The Closure Operation as the Foundation of Topology

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June 1, 2023

1 Introduction

In the early 1900s, axiomatizing different mathematical disciplines was all the rage. While a discipline like geometry was well established by that time, topology was still quite new. Hence, different ways to approach its axiomatization were still being explored. We will study the work of two mathematicians who contributed to that effort: Felix Hausdorff and Kazimierz Kuratowski.

Born to Jewish parents, Felix Hausdorff (1868–1942) is known today as one of the founders of modern topology. He also made significant contributions to set theory, descriptive set theory, measure theory, function theory, and functional analysis. Hausdorff studied mathematics and astronomy, mainly in the city of Leipzig. He graduated from the University of Leipzig and became a lecturer there as well. In 1901, he was appointed as an adjunct professor at the University of Leipzig, where he taught until 1910 when he accepted a position at the University of Bonn. Hausdorff continued to teach as a university professor, primarily in Bonn, until he was forced to retire by the Nazi regime in 1935. After being informed in 1942 that they were to be placed in an internment camp, Hausdorff committed suicide, together with his wife and his wife's sister. His pioneering contributions to the field of topology during his lifetime included development of the concept known today as Hausdorff spaces, as well as the concepts of metric and topological spaces. The development of the idea of closeness independent of the ability to be measured also interested Hausdorff.

A Polish mathematician and logician, Kazimierz Kuratowski (1896–1980) was born in Warsaw to a well-known lawyer. In 1913, he enrolled as an engineering student at the University of Glasgow, evidently because he did not wish to study in Russian. Two years later, he began studying mathematics in Poland after the first World War forced him to return there. In 1933, he became a professor at Warsaw University and began his studies on applications of topological spaces in other areas of mathematics. One of his most significant contributions to general topology was an axiomatization of the closure operator in which he used Boolean algebra to characterize the topology of an abstract space without relation to the notion of points. This latter contribution is the main focus of this project.

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2 Closed Sets and Closure

In his foundational textbook on topology entitled *Set Theory*[1] [Hausdorff, 1957], Hausdorff wrote:

The mathematical discipline concerned with [topological invariance] is called Topology or Analysis Situs. (The latter term, due to Leibniz, was re-introduced by Riemann.) ... This seems like a suitable occasion to touch, in all brevity, on those point-set theories that emphasize the topological point of view from the very beginning and work only with the topologically invariant concepts. ... ...

What are primary in the topological space X are the sets that are *closed* (in X) and their complements, the *open* sets; ... ... The closed or open sets can be taken as our starting point and left undefined, or they can be defined from related concepts (limit point, neighborhood), but always derived in such a way as to keep invariant their topological character; the more detailed nature of the space is then determined by *axioms* ... .

He then went on to give the following set of axioms for closed sets.

[Closed] Sum and Intersection Axioms. The closed sets must, regardless of anything else, satisfy the following conditions:

1. The space X and the null set $\emptyset$ are closed.
2. The union of two closed sets is closed.
3. The intersection of any number of closed sets is closed.

As a consequence, the closure $\overline{A}$ can be defined as the intersection of all the closed sets containing A ... it has the following properties:

Closure Axioms.

1. $\overline{\emptyset} = \emptyset$
2. $A \subseteq \overline{A}$
3. $\overline{\overline{A}} = \overline{A}$
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

It would also be possible to make these properties our starting point (*à la* Kuratowski) by taking them as our axioms for the set function $\overline{A}$.

[1] Hausdorff's first book-length treatment of topology was entitled *Grundzüge der Mengenlehre* (*Fundamentals of Set Theory*) [Hausdorff, 1914], and published prior to Kuratowski's work. Later, Hausdorff published a significantly revised version of his 1914 textbook—essentially an entirely new book—that appeared in two editions (in 1927 and 1935) under the title *Mengenlehre* (*Set Theory*). In this project, we use excerpts from the published English translation [Hausdorff, 1957, pp. 257–258] of the 1935 German edition, with minor changes by the project author. For instance, throughout this project, we use today's set notation $\cup, \cap$ in place of the boolean algebra notation $+, \times$ that both Hausdorff and Kuratowski employed for the operations of union and intersection, respectively.

The goal of this project is to prove the last claim made by Hausdorff concerning the possibility of making Kuratowksi's four Closure Axioms "our starting point." We will also show, in Task 6, that this axiomatization is equivalent to the "open set axioms" with which you are familiar. In other words, we will show that the **Open Sum and Intersection Axioms (OSIA)**, which can be found in any modern introductory textbook on topology, are equivalent to the **Closure Axioms (CA)**. To set the stage for this work, we begin by examining Hausdorff's assertion that "the closed or open sets can be taken as our starting point and left undefined" and his definition of closure.

Task 1 Prove that Hausdorff's **Closed Sum and Intersection Axioms (CSIA)** are equivalent to today's OSIA. When starting with CSIA, "closed" is an undefined term and $A \subseteq X$ is defined to be open if $X - A$ is closed. Similarly, when starting with OSIA, "open" is undefined and $A \subseteq X$ is defined to be closed if $X - A$ is open. *Hint*: Use De Morgan's laws.

Task 2 Let A be any set, and assume CSIA. Using Hausdorff's definition of $\overline{A}$, prove that $\overline{A}$ is closed.

3 Kuratowski on the Closure Axioms

We now investigate a selection from a paper by Kuratowski, entitled "Sur l'opération $\overline{A}$ de l'Analysis Situs" ("On the closure operation in topology") [Kuratowski, 1922]. As Kuratowski described in a footnote to its title, this paper was "the first part—slightly modified—of my thesis presented on 12 May 1920 at the University of Warsaw for the degree of Doctor of Philosophy." He explained the purpose of his paper, as well as the role of the four statements referenced by Hausdorff as the Closure Axioms (CA), as follows.[2]

This Note is devoted to the analysis of these propositions [CA] and their consequences. We proceed axiomatically: we assume as given an arbitrary set[3] X and a function $\overline{A}$ such, for each [set] A contained in X, [the set] $\overline{A}$ is also contained therein and fulfills axioms 1–4.

It is worth pondering exactly what Kuratowski has set up here. As mentioned above, axiomatic systems were all the rage at this time, and Kuratowski was right in the thick of it. Essentially, he was declaring that the four stated properties are so fundamental and so robust, that we can decree that we have an abstract set operation, denoted $\overline{A}$, that satisfies them as axioms. From there, we can build the theory of this operation and prove many things within it. With this in mind, we turn back to Kuratowski.

[2] All translations of Kuratowski excerpts in this project were prepared by Janet Heine Barnett, Colorado State University Pueblo, 2022, using the set notation $\cup, \cap, X$ in place of the boolean algebra notation $+, \times, 1$ that Kuratowski employed to represent union, intersection, and the universal set, respectively.

[3] In his 1922 paper, Kuratowski used an arbitrary subset of Euclidean n-space, not an arbitrary set. Because this was a transitional period between metric spaces and more general topological spaces, it is not clear whether Kuratowski realized at the time that he wrote this paper that his axiomatization applies to general topological spaces. We have chosen to drop Kuratowski's assumption of Euclidean n-space throughout this project, as the mathematics is exactly the same but the generality is much greater.

§1. General properties of $\overline{A}$.

We now establish the ... fundamental properties[4] of $\overline{A}$.

Theorem 1. $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.

Theorem 2. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Theorem 3. $\overline{A} - \overline{B} \subseteq \overline{A - B}$.

Theorem 4. $\overline{X} = X$.

We let Kuratowski prove Theorem 1 (but using Hausdorff's numbering of the Closure Axioms for consistency).

In effect, the inclusion $A \subset B$ means that $B = A \cup B$, from which [we have] $\overline{B} = \overline{A \cup B}$, and, by CA 4, $\overline{B} = \overline{A} \cup \overline{B}$; thus $\overline{A} \subset \overline{B}$.

Task 3 Prove Theorems 2–4 assuming CA. *Hint*: The following identities from set theory may prove useful in your proof: $A \cap B \subseteq A$, $A \cap B \subseteq B$, $A \subseteq A \cup B$, $A \cup B = (A - B) \cup B$.

Kuratowksi also stated the following generalization of Theorem 2.

Theorem 2a. A_i denoting an arbitrary family of sets with i an index variable, we have

$$\overline{\bigcap_i A_i} \subseteq \bigcap_i \overline{A_i} \text{ and } \bigcup_i \overline{A_i} \subseteq \overline{\bigcup_i A_i}.$$

Task 4 Prove Theorem 2a.

[4] Kuratowksi stated and proved six fundamental properties in his paper, two of which we have omitted since they are not relevant to the goals of this project.

After proving several other generalized properties of the closure operation, Kuratowski turned his attention to other topological concepts.

§2. Fundamental notions of Analysis Situs

We define in this section several notions of Analysis Situs, by relying on the operation $\overline{A}$.

The set A is said to be *closed* when $\overline{A} = A$.

... the union of two closed sets is closed; ... the intersection of [any collection of] closed sets is closed.

Task 5 Kuratowski has claimed here that his definition of closed set satisfies CSIA2 and CSIA3. Using what you and Kuratowski have shown so far, prove this claim.

Before turning to our main result, it may help to think about the following. We have two sets of axioms (CSIA and CA) and two terms (closed and closure). If we assume the CSIA, which of the two terms is undefined and which one needs a definition? If we assume the CA, which of the two terms is undefined and which one needs a definition? Both definitions are located somewhere in this project. Make sure you locate them and have them on hand for the final question.

Task 6 Finally, prove that a collection of sets satisfies CA if and only if it satisfies CSIA. Explain why this also shows that CA is equivalent to OSIA.

References

F. Hausdorff. *Grundzüge der Mengenlehre (Fundamentals of Set Theory)*. Von Veit, Leipzig, 1914.

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F. Hausdorff. *Set Theory*. Chelsea Publishing Company, New York, 1957. English translation of [Hausdorff, 1935] by John R. Aumann et al.

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Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed for use in an introductory course in topology. It is intended for students who have already seen the standard open set axioms and provides an equivalent axiomatic system for grounding all of point-set topology based on the work of Kuratowski. Other than exposure to and familiarity with the open set axioms, this is a self-contained project which can easily be done during a single class period with any unfinished tasks assigned for homework. Because most of the tasks have several claims to prove, this allows the instructor the ability to show one claim, another claim to be worked on in groups during class, and other claims to be done as homework tasks. In addition to the overarching goals of familiarizing students with an axiomatic system and showing them that multiple systems are equivalent, this project has the further content goal of familiarizing the student with the closure operation.

The closure axioms are not a standard topic in a first course in topology. As such, the project provides some interesting opportunities. For example, I have used this project as part of a final exam in a topology course. Another option is to use this project as a jumping off point to delve deeper into the nature of axiomatic systems. Different axiomatic systems are a huge topic in a course in geometry, but they are given little attention in other courses. This project allows one to see that there are multiple sets of axioms that are equivalent, yet look very different. This raises the question as to whether or not one could come up with a different, non-equivalent set of axioms for doing topology. Just as there are different geometries based on the axioms one chooses, can one create different notions of topologies based on the choice of axioms? What if one of the closure axioms was eliminated or changed? What if another axiom was added? These can be sources of exploration.

Student Prerequisites

This project can be done without any prior knowledge of topology. In that sense, this project could be a project on day 1 of a topology course, with the instructor simply providing a list of the open set axioms to the students as a companion to the PSP. However, it is recommended that students have been exposed to some concepts in topology, particularly that they have some familiarity with the open set axioms. The reason is because without that context, the material in this PSP will seem unmotivated and abstract. The fact that the closure operation axioms are equivalent to the open and closed set axioms will be appreciated more if students are aware of the fact that all of topology is built up from the open set axioms.

PSP Design and Task Commentary

Because the nature of this project is axiomatic, the tasks are fairly straightforward and clean. Task 1, for example, is a simple application of De Morgan's laws and a standard exercise in any first course in topology.

LaTeX code of the entire PSP is available from the author by request to facilitate preparation of advanced preparation/reading guides or "in-class worksheets" based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Suggestions for Classroom Implementation and Sample Schedule

This project can easily be completed in two class periods, possibly even one. Assign reading up to Task 2 and completion of the first two tasks as advance preparation for the first day. Class can then begin with a discussion about the reading, the nature of the problem that is under consideration, and the nature of axioms. Students can share their solutions for the first two tasks on the board, and a whole-class discussion can ensue to ensure that students understand what was done in the first task and why it is important. Students can be put into small groups to spend the rest of the class period working on the remaining exercises. These can either be written up and handed in the next day, or the class can go over them together the next day.

Connections to other Primary Source Projects

The following additional primary source-based projects by the author are also freely available for use in teaching courses in point-set topology. The first two projects listed are full-length PSPs that require 10 and 5 class periods respectively to complete. All others are designed for completion in 2 class periods.

- *Nearness without Distance*
- *Connectedness: Its Evolution and Applications*
- *From Sets to Metric Spaces to Topological Spaces*
  Possibly also suitable for use in Introductory Analysis courses
- *Topology from Analysis*
  Also suitable for use in Introductory Analysis courses
- *The Cantor set before Cantor*
  Also suitable for use in Introductory Analysis courses
- *Connecting Connectedness*
- *A Compact Introduction to a Generalized Extreme Value Theorem*

Classroom-ready versions of these projects can be downloaded from https://digitalcommons.ursinus.edu/triumphs_topology. They can also be obtained (along with their LaTeX code) from the author.

Acknowledgments

The development of this student project has been partially supported by the TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) Program with funding from the National Science Foundation's Improving Undergraduate STEM Education Program under Grant Nos. 1523494, 1523561, 1523747, 1523753, 1523898, 1524065, and 1524098. Any opinions, findings, and conclusions or recommendations expressed in this project are those of the author and do not necessarily represent the views of the National Science Foundation.

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