continued small-group work on Task 8; if advance preparation work was assigned on this task, this could consist simply of a comparison of individual student answers.
  - Concluding whole group discussion of the mathematical ideas in the project.
- **Homework (optional):** A complete formal write-up of student work on Task 8 could be assigned, to be due at a later date (e.g., one week after completion of the in-class work).

L<sup>A</sup>T<sub>E</sub>X code of the entire PSP is available from the author by request to facilitate preparation of reading guides or ‘in-class task sheets’ based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

## Connections to other Primary Source Projects

Links to all three “Gaussian Guesswork” PSPs (described earlier in these Notes) are as follows:

- Gaussian Guesswork: Elliptic Integrals and Integration by Substitution  
[https://digitalcommons.ursinus.edu/triumphs\\_calculus/8/](https://digitalcommons.ursinus.edu/triumphs_calculus/8/)
- Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve  
[https://digitalcommons.ursinus.edu/triumphs\\_calculus/3/](https://digitalcommons.ursinus.edu/triumphs_calculus/3/)
- Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean  
[https://digitalcommons.ursinus.edu/triumphs\\_calculus/2/](https://digitalcommons.ursinus.edu/triumphs_calculus/2/)

The following additional projects based on primary sources are also freely available for use in teaching standard topics in the calculus sequence. The PSP author name of each is given (together with the general content focus, if this is not explicitly given in the project title). Each of these can be completed in 1–2 class days, with the exception of the three projects followed by an asterisk (\*) which require 3, 4 and 6 days respectively for full implementation. Classroom-ready versions of these projects can be downloaded from [https://digitalcommons.ursinus.edu/triumphs\\_calculus](https://digitalcommons.ursinus.edu/triumphs_calculus).

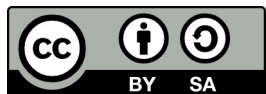
- Investigations Into d’Alembert’s Definition of Limit (calculus version), Dave Ruch
- L’Hôpital’s Rule, Danny Otero
- The Derivatives of the Sine and Cosine Functions, Dominic Klyve
- Fermat’s Method for Finding Maxima and Minima, Kenneth M Monks
- Beyond Riemann Sums: Fermat’s Method of Integration, Dominic Klyve
- How to Calculate  $\pi$ : Buffon’s Needle (calculus version), Dominic Klyve (integration by parts)
- How to Calculate  $\pi$ : Machin’s Inverse Tangents, Dominic Klyve (infinite series)



- Euler’s Calculation of the Sum of the Reciprocals of Squares, Kenneth M Monks (infinite series)
- Fourier’s Proof of the Irrationality of  $e$ , Kenneth M Monks (infinite series)
- Jakob Bernoulli Finds Exact Sums of Infinite Series (Calculus Version),\* Danny Otero and James Sellars
- Bhāskara’s Approximation to and Mādhava’s Series for Sine, Kenneth M Monks (approximation, power series)
- Braess’ Paradox in City Planning: An Application of Multivariable Optimization, Kenneth M Monks
- Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green’s Theorem,\* Abe Edwards
- The Radius of Curvature According to Christiaan Huygens,\* Jerry Lodder

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For more information about the NSF-funded project TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS), visit:

<https://blogs.ursinus.edu/triumphs/>



## APPENDIX I

This appendix provides a ‘Sample Reading Guide’ that illustrates the author’s method for assigning advance preparation work in connection with classroom implementation of primary source projects. More detail concerning these guides is included in the subsection “Suggestions for Classroom Implementation” of the Notes to Instructors for this project.

## SAMPLE READING GUIDE: Advance Preparation Work for Day 1

Background Information: The goal of the advance reading and tasks assigned in this 3-page reading guide is to lay the groundwork for in-class work on the tasks in Section 3 of this project, by having students first consider possible techniques for approaching the integral in question (Tasks 1 & 2) and introducing them to the basic idea of the arithmetic-geometric mean (Tasks 3 & 4) which is central to the substitution used by Gauss in the proof that forms the core of Section 3.

\*\*\*\*\*

**Reading Assignment:** *Gaussian Guesswork: Elliptic Integrals and Integration by Substitution*, pp. 1–4

1. Read the introduction on page 1.

*Any questions or comments?*

2. In Section 1, read pages 1–2, stopping at Task 1.

*Any questions or comments?*

3. **Class Prep:** Complete **Task 1** here:

**Task 1** In this task, we consider the integral  $\int \frac{1}{\sqrt{1-x^4}} dx$ .

- (a) Consider the substitution  $x^2 = \sin \theta$ . Explain why this is a natural substitution to try here.

- (b) Now verify that the substitution from part (a) gives the integral result that Gauss noted in his diary entry of January 7, 1797. What strategies might we use in order to evaluate the resulting  $d\theta$  integral? What prevents these strategies from going through?

4. Read the paragraph between Task 1 and Task 2.

*Any questions or comments?*

5. **Class Prep:** Complete **Task 2** here:

**Task 2**

(a) Let  $x = \sin \theta$ . Show  $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\sin^2 \theta}}$ .

(b) Now re-write  $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\sin^2 \theta}}$  as an integral of the form  $\int_0^{\pi/2} \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}}$ .  
State the values of  $m$  and  $n$  clearly.

6. In the beginning of Section 2, stopping at Task 3.

*Any questions or comments?*

7. **Class Prep:** Complete **Task 3** here:

**Task 3**

Write (at least) two comments and (at least) two questions about what Gauss' description of the *arithmetic-geometric mean* in the first excerpt on page 4.

8. **Class Prep:** Complete **Task 4** here:

**Task 4** This task examines examples from Gauss' 1799 paper on the arithmetic-geometric mean.

- (a) Verify that the values given by Gauss in Example 1 (in the second excerpt on page 4) are correct. Are you able to use your calculator to obtain the same degree of accuracy (21 decimal places!) that Gauss obtained by hand calculations?
- (b) In Example 1 above, notice how quickly the two sequences converge to the same limiting value, thereby allowing us to assert that  $\mu(1, 0.2) = 0.520801638106187\dots$ . Compile some additional numerical evidence concerning Gauss' claim that 'the sequences  $m, m_1, m_2, m_3$ , etc., and  $n, n_1, n_2, n_3$ , etc. converge rapidly to a common limit' by computing the first few terms of the sequences  $(m_k), (n_k)$  in the following two examples from Gauss' 1799 paper. For each, compute a sufficient number of terms to approximate the value of the arithmetic-geometric mean  $\mu(m, n)$  to at least 10 decimal places.
- (i)  $m = 1, n = 0.8$
- (ii)  $m = \sqrt{2}, n = 1$
- (c) Based on these three examples, how convincing do you find Gauss' assertion concerning the rapidity of the convergence of the two sequences to the value of  $\mu(m, n)$ ?

## APPENDIX II

This appendix provides an (8-page!) in-class worksheet for Task 8, which is designed to guide students through the details of the substitution used by Gauss in his proof concerning the evaluation of integrals of the form  $\int_0^{2\pi} \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}}$ . Closely supervised small-group work interspersed with whole-class discussion is recommended in order to help students keep track of the overall direction and goal of this task.

## Task 8 Worksheet

In this task, we examine the details of Gauss' substitution, restated here for ease of reference:

$$\mathbf{GIVEN:} \quad \sin \theta = \frac{2m \sin \theta_1}{(m+n) \cos^2 \theta_1 + 2m \sin^2 \theta_1}$$

The series of steps outlined below<sup>1</sup> show how to get from this substitution to the following equality:<sup>2</sup>

$$\mathbf{GOAL:} \quad \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}} = \frac{d\theta_1}{\sqrt{m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1}}$$

In each part of this task, you should complete the details that appear in bold font.

(a) Use the Pythagorean identity  $\cos^2 \theta_1 = 1 - \sin^2 \theta_1$  to show that the following holds:

$$\mathbf{Part (a) End Result} \quad (m+n) \cos^2 \theta_1 + 2m \sin^2 \theta_1 = (m+n) + (m-n) \sin^2 \theta_1$$

---

<sup>1</sup>If you enjoy algebraic challenges, you could first try your hand at carrying the substitution through on your own first. But even if you decide not to take the full plunge, get ready for a bit of a workout! The particular outline given in this task is based on details provided by Carl Gustav Jacob Jacobi (1804–1851) in his own celebrated book on elliptic functions, written in 1829. Gauss himself left no record of his reasoning.

<sup>2</sup>The key to setting up the new integrand in  $\theta_1$  will be to find two different (but equal) expressions in  $\theta_1$  for the differential  $d(\sin \theta)$ . We first do this using the (original) constants  $m, n$  from the  $d\theta$  integral throughout; the end result of this work will come from Differential Equality #2 in part (e) below. Part (f) will then look at how to bring the (new) constants  $m_1, n_1$  into the  $d\theta_1$  integrand.

(b) Using the end result of part (a), we can now re-write the given substitution as follows:

$$\mathbf{GIVEN (re-written):} \quad \sin \theta = \frac{2m \sin \theta_1}{(m+n) + (m-n) \sin^2 \theta_1}$$

Use the quotient rule to differentiate the right-hand side of this expression, then simplify in order to show that the following holds:

$$\mathbf{Differential Equality \#1:} \quad \cos \theta d\theta = \frac{2m \cos \theta_1 [(m+n) - (m-n) \sin^2 \theta_1]}{[(m+n) + (m-n) \sin^2 \theta_1]^2} d\theta_1$$

- (c) Our next step will be to re-write  $\cos \theta$  in terms of the variable  $\theta_1$ .  
 (Eventually, we will substitute this into the left-hand side of Differential Equality #1.)

- (i) We start by again using the Pythagorean Identity  $\cos^2 \theta = 1 - \sin^2 \theta$  and the re-written form of the given substitution from part (b) as follows. *Be sure you see why each step holds!*

$$\begin{aligned} \cos^2 \theta &= 1 - \sin^2 \theta = && 1 - \left[ \frac{2m \sin \theta_1}{(m+n) + (m-n) \sin^2 \theta_1} \right]^2 \\ &= && \frac{[(m+n) + (m-n) \sin^2 \theta_1]^2 - 4m^2 \sin^2 \theta_1}{[(m+n) + (m-n) \sin^2 \theta_1]^2} \\ &= && \frac{\overbrace{(m+n)^2 + 2(m+n)(m-n) \sin^2 \theta_1 + (m-n)^2 \sin^4 \theta_1 - 4m^2 \sin^2 \theta_1}}{\text{Set this equal to } V.} \\ &= && \frac{(m+n)^2 + 2(m+n)(m-n) \sin^2 \theta_1 + (m-n)^2 \sin^4 \theta_1 - 4m^2 \sin^2 \theta_1}{[(m+n) + (m-n) \sin^2 \theta_1]^2} \end{aligned}$$

**Simplify the numerator  $V$  of the right-hand side of this last expression to show that the following holds:**

$$V = (m+n)^2 - 2(m^2 + n^2) \sin^2 \theta_1 + (m-n)^2 \sin^4 \theta_1 \quad (\diamond)$$



**Part (c-i) Continued**

**Recall from preceding page:**  $V = (m + n)^2 - 2(m^2 + n^2) \sin^2 \theta_1 + (m - n)^2 \sin^4 \theta_1$  ( $\diamond$ )

Next show that the expression for  $V$  given in ( $\diamond$ ) can be re-written to get:<sup>3</sup>

$$V = \cos^2 \theta_1 [(m + n)^2 - (m - n)^2 \sin^2 \theta_1] \quad (\heartsuit)$$

Finally, substitute the expression given for  $V$  in ( $\heartsuit$ ) back into the numerator of the expression we obtained above for  $\cos^2 \theta_1$  in order to get the following:

**Part (c-i) End Result**  $\cos^2 \theta = \frac{\cos^2 \theta_1 [(m + n)^2 - (m - n)^2 \sin^2 \theta_1]}{[(m + n) + (m - n) \sin^2 \theta_1]^2}$

- (ii) Now use the result of part (i) to write an expression for  $\cos \theta$ . **Box your answer, and label it “Part (c-ii) End Result”.** (*There will be a square root in this expression!*)

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<sup>3</sup>A sophisticated way to do this is to use the equality  $2(m^2 + n^2) = (m + n)^2 + (m - n)^2$  to rewrite the middle term of ( $\diamond$ ), then factor ( $\diamond$ ) by grouping. Another option is to start with the right-hand side of ( $\heartsuit$ ) and work backwards to get ( $\diamond$ ). Either way, the Pythagorean Identity will be useful.

(d) It will be helpful to also re-write the expression  $\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}$  in terms of  $\sin \theta_1$ .

Dropping the square root for now, let's start by substituting the expressions for  $\sin \theta$  and  $\cos^2 \theta$  from parts (b) and (c) respectively into this expression, simplifying a bit, and using the Pythagorean Identity once more. *Again, be sure you see why each step holds!*

$$\begin{aligned}
 m^2 \cos^2 \theta + n^2 \sin^2 \theta &= m^2 \left[ \frac{\cos^2 \theta_1 [(m+n)^2 - (m-n)^2 \sin^2 \theta_1]}{[(m+n) + (m-n) \sin^2 \theta_1]^2} \right] + n^2 \left[ \frac{2m \sin \theta_1}{(m+n) + (m-n) \sin^2 \theta_1} \right]^2 \\
 &= \frac{m^2 \cos^2 \theta_1 [(m+n)^2 - (m-n)^2 \sin^2 \theta_1] + 4m^2 n^2 \sin^2 \theta_1}{[(m+n) + (m-n) \sin^2 \theta_1]^2} \\
 &= m^2 \overbrace{\frac{(1 - \sin^2 \theta_1) [(m+n)^2 - (m-n)^2 \sin^2 \theta_1] + 4n^2 \sin^2 \theta_1}{[(m+n) + (m-n) \sin^2 \theta_1]^2}}^{\text{Set this equal to } W}
 \end{aligned}$$

**Expand, then simplify, the expression  $W$ ; then show that the following holds:**

$$W = [(m+n) - (m-n) \sin^2 \theta_1]^2$$

**Conclude that the following holds:**

$$m^2 \cos^2 \theta + n^2 \sin^2 \theta = \left[ m \frac{(m+n) - (m-n) \sin^2 \theta_1}{(m+n) + (m-n) \sin^2 \theta_1} \right]^2,$$

or equivalently,

<b>Part (d) End Result</b>	$\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta} = m \frac{(m+n) - (m-n) \sin^2 \theta_1}{(m+n) + (m-n) \sin^2 \theta_1},$
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(e) We're now ready to go back to the differential equality that we found in part (b):

$$\textbf{Differential Equality \#1:} \quad \cos \theta d\theta = \frac{2m \cos \theta_1 [(m+n) - (m-n) \sin^2 \theta_1]}{[(m+n) + (m-n) \sin^2 \theta_1]^2} d\theta_1$$

Substituting the expression for  $\cos \theta$  from part (c-ii) into Differential Equality #1, we get:

$$\textbf{Differential Equality \#2:}$$
$$\frac{\cos \theta_1 \sqrt{(m+n)^2 - (m-n)^2 \sin^2 \theta_1}}{(m+n) + (m-n) \sin^2 \theta_1} d\theta = \frac{2m \cos \theta_1 [(m+n) - (m-n) \sin^2 \theta_1]}{[(m+n) + (m-n) \sin^2 \theta_1]^2} d\theta_1$$

Use the end result of part (d) and some simplification to show that we can re-write Differential Equality #2 as follows:

$$\textbf{Part (e) End Result} \quad \sqrt{(m+n)^2 - (m-n)^2 \sin^2 \theta_1} d\theta = 2\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta} d\theta_1$$

(f) Rearranging the result of part (e) gives us the following:

$$\text{Differential Equality \#3: } \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}} = \frac{2d\theta_1}{\sqrt{(m+n)^2 - (m-n)^2 \sin^2 \theta_1}}$$

This is very close to what we want:

$$\text{GOAL: } \frac{d\theta}{\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}} = \frac{d\theta_1}{\sqrt{m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1}}$$

All that remains is to see how the expression involving the constants  $m, n$  on the right-hand side of Differential Equation #3 relates to the constants  $m_1, n_1$  on the right-hand side of our goal equality.

**The key to doing this is to recall that  $m_1 = \frac{m+n}{2}$  and  $n_1 = \sqrt{mn}$ .**

**Use these facts to first show the following:**

$$\text{Part (f) Result \#1 } 4(m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1) = (m+n)^2 - (m-n)^2 \sin^2 \theta_1$$

*Hint: The Pythagorean Identity will be useful once more!*

**Then use this Part (f) Result #1 to show that the right-hand side of Differential Equation #3 can be re-written as follows:**

$$\text{Part (f) Result \#2 } \frac{2d\theta_1}{\sqrt{(m+n)^2 - (m-n)^2 \sin^2 \theta_1}} = \frac{d\theta_1}{\sqrt{m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1}}$$

- (g) Substituting Result #2 of Part (f) into Differential Equation #3 leaves us with just one last piece to explain about Gauss' final conclusion that

$$\int_0^{2\pi} \frac{d\theta}{2\pi\sqrt{m^2 \cos^2 \theta + n^2 \sin^2 \theta}} = \int_0^{2\pi} \frac{d\theta_1}{2\pi\sqrt{m_1^2 \cos^2 \theta_1 + n_1^2 \sin^2 \theta_1}}.$$

**Complete the proof by explaining why the limits of integration do not change under Gauss' substitution.** (*Gauss commented on these limits in the first sentence of his proof.*)