Fourier's Infinite Series Proof of the Irrationality of e

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We begin with a short passage from the ancient Greek philosopher Aristotle[1] (384 BCE–322 BCE), taken from his *Prior Analytics*.[2]

§I.23. For all who effect an argument *per impossibile* infer syllogistically what is false, and prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory; e.g. that the diagonal of a square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate.

The goal of this project is to work through a proof of the irrationality of the number e due to Joseph Fourier. This number would not have even been *defined* in any publication for another two millennia[3] (plus a few years) after the writing of *Prior Analytics*! So, the reader may wonder why we are rewinding our clocks so far back. Well, it turns out that the key patterns of logic in Fourier's proof of the irrationality of e can be traced right back to this passage from Aristotle.

In Section 1, we extract the critical bit of Aristotelian logic needed to understand Fourier's proof and give it more of a modern formulation. In Section 2, we establish common language regarding important sets of numbers that have come to characterize much of modern mathematics to be sure we know exactly what we mean when we say "irrational." In Section 3, we follow Aristotle's logic

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[1]Aristotle was born in northern Greece. His father, a doctor, likely wanted him to go into medicine. However, both of Aristotle's parents died when he was quite young and he ended up enrolling at the age of seventeen at the Academy in Athens, a selective school founded by Plato (428/427 BCE–348/347 BCE). There he received an education from, among others, the mathematician Eudoxus (c. 400 BCE–c. 350 BCE), whose work was incorporated into Euclid's *Elements*. Aristotle eventually became a teacher at the Academy, a position he held for twenty years. He later served for a time as a tutor to Alexander the Great before founding his own school, the Lyceum, in Athens in 355 BCE. In addition to founding the formal study of Western logic, Aristotle's works were highly influential in the western study of philosophy, ethics, physics, and mathematics for centuries (O'Connor and Robertson, 1999a).

[2]Written or dictated by Aristotle in roughly 350 BCE, *Prior Analytics* was most likely a collection of notes on his lectures at the Lyceum. It is today considered the first writing on pure logic, dealing largely with syllogisms and how statements about particulars relate to statements about universals. A translation of the *Prior Analtyics* by Arthur J. Jenkinson appears in the compilation *The Basic Works of Aristotle* (McKeon, 1941, pp. 65–107); this particular passage is found in Section 41a, lines 23ff of that translation.

[3]Early definitions of e were, perhaps unsurprisingly, associated with the natural logarithm, which was then called the "hyperbolic logarithm." An application of the number e that was independent of logarithms was first formulated by Jakob Bernoulli (1655–1705) in the context of a compound interest problem in the late 1600s. Since then, it has come up in a shockingly broad spectrum of situations! For a nice article describing these, see (Reichert, 2019).

template to prove that the square root of two is irrational, a good warm-up for the more intricate task that lies ahead in Section 4, where we examine Fourier's proof of the irrationality of e. For a lovely epilogue (epi-natural-log?), we take a look in Section 5 at how Liouville extended Fourier's work to learn a bit more about just how interesting a number e is, briefly exploring the idea of *transcendental numbers*.

1 Proof by Contradiction

Let us revisit just the first part of the Aristotle passage.

> For all who effect an argument *per impossibile* infer syllogistically what is false, and prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory . . . .

For our purposes here, the phrase "infer syllogistically" can be simply taken to mean that one concludes a statement from two or more prior statements. The other key phrases in this passage are the following:

- "original conclusion," meaning what is desired to be proven,
- "its contradictory," meaning the negation of what is desired to be proven, and
- "what is false," meaning some statement that is known to be false.

Analyzing what Aristotle said above about how to put these items together in a *per impossibile* argument, we see that the idea is to prove a statement is true by assuming its negation and then deducing a known falsehood.

Today, this process is most commonly called "proof by contradiction"[4] and remains one of the most powerful tools in the mathematician's toolbox. Here's how we might write the basic outline of a proof by contradiction today.

To prove the statement P:

- Assume the negation of P.[5]
- Deduce a statement Q.
- Deduce the negation of Q.
  (Since Q and its negation are contradictory, we now know our assumption is false.)
- Conclude that P is true.

[4] Note that the translator's choice of the Latin phrase *reductio per impossibile* in the Aristotle passage is one way to describe contradiction: reducing one's hypothesis to an impossible conclusion. However, it is common today to instead call proof by contradiction by a different Latin phrase that corresponds to another of Aristotle's argument forms, namely *reductio ad absurdum*: reducing one's hypothesis to an absurd conclusion. The difference is subtle but sometimes incredibly important!

[5] This is what Aristotle called "its contradictory." For example, the negation of the statement "7 is prime" is the statement "7 is not prime," and vice versa.

Let us digest this proof strategy in a small example.

$\boxed{\textbf{Task 1}}$ Consider the following proof by contraction.

> *Assume there are only finitely many whole numbers.*
> *Then, there must exist a largest whole number. Call this number n.*
> *By definition of "largest," there are no whole numbers m such that $m > n$.*
> *However, $n + 1$ is a whole number (since n is), and $n + 1 > n$.*
> *Thus, there are infinitely many whole numbers.*

(a) In the argument above, identify the statements P and Q that were used in our modern formulation of proof by contradiction.

(b) Likewise, identify the components of an argument *per impossibile*, as described by Aristotle.

- "original conclusion,"
- "its contradictory," and
- "what is false."

Our goal in this project is to prove that e is irrational, following Fourier's proof by contradiction. Before we can embark on this journey, we need to make sure we have precise definitions of our number systems so we know exactly what it means to be irrational!

2 Some Fundamental Sets of Numbers

Though by no means an exhaustive list, we now present a few fundamental sets of numbers.[6] Mathematicians use these particular number systems so frequently that there is a standard notation that has been adopted to refer to them, which we show below.

- **Natural Numbers.** The set $\mathbb{N}$ of natural numbers is the set of all positive whole numbers, along with zero.[7] That is,
$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}.$$

- **Integers.** The set of integers $\mathbb{Z}$ is the set of all whole numbers, whether they are positive, negative, or zero. That is,
$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

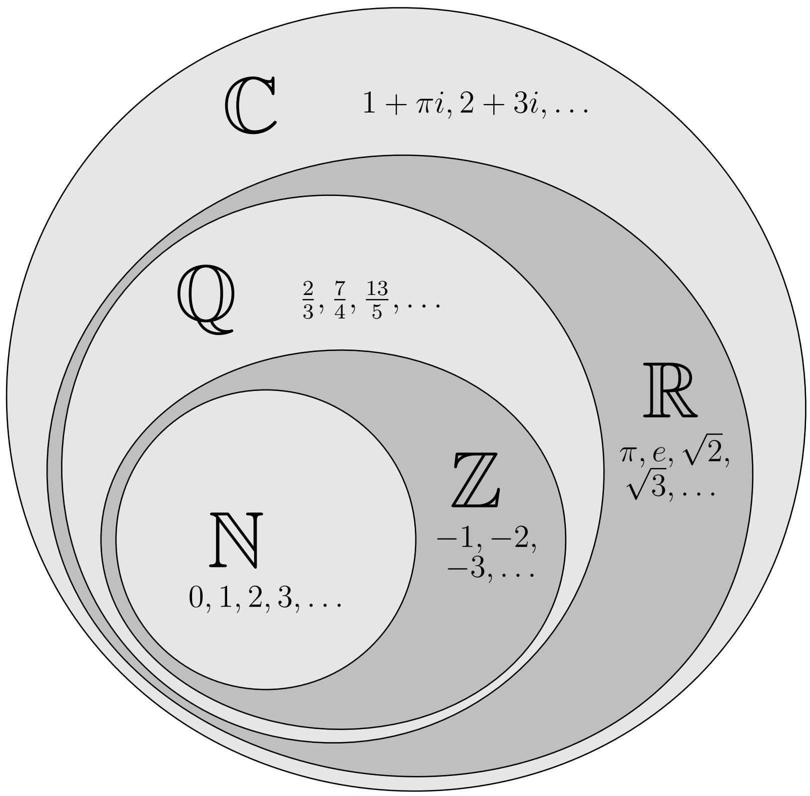
- **Rational Numbers.** The set of rational numbers $\mathbb{Q}$ is the set of all numbers expressible as a fraction whose numerator and denominator are both integers. (Of course, the denominator will have to be non-zero.)

[6] There are a great many more number systems that mathematicians work with, such as quaternions and integers mod n. However, they do not come up in the primary sources we include in this project. For that matter, neither do complex numbers arise in the mathematics of this project, but we include them in the diagram in this section just to help place this work in context with the number systems you likely worked with in your precalculus class.

[7] Some mathematicians do not include zero in the set of natural numbers. Here, we do.

- **Real Numbers.** The set of real numbers $\mathbb{R}$ is the set of all numbers expressible as a decimal expansion (finite or infinite).
- **Complex Numbers.** The set $\mathbb{C}$ of complex numbers is the set of all numbers expressible as $a + bi$, where a and b are real numbers, and i is a symbol such that $i^2 = -1$.

The figure below illustrates the relationships among these number systems, each labelled with the corresponding blackboard bold letter,[8] along with a few examples from each set of numbers. Note the inclusion of each number system in the next: every natural number is also an integer, every integer is rational, and so on. For example, the number 2 is in the set of complex numbers because the set of complex numbers contains all of the other sets shown here.[9]



We define one last term before proceeding, critical to our work in this project. Visually, we are trying to identify what numbers lie outside the region marked with $\mathbb{Q}$ but inside the region marked with $\mathbb{R}$ in the diagram above. Such numbers are called *irrational*, and the diagram shows a few famous ones: $\pi, e, \sqrt{2}$, and $\sqrt{3}$. Verbally, we get the following definition.

- A number is called *irrational* if it is real but not rational. That is, r is irrational if and only if $r \in \mathbb{R}$ but $r \notin \mathbb{Q}$.

[8] If one wants an easy way to remember this notation: we have simply $\mathbb{N}$ for Natural, $\mathbb{R}$ for Real, and $\mathbb{C}$ for Complex. The two that don't seem to match their leading letter also have good reasons for their naming: $\mathbb{Z}$ for Zahl, which is German for "number," and $\mathbb{Q}$ for Quotient.

[9] The author would like to thank his former student Jenna Allen for creating the Venn diagram representing the number systems.

Following the definition given above, we see that to prove a number r is rational, one must simply find a pair of integers m and n such that $r = \frac{m}{n}$.

Task 2 Prove that each of the following quantities is rational.

(a) 2

(b) 2.5

(c) $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots$

To prove a number is irrational, however, is typically harder (and sometimes *much* harder!). After all, if a number is irrational, it is not possible to just check every possible pair of integers m and n to see whether the given number equals their ratio. Instead, such a claim generally requires proof by contradiction. Specifically, in order to use proof by contradiction to show that a real number r is irrational, one can perform the following steps:

- **Step 1.** Assume r is rational. (This is the negation of the statement P that we wish to prove.)
- **Step 2.** Thus, there exist some integers m and n with $r = \frac{m}{n}$.
- **Step 3.** Use the equation $r = \frac{m}{n}$ and known properties of the number r to arrive at a contradiction (e.g., by establishing that, for some statement Q, both Q and the negation of Q hold at the same time). *This step can take a bit of work!*
- **Step 4.** Conclude that our assumption of r being rational must have been false, so r is in fact irrational. (This is our desired conclusion, or what Aristotle called the "original conclusion".)

In the next section, we will follow the above 4-step method to do a warm-up example of proving the irrationality of a different number, before we try to understand how Fourier used it to prove the irrationality of e.

3 A Warm-up Irrationality Proof

We now return to the original Aristotle passage, where he claimed that an argument *per impossibile* could be used to show

> ... that the diagonal of a square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate.

A modern idea that corresponds to Aristotle's word "commensurate" is that of being a "rational multiple of."[10] For example, consider a line segment of length 12 and a line segment of length 18, and notice that the latter has a measurement that is 3/2 times the former. Since 3/2 is a rational number, this modern calculation corresponds to saying that these line segments are commensurate.

[10]An extended version of this project that explores irrationality proofs using the Greek notion of commensurability more extensively is available at https://digitalcommons.ursinus.edu/triumphs_calculus/22/.

Task 3 (a) Explain why the perimeter of a circle of radius 2 and the perimeter of a circle of radius 3 are commensurate.

(b) Given that π is an irrational number,[11] explain why the perimeter of a circle and the diameter of that same circle are incommensurate.

Task 4 (a) Suppose a square has side length equal to 1. What is the length of the diagonal? Draw a diagram to support your answer.

(b) Aristotle's claim that "the diagonal of a square is incommensurate with the side" is equivalent to claiming some number is irrational. Based on your answer to part (a), what number is this?

To show that the number associated with the commensurability of the side and diagonal of a square is indeed irrational, we present an argument below that captures the same essence as the argument used during Aristotle's time: proving a claim *per impossibile* by showing that otherwise, odd numbers would be equal to evens!

Assume $\sqrt{2}$ is rational.

Then there exist integers m and n such that

$$\sqrt{2} = \frac{m}{n}.$$

We can assume that at least one of m or n is odd, since if both were even, the fraction could be reduced to make one of them odd.

Squaring both sides of our equation produces

$$2 = \frac{m^2}{n^2},$$

which implies

$$2n^2 = m^2.$$

Thus, m^2 is even, since it is twice a whole number. This means m is even as well (since the square of an odd would not be even).

Since m is even, then $m = 2k$ for some integer k. Substituting $m = 2k$ into $2n^2 = m^2$, we have

$$2n^2 = (2k)^2,$$

which implies

$$2n^2 = 4k^2,$$

[11] Although the constant ratio between the circumference and diameter of a circle was well-known to the ancient Greeks, the irrationality of π took mathematicians even longer to prove than the irrationality of e. Johann Heinrich Lambert (1728–1777) was the first to do so, using an argument based on an infinite continued fraction expansion of $\tan(x)$ that appeared in his paper (Lambert, 1768). The earliest proof that e is irrational also used an infinite continued fraction expansion; it was published by Leonhard Euler (1707–1783) in 1737, some 100 years before the infinite series proof for the irrationality of e given by Fourier that we'll look at in the next section. For a guided tour of Euler's work on the irrationality of e, see the article (Sandifer, 2006).

and thus

$$n^2 = 2k^2.$$

This means n is even, by the same reasoning as above.

We have thus established that, on one hand, m and n are not both even (by our assumption), but on the other hand, both m and n are even, which is a contradiction.[12]

Therefore, $\sqrt{2}$ is irrational.

Task 5 (a) In our formulation of proof by contradiction, identify the statements P and Q that were used in the argument above.

(b) Likewise, in the argument above, identify the components of an argument *per impossibile*, as identified by Aristotle:

- "original conclusion,"
- "its contradictory," and
- "what is false."

Task 6 (a) Take the argument for the irrationality of $\sqrt{2}$ and adapt it to write a proof of the irrationality of $\sqrt{3}$. (**Hint.** Replace the notions of "even" and "odd" by "divisible by 3" and "not divisible by 3", respectively. From there, essentially the same argument should work!)

(b) Suppose you try to adapt it to prove the irrationality of $\sqrt{4}$. Where does the argument break down?

Notice how strong of a claim was proven above: no matter how hard we try, scouring infinitely many possibilities for m and infinitely many possibilities for n, we can never find a pair whose ratio represents the measure of a diagonal of a unit square. Aristotle himself commented on this in his important philosophical work *Metaphysics*,[13] declaring that

> ... it seems wonderful to all who have not yet seen the reason, that there is a thing which cannot be measured even by the smallest unit. ... [yet] there is nothing which would surprise a geometer so much as if the diagonal turned out to be commensurable.

Task 7 How surprising do you find it that the square and diagonal of a square are incommensurable—that is, that $\sqrt{2}$ is irrational—and why? What about π? Or e?

[12] This conclusion corresponds to Aristotle's statement that "odd numbers are equal to evens" if the side and diagonoal of a square were commensurate.

[13] A translation of the *Metaphysics* by W. D. Ross appears in *The Basic Works of Aristotle*, (McKeon, 1941, pp. 689–934); this particular passage is found in section 983a, lines 17ff of that translation.

4 Fourier's Proof of the Irrationality of e

Joseph Fourier (1768–1830) was born into a working-class family in Auxerre, France. He quickly entered unfortunate circumstances: at the age of eight he became an orphan. Luckily, he obtained admission to a local military school, where he received an education from the Benedictine monks of Saint-Maur. In 1790, they gave him a mathematics teaching appointment at their school in Auxerre, where he also taught rhetoric, history, and philosophy. He later became a founding faculty member at the Ècole Polytechnique in Paris,[14] where Napoleon sometimes attended lectures. This led to Napoleon's request for Fourier's help in the administration of Egypt after its occupation by France in 1798. Upon his return to France, Fourier served as the prefect of the Department of Isère, where he led extensive infrastructure projects to quell chronic infections that were emanating from marshes in the area. In 1817, he was elected to the Académie des Sciences and five years later he became its perpetual secretary.[15]

Thus, Fourier was quite the busy person, not only as an academic but also as a civil servant. Perhaps, then, it is not terribly surprising that he never published his proof that e is irrational himself! Rather, it appears in the book *Mélanges d'analyse algébrique et de géométrie* (*Miscellany of algebraic analysis and geometry*), written in 1815 by Janot de Stainville (1783–1828).[16] De Stainville explained how Fourier's proof was communicated to him as follows (de Stainville, 1815, p. 341).[17]

> Note: this demonstration has been shared with me by Mr. Poinsot, who had it from Mr. Fourier.

The "Mr. Poinsot" to whom de Stainville referred was Louis Poinsot (1777–1859).[18] Poinsot and Fourier share a particular honor: they are both included among the seventy-two names of prominent mathematicians and scientists engraved into the Eiffel Tower!

De Stainville prefaced his presentation of Fourier's proof of the irrationality of e by giving an approximate value for that number. He proclaimed the following:

> After having found an approximate value for the number e, it is good to consider it in itself, and to demonstrate that not only is it comprised between 2 and 3, but that no rational fraction comprised between these two numbers can represent it; . . . .

[14] Founded during the French Revolution in 1794 (the same year as Fourier's arrest for having defended a member of a particular political faction), the École Polytechnique remains one of the most well-respected institutions of mathematics in the world today.

[15] The title "perpetual secretary" implies being chairperson and chief representative of the Académie for life, with the option to step down, after which one becomes known as the honorary perpetual secretary. For more on Fourier's life, see (O'Connor and Robertson, 1998).

[16] Nicolas Dominique Marie Janot de Stainville was a member of the École Polytechnique class of 1802. He was then hired back by his alma mater to work as a tutor in 1810 (Verdier, 2008).

[17] All translations of excerpts of de Stainville's *Mélanges* were prepared by Diane Van Tiggelen, an undergraduate student and Learning Assistant at Front Range Community College, in 2020.

[18] Louis Poinsot was also a student and then later a professor at the École Polytechnique in Paris. He is perhaps best remembered for having written *Eléments de statique*, a work which is today considered to be the founding work on geometric mechanics.

To prove both these assertions, Fourier (and, hence, de Stainville) used the following series representation for the number e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

This representation was well-known by that time. Not only did Jakob Bernoulli publish a general form of this infinite series in connection with the solution to a financial problem involving compounded interest in (Bernoulli, 1690, pp. 219–222), but the exceptionally talented and indescribably influential Swiss mathematician Leonhard Euler (1707–1783) later derived it in his famous 1748 precalculus book *Introductio Anylysum Infinitorum* (*Introdcution to the Analysis of the Infinite*). Let us now see how Fourier put this series to work by walking through Fourier's proof (as presented in (de Stainville, 1815, pp. 339–341), beginning with his demonstration that e is a real number between 2 and 3.[19]

First it [the number e] is greater than 2, because the two first terms of the series

$$1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \text{etc.},^{20}$$

are both equal to one, and the sum of the other terms is positive, but as this sum is less than the sum of the terms of the equation

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \text{etc.},$$

which is equal to one, because it results from the division of 1 by $2-1$, it follows that the sum of the fractions

$$\frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \text{etc.},$$

is necessarily less than one, and accordingly, that the number e is less than 3.

Task 8 Although this is a very nicely written argument, a few steps could benefit from more detail. To this end, explain carefully why each of the following claims is true.

(a) "… this sum is less than the sum of the terms of the equation …."

(b) "… which is equal to one, because it derives from the division of 1 by $2-1$ …." (In particular, be sure to identify which famous formula is being applied on that step!)

(c) "… the number e is less than 3."

[19] Note that here we present de Stainville's write-up since it was his introduction to Fourier's proof, but an argument that shows $2.5 < e < 3$ actually dates back to Jakob Bernoulli's 1690 publication (Bernoulli, 1690, pp. 219–222) in which he used an infinite series to study a problem related to compound interest.

[20] Note that we are reproducing the original notation symbol for symbol. The lower dots are used to indicate multiplication. For example, de Stainville uses "2.3" to represent "2 times 3" rather than a decimal form of 23/10. Furthermore, note that de Stainville's order of operations had the lower dot evaluated *after* addition, which is the opposite of what we typically do with multiplication vs. addition.

Having established that e is in fact some real number between 2 and 3, de Stainville moved on to present Fourier's proof of irrationality.

I also affirm that no rational fraction can represent it [the number e], because if an irreducible fraction m/n was equal to it, we would have

$$\frac{m}{n} = 2 + \frac{1}{2} + \frac{1}{2.3} + \cdots + \frac{1}{2\ldots n} + \frac{1}{2\ldots n.n+1} + \text{etc.};$$

but if we multiply the two sides of this equation by the product $1.2\ldots n$ of the sequence of natural numbers, up to the one that indicates the denominator of the fraction of the first side, we will have

$$\{1.2\ldots n-1\}m = \text{ an integer } + \frac{1}{n+1} + \frac{1}{n+1.n+2} + \frac{1}{n+1.n+2.n+3} + \text{etc.},$$

or

$$\frac{1}{n+1} + \frac{1}{n+1.n+2} + \frac{1}{n+1.n+2.n+3} + \text{etc.}$$

[which] is smaller than

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \text{etc.},$$

and since this last quantity is equal to

$$\frac{1}{(n+1)-1},$$

and the first side is a whole number, it will follow that if to a whole number one adds a fraction lesser than $1/n$, the result would be a whole number, which is absurd; therefore it is equally absurd to suppose that the number e is rational, thus it is irrational.

Let us process this proof by rewriting it in a more modern form, updating our language and notation a bit.

Task 9 Use de Stainville's write-up of Fourier's proof to fill in the missing parts of the following proof that e is irrational. The blanks are labelled (A), (B), (C), …, (L).

Proof. First let's write e as an infinite series:

$$e = \underline{\quad (A) \quad}.$$

We proceed by using the classic proof technique called $\underline{\quad (B) \quad}$. Accordingly, we assume e is rational and then show that it leads to an impossible statement.

Proceeding, we assume e is rational. Then, there exist some $m, n \in \mathbb{N}$, with $n > 1$, such that

$$\underline{\quad\text{(C)}\quad} = e.$$

We now identify the statement that will produce our contradiction. We will prove both of the following:

1. The quantity $\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$ **is** an integer.
2. The quantity $\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$ **is not** an integer.

The first statement is demonstrated as follows. We multiply both sides of the equation from part (C) by the integer $\underline{\quad\text{(D)}\quad}$ to obtain

$$(n-1)!m = n!e.$$

Notice that the left-hand side is an integer because $\underline{\quad\text{(E)}\quad}$. Thus, the right-hand side, $n!e$, must also be an integer. Notice, however, that the right-hand side can be decomposed as follows by substituting the infinite series for e and applying the distributive law:

$$n!e = n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} + \cdots\right)$$

$$= n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) + n!\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots\right).$$

The first term, $n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$, is an integer because $\underline{\quad\text{(F)}\quad}$. Subtracting that term from both sides, we can rewrite the above equation as

$$n!e - n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$$
$$= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots.$$

Since the left-hand side remains an integer, this completes the proof of the first statement: that the quantity of interest is an integer.

We now proceed to show the second statement: that the quantity of interest is not an integer. In particular, we will show that

$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$$

lies in an interval that contains no integers. Specifically, we will show that this quantity lies between $\frac{1}{n+1}$ and $\frac{1}{n}$. Proceeding, we have

$$
\begin{align}
\frac{1}{n+1} &< \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \tag{1}\\
&< \frac{1}{n+1} + \frac{1}{(n+1)(n+1)} + \frac{1}{(n+1)(n+1)(n+1)} + \cdots \tag{2}\\
&= \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots \tag{3}\\
&= \frac{\frac{1}{n+1}}{1-\frac{1}{n+1}} \tag{4}\\
&= \frac{1}{n}. \tag{5}
\end{align}
$$

The above steps are justified as follows. The inequality on line (1) is true because ___(G)___. To get from line (1) to line (2), we use the fact that ___(H)___. The link between line (2) and line (3) is simply algebra. To get from line (3) to line (4), we sum an infinite geometric series with common ratio ___(I)___ and initial term ___(J)___. The transition from line (4) to line (5) again follows from ordinary algebraic simplification.

Thus, we have demonstrated that

$$\frac{1}{n+1} < \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right) < \frac{1}{n},$$

as desired. Since $\frac{1}{n+1}$ and $\frac{1}{n}$ are strictly between 0 and 1, the quantity ___(K)___ must lie strictly between 0 and 1 as well. However, there are no integers between 0 and 1, so that quantity cannot be an integer.

Thus, if our assumption that e is rational were true, we would be able to prove the existence of a quantity that both is and is not an integer at the same time. This is a contradiction. Therefore, we conclude that

__________(L)__________.

$\square$

After reading a long and complicated argument, some small "sanity check" kinds of questions are often helpful with regards to moving the argument from a place of "I didn't disagree with that at any particular step" to the much better place of "ok, that argument feels intuitive to me." The following tasks hopefully help with that!

Task 10 First, let's make sure we understand the logic of the above argument.

(a) Identify Aristotle's key components in this argument. Specifically, identify each of the "original conclusion," "its contradictory," and "what is false." At the end of all of this work, what have we successfully demonstrated?

(b) The contradiction was established by using the assumption of the rationality of e to prove two statements (labelled "1." and "2." in the proof) that were in direct opposition to each other. Which one was actually true, and why?

Task 11 To help visualize what exactly happened in the argument above, plot the following five quantities in order on a number line: $0, 1, \frac{1}{n}, \frac{1}{n+1}$, and $\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$.

Task 12 Why can we assume that $n > 1$? (**Hint.** Revisit the first primary source passage from de Stainville!) Furthermore, why was that important? Where was that fact used in the proof?

5 Transcendence of e

In this project, we've demonstrated the irrationality of two different real numbers: e and $\sqrt{2}$. In a certain sense, however, e is even *more* irrational than $\sqrt{2}$. In a paper published 25 years after that of de Stainville, Joseph Liouville (1809–1882)[21] adapted Fourier's methods to prove that e^2 is also irrational.[22] In other words, if we square the irrational number e, we get another irrational number. On the other hand, if we square the irrational number $\sqrt{2}$, we instead get a rational number (i.e., 2).

The observations in the preceding paragraph hint at the idea of a *transcendental number*: a real number that can *not* be obtained as a root of a polynomial with integer coefficients.[23] In contrast, an *algebraic number* is one that *can* be obtained as a root of a polynomial with integer coefficients. While the square root of 2 is irrational, it is still algebraic, since it is a root of a polynomial with integer coefficients (namely, $x^2 - 2$). However, it turns out that e is transcendental as well as irrational. This fact turns out to be much more difficult to prove than the irrationality of e.

The first proof of the transcendence of any number came from Liouville himself, about thirty years after Fourier's proof of e's irrationality, when he considered the number

$$0.110001000000000000000001000\ldots.$$

It has a 1 in each decimal place given by $n!$ for some natural number n, and 0 everywhere else. Liouville proved this number was transcendental in the landmark paper "Sur les classes très étendues de quantités dont la valeur n'est ni algébrique ni même réductible à des irrationelles algébriques" ("On the very extensive classes of quantities whose value is neither algebraic nor even reducible to algebraic irrationals") (Liouville, 1851).[24]

It took yet another thirty years to prove the transcendence of e, when Charles Hermite[25] (1822–1901) accomplished this challenging task in his paper (Hermite, 1873). Though the argument for

[21] Liouville's father, like Fourier, worked under Napoleon during wartime. Liouville himself began to study at the École Polytechnique in 1825 and later founded an important mathematics journal, the *Journal de Mathématiques Pures et Appliquées*. For more about Liouville's life, see (O'Connor and Robertson, 1997).

[22] An extended version of this project that includes Liouville's argument for the irrationality of e^2 from his paper (Liouville, 1840) is available at `https://digitalcommons.ursinus.edu/triumphs_calculus/22/`.

[23] The word "transcendental" comes from the medieval Latin word "transcdendere," meaning "to surmount, rise above."

[24] As the title of Liouville's paper suggests, the set of transcendental numbers is indeed very extensive. Using ideas about different sizes of infinity that were developed in the late 19th century, it's possible to prove that the real numbers contain far more transcendental numbers than algebraic numbers, and also that there are far more irrational numbers than rational numbers!

[25] Charles Hermite was born in Dieuze, Lorraine, France. He became known not only for his contributions to number theory, analysis, linear algebra, and differential equations, but also for his spectacular teaching (O'Connor and Robertson, 1999b)!

transcendence proved more difficult, it had something in common with all of the irrationality arguments in this project: Hermite's proof still proceeded *per impossibile*!

Task 13 Based on the discussion above, classify each of the real numbers given below as either:

- rational (and thus also algebraic),[26]
- irrational but still algebraic, or
- irrational and transcendental.

What makes you think so in each case?

(a) $\sqrt{3}$ (b) $1+\sqrt{2}$ (c) $1/e$

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[26]To see why all rational numbers m/n (with m, n integers) are also algebraic, just write down a polynomial with integer coefficients that has m/n as a root!

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6 Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is intended to show students how the methods of series and their analysis are not only useful for computation, but also for proving theoretical results. The key competencies that come up in this project are as follows:

- Infinite series for e
- Infinite geometric series formula
- Comparison test arguments

This project also introduces students to proof by contradiction, hopefully preparing them for more rigorous proof courses in the future! It (or the extended version of it described below) could be used in an Introduction to Proofs course as well. In particular, it offers an example of proving the irrationality of a number that involves more than the typical argument found in standard textbooks for such a course.

Student Prerequisites

In this project, we assume the student is familiar with comparison test arguments for infinite series as well as the infinite geometric series formula.

It should be noted that it is **not** necessary that a student has seen the power series for the exponential function (nor has had any exposure to power series whatsoever) to do this project. Even the power series for e^x is not actually used here; the infinite series for e that Fourier used is provided to the student in this project. This project could in fact be used to mark the end of a chapter on infinite series before transitioning to power series!

PSP Design and Task Commentary

This PSP will expose the student to arguments that extensively use the infinite series for the number e and geometric series, but in the context of proving the irrationality of e. This serves as a fabulous warm-up for a student who later takes an introduction to proof course; almost all arguments in this PSP use proof by contradiction!

More specifically, the sections of this PSP entail the following:

- **Section 1 (Proof by Contradiction).** In this section, the student learns the general form of a proof by contradiction from a passage in Aristotle's *Prior Analytics* (McKeon, 1941, pp. 65–107). A student task then analyzes a simple contradiction argument proving there are infinitely many natural numbers using the well-ordering principle.
- **Section 2 (Some Fundamental Sets of Numbers.)** Here the project makes sure the student understands exactly what is meant by the words *rational* and *irrational* before attempting to prove statements involving these words! The template for an irrationality proof via contradiction is also given.

- **Section 3 (A Warm-up Irrationality Proof).** The project returns to the passage from Aristotle, in which he claimed that the side length and diagonal of a square are not *commensurate* since otherwise "odd numbers are equal to evens." The Greek geometers' notions of commensurability/incommensurabilty are briefly related to the rational and irrational numbers. The student then works through a proof of the irrationality of $\sqrt{2}$ corresponding to Aristotle's claim—a much easier warm-up before the main event in the next section!
- **Section 4 (Fourier's Proof of the Irrationality of e).** Here the student works through de Stainville's argument, which compares the series representation for e against a geometric series to show that e is a number between 2 and 3. Afterwards, the student works through Fourier's proof by contradiction that proves e is irrational (as communicated by de Stainville), which again uses a comparison to a geometric series.
- **Section 5 (Transcendence of e).** As a brief epilogue, the student explores the idea of transcendental numbers as an extension of irrationality, comparing the behavior of $\sqrt{2}$ with that of e.

The project also provides a short biography of Fourier, whose remarkable life serves as an inspiration to students suffering from imposter syndrome as a result of humble origins.

Suggestions for Classroom Implementation

Instructors are strongly encouraged to work the entire PSP before using it in class: although only simple techniques are employed, the proofs are a bit subtle.

Copies of this PSP are available at the TRIUMPHS website (see URL in Acknowledgements). The author is happy to provide LaTeX code for this project. It was created using Overleaf which makes it convenient to copy and share projects and can allow instructors to adapt this project in whole or in part as they like for their course.

Sample Implementation Schedule (based on a 50-minute class period)

The author recommends two full 50-minute class periods for implementation of this PSP.

- The readings and tasks of the PSP up to and including Section 2 can be assigned as preparation for class.
- Start class with 20 minutes of follow-up discussion on the first two sections. In particular, make sure the students are clear on all vocabulary involved.
- The next 30 minutes could consist of students working in small groups, working to understand the argument in Section 3.
- Between class sessions 1 and 2, students should aim to finish any unfinished tasks in Sections 1–3 and read the first part of Section 4, stopping just above Task 9. Students could also be asked to make an attempt to answer Task 8 and/or Task 9 before class.
- During the first 20 minutes of the following class, the instructor could lead a discussion on the key ideas of Section 4.
- The remainder of the second class can be devoted to finishing the tasks in Section 4.
- Reading and completing the tasks in Section 5 can be assigned for homework (along with any other unfinished tasks from the previous sections).

Connections to Other Primary Source Projects

An extended version of this primary source project, entitled *Why $\sqrt{2}$ is a Friendlier Number than e: Irrational Adventures with Aristotle, Fourier, and Liouville*, is available at https://digitalcommons.ursinus.edu/triumphs_calculus/22/. In addition to providing the details of Liouville's argument for the irrationality of e^2 through a series of student tasks, that version explores the Greek concept of commensurate figures and its relation to the modern concept of rational numbers in more detail. While that longer version has been used in Calculus 2 courses, it is especially well-suited for use in transition to proof courses or as part of a capstone experience for prospective secondary mathematics teachers. Its implementation requires four 50-minute class periods, depending on the audience and mode of implementation.

The following projects based on primary sources are also freely available for use in teaching standard topics in the calculus sequence. The PSP author name of each is given (together with the general content focus, if this is not explicitly given in the project title). Each of these projects can be completed in 1–2 class days, with the exception of the four projects followed by an asterisk (*) which require 3, 4, 3, and 6 days respectively for full implementation. Classroom-ready versions of these projects can be downloaded from https://digitalcommons.ursinus.edu/triumphs_calculus/.

- *Investigations Into d'Alembert's Definition of Limit (Calculus version)*, by Dave Ruch
- *L'Hôpital's Rule*, by Daniel E. Otero
- *The Derivatives of the Sine and Cosine Functions*, by Dominic Klyve
- *Fermat's Method for Finding Maxima and Minima*, by Kenneth M Monks
- *Gaussian Guesswork: Elliptic Integrals and Integration by Substitution*, by Janet Heine Barnett
- *Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve*, by Janet Heine Barnett
- *Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean*, by Janet Heine Barnett
- *Beyond Riemann Sums: Fermat's Method of Integration*, by Dominic Klyve (uses geometric series)
- *How to Calculate π: Machin's Inverse Tangents*, by Dominic Klyve (infinite series)
- *Euler's Calculation of the Sum of the Reciprocals of Squares*, by Kenneth M Monks (infinite series)
- *Jakob Bernoulli Finds Exact Sums of Infinite Series (Calculus Version)*,* by Daniel E. Otero and James A. Sellars
- *Bhāskara's Approximation to and Mādhava's Series for Sine*, by Kenneth M Monks (approximation, power series)
- *Braess' Paradox in City Planning: An Application of Multivariable Optimization*, Kenneth M Monks
- *Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green's Theorem*,* by Abe Edwards
- *The Fermat-Torricelli Point and Cauchy's Method of Gradient Descent*,* by Kenneth M Monks (partial derivatives, multivariable optimization, gradients of surfaces)
- *The Radius of Curvature According to Christiaan Huygens*,* by Jerry Lodder

Another PSP that connects very nicely to this one is *Euler's Rediscovery of e* by David Ruch (Ruch, 2017), which explores the origin of the infinite series for e on which Fourier's proof depends. Although that PSP is intended for use in an introductory course in analysis, it is quite appropriate for a second-semester calculus classroom if one simply stops at Task 5. It is available at https://digitalcommons.ursinus.edu/triumphs_analysis/3/.

Recommendations for Further Reading

The articles mentioned in the various footnotes of this project are all suitable for student reading. Charles Hermite's paper (Hermite, 1873), in which e is proven to be transcendental, would be a fabulous (though challenging) follow-up for the advanced student.

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For more information about the NSF-funded project TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS), visit

http://blogs.ursinus.edu/triumphs/ .