Beyond Riemann Sums: Fermat's Method of Integration

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Finding areas of shapes, especially shapes with curved boundaries, has challenged mathematicians for thousands of years. Today calculus students learn to graph functions, approximate the area with rectangles of the same width, and use "Riemann sums" to approximate the area under a curve (or, by taking limits, to find it exactly). Centuries before Bernhard Riemann (1826–1866) used this method, and a generation before Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) discovered calculus, seventeenth-century lawyer (and math fan) Pierre de Fermat (1607–1665) used a similar method, with an interesting twist that let the widths of the rectangles get larger as x grows. In this project, we will explore Fermat's method.

1 On "Quadrature"

It's worth noting that Fermat thought of area slightly differently than we usually do today. Following the work of ancient Greek mathematicians, his goal was "quadrature"—finding or constructing a rectangular shape with the same area as that of the shape he wanted to measure. In English, this is often called "squaring," so that "squaring the circle" means to construct, using the tools of geometry, a square with the same area as a given circle.

During Fermat's time, there were not many curves under which mathematicians could find the area precisely. Fermat was particularly interested in exploring not individual curves, but whole classes of them that could be studied with the same means. In his "Treatise on Quadrature"[1] (written c. 1659), one such class that he studied were curves that today we would write as functions of the form $y = \frac{1}{x^n}$. His primary method of finding these areas closely resembled the method of Riemann sums, which was not introduced until almost two centuries later.[2]

Rather than use rectangles of equal width, Fermat wanted to approximate the area under these curves with a series of rectangles whose areas lay in geometric proportion. First, however, he needed to establish some basic facts about such a series:[3]

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[1]As was not unusual for 17th-century works, the full title of Fermat's treatise is quite long: "De aequationum localium transmutatione, & emendatione, ad multimodam curvilineorum inter se, vel cum rectilineis comparationem, cui annectitur proportionis geometricae in quadrandis infinitis parabolis et hyperbolis usus," which Mahoney translates as "On the transformation, and alteration, of local equations for the purpose of variously comparing curvilinear figures among themselves, or to rectilinear figures, to which is attached the use of geometric proportions in squaring an infinite number of parabolas and hyperbolas" [Mahoney, 1994, p. 245].

[2]Augustin-Louis Cauchy (1789–1857) is generally credited with being the first to define the definite integral in our modern formulation. Riemann later developed the notion of what we now call "Riemann sums", but they didn't originally look like the version we teach in calculus today.

[3]All translations of Fermat excerpts in this project were prepared by the author.

Given any geometric proportion the terms of which decrease indefinitely, the difference between two consecutive terms of this progression is to the smaller term as the greater one is to the sum of all following terms.

Let's try to make sense of this. In a *geometric proportion*, any two consecutive terms have a constant ratio. Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{r^n} = 1 + \frac{1}{r} + \frac{1}{r^2} + \cdots .$$

Task 1 Is this a " geometric proportion the terms of which decrease indefinitely," as Fermat required? Why or why not?

Task 2 The latter part of Fermat's claim is an equality of two ratios. Write it out, and then solve for the the value of the infinite series. Does the solution look familiar?

2 Squaring a Hyperbola

Having established this crucial bit of mathematics, Fermat turned to his primary goal: finding the area under a hyperbola between a given starting point and infinity. Many different (and equivalent) definitions of hyperbolas have been given over history. Fermat's definition follows. His original picture is shown in Figure 1.

I define hyperbolas as curves going to infinity, which, like $DSEF$ [Fig. 1], have the following property. Let BA and AC be asymptotes which may be extended indefinitely; let us draw parallel to the asymptotes any lines EG, HI, NO, MP, RS, etc. We shall then always have the same ratio between a given power of AH and the same power of AG on one side, and a power of EG (the same as or different from the preceding) and the same power of HI on the other.

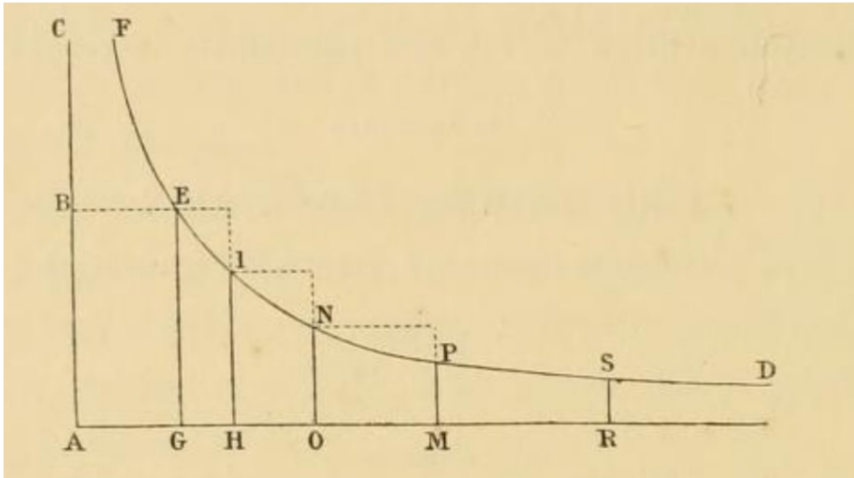

Figure 1: Fermat's setup to find the area under a hyperbola.

Modern ways of writing functions had not yet been developed; today, for example, we might guess that Fermat is referring to a curve defined by the equation $x^n y^m = k$ (n, m positive integers, and k constant).

Task 3 Use Fermat's picture and description to check whether he is describing the curve defined by $x^n y^m = k$. If yes, why? If no, how are the definitions different?

Fermat wanted to find the area under this curve from a given starting point, say the line GE. He suspected that even though the length of the curve was infinite, the area under it may be finite. In fact, Fermat made what may be a rather startling claim:

I say that all these infinite hyperbolas except the one of Apollonius [the simplest hyperbola, described by $y = 1/x$] ..., may be squared by the method of geometric progression according to a uniform and general procedure.

Task 4 Why did Fermat exempt the curve given by $y = 1/x$ from his claim?

Task 5 Are there any other restrictions that he should have put on functions of the form $y = 1/x^n$ if the area under the curve is to be finite?

Fermat started to set up his solution as follows:

Imagine the terms of a geometric progression, extended *in infinitium*, of which the first is AG, the second AH, the third AO, etc., *in infinitum*,

Task 6 Compare Fermat's geometric progression with Figure 1. What do the terms correspond to? Does it look as if the figure is drawn to scale? Explain your answer.

Having established lengths on the x-axis in geometric progression, Fermat next turned to the areas of the rectangles seen in the figure.

Being given this, together with the fact that

as AG is to AH, so AH is to AO,
and so AO is to AM,

it will be equally that

as AG is to AH, so the interval GH is to HO, and so the interval HO is to OM, etc.

Moreover, the parallelogram [formed by] EG and GH will be to the parallelogram formed by HI and HO as the parallelogram formed by HI and HO is to the parallelogram formed by NO and OM: that is to say the ratio of the parallelograms on GE and GH to the parallelogram on HI and HO is constructed from the ratio of lines GE to HI and from the ratio of lines GH to HO, it will be moreover that

as GH is to HO, so AG is to AH, and so on.

In the next few tasks, let's look at a specific example that will allow us to better understand Fermat's claim.

Task 7 Let's assume that the curve represents the function $f(x) = \frac{1}{x^2}$.

(a) For our example, we can choose values of A, G, H, O, M, etc., so that Fermat's statement that "as AG is to AH, so AH is to AO, and so AO is to AM" holds. Set the x-coordinate of A to be $x_A = 0$, of G to be $x_G = 1$, and of H to be $x_H = 2$. What should the values of x_O, x_M, and x_R be?

(b) Explain why it's true that "as AG is to AH, so the interval GH is to HO," and also why "as AG is to AH, so the interval HO is to OM."

(c) The most important step for Fermat was probably his claim that the areas of the parallelograms form a geometric progression, which he described (as above) as "Moreover, the parallelogram [formed by] EG and GH will be to the parallelogram formed by HI and HO as the parallelogram formed by HI and HO is to the parallelogram formed by NO and OM." Find the areas of the first few parallelograms in our example. Is Fermat's claim true? If yes, explain why this progression will continue to hold. If not, explain why not.

At this point, Fermat began to wrap up his argument:

But the lines AO, AH, AG, which form the ratios of the parallelograms, define by their construction a geometric progression; hence the infinitely many parallelograms $EG \times GH, HI \times HO, NO \times OM$, etc., will form a geometric progression, the ratio of which will be AH/AG. Consequently, according to the basic theorem of our method, GH, the difference of two consecutive terms, will be to the smaller term AG as the first term of the progression, namely, the parallelogram $GE \times GH$, to the sum of all the other parallelograms in infinite number ... this sum is the infinite figure bounded by HI, the asymptote HR, and the infinitely extended curve IND.

Task 8 Draw a new picture of the hyperbola defined by $f(x) = \frac{1}{x^2}$ and the rectangles that Fermat described in the previous excerpt. Lightly shade in the rectangles. Do you agree that the area under the curve is bounded by the sum of the areas of the rectangles? Why or why not?

Task 9 Use your calculations and the results you found on the geometric progression of the areas of the rectangles in Task 7(c) to explain why the infinite sum of the rectangles' area for the function $f(x) = \frac{1}{x^2}$ converges, and that the area under the hyperbola is finite.

Task 10 Let's try to use Fermat's ideas to put an upper bound on the area under the curve $f(x) = \frac{1}{x^2}$ to the right of the line defined by $x = a$ for some real number a.[4] Instead of choosing specific values for our geometric series, we'll describe the x-coordinates of all the points in terms of a starting value ($x_G = a$), and a ratio ($x_H = ar$).

(a) Find the "x-coordinates" of the points H, O, and M in terms of a and r.

(b) Find the area of the rectangle formed by GE and GH in terms of a and r.

(c) Find the area of the rectangle formed by HI and HO in terms of a and r.

(d) Find the area of the rectangle formed by ON and OM in terms of a and r.

(e) Find the area of the nth rectangle in terms of n, a, and r.

(f) Now write down and evaluate the infinite series obtained by summing the areas of the rectangles. If you have learned to evaluate improper integrals, how does this area compare to the true area under the curve?

[4] Once again, we're going to treat the line with points A, G, H, O, M, R as the x-axis.

3 A More Precise Measurement

So far we have used Fermat's method to put an upper bound on the area under the hyperbola defined by $f(x) = \frac{1}{x^2}$. It would, of course, be more satisfying to find the answer precisely. Fermat had an idea of how this could be done, as well:

. . . let them [the lengths of the intervals], by approximation, approach each other as closely as is necessary in order that, by the Archimedean method,[5] the rectilinear parallelogram $GE \cdot GH$ is almost equal to the fixed quadrilateral figure $GHIE$; and further, such that the first of the rectilinear differences of the proportionals, GH, HO, OM, and so on, are almost equal to one another . . . .

Task 11 Let r be the ratio of the lengths of consecutive intervals. In modern terms, what is Fermat suggesting about the limiting value of r?

Task 12 In practice, could one estimate the area under this curve as closely as one would like using values of r really close to this value? Why or why not?

References

Pierre de Fermat. De aequationum localium transmutatione, & emendatione, ad multimodam curvilineorum inter se, vel cum rectilineis comparationem, cui annectitur proportionis geometricae in quadrandis infinitis parabolis et hyperbolis usus (On the transformation, and alteration, of local equations for the purpose of variously comparing curvilinear figures among themselves, or to rectilinear figures, to which is attached the use of geometric proportions in squaring an infinite number of parabolas and hyperbolas). In *Varia opera mathematica D. Petri de Fermat*, volume 3, pages 44–57. Johannes Pech, Toulouse, 1679. Also in Fermat's *Oeuvres*, volume 1, pages 255–288, Gauthier-Villars et fils, Paris, 1891. Written c. 1659.

Michael Sean Mahoney. *The Mathematical Career of Pierre de Fermat, 1601–1665.* Princeton University Press, 1994.

[5] Archimedes (c. 287 BCE–c. 212 BCE) used a method very much like what Fermat described in this section. The idea is to break a complicated area into small shapes (say, rectangles) that one can easily find the area of, and then repeat the process with even smaller shapes that form a better approximation to the original area, and think about the limit of the result.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) attempts to strengthen students' intuition for Riemann integration by presenting a path to finding areas reminiscent of, but distinct from, Riemann's methods. By approximating area with rectangles of increasing base length, Pierre de Fermat became the first mathematician to be able to find the area under any (convergent) curve described by $y^m = x^n$. In this project, students work through the type of calculation used by Fermat, with a focus on the area under $y = 1/x^n, n > 1$.

Student Prerequisites

Fermat's solution to the sum of a geometric series is provided at the beginning of this PSP, but it's expected that students will already have seen a modern formulation. It would be useful for students to have been introduced to the limit concept for some of the later exercises. Another optional exercise requires improper integrals. No other background outside of algebra is required.

PSP Design and Task Commentary

The PSP has only one primary source: Pierre de Fermat's "De aequationum localium transmutatione, & emendatione, ad multimodam curvilineorum inter se vel, cum rectilineis comparationem, cui annectitur proportionis geometricae in quadrandis infinitis parabolis et hyperbolis usus," [Fermat, 1679], better known to history as the "Treatise on Quadrature." Fermat wrote a generation before the calculus of Newton and Leibniz. He was interested in many of the questions that we still teach in calculus today (especially rates of change and areas), but lacked any of the machinery available to the modern student.

The "Treatise" includes a variety of methods for finding the area under different curves. Many of these are difficult to understand from a modern point of view; Fermat wrote, for example, before the function concept, and needed to approach these problems geometrically. This project focuses only on the area under the "hyperbola" $y = 1/x^n$, which provides both useful insight into infinite series, and a clever trick that provides a nice example of mathematical invention.

Some comments on select tasks follows.

- Task 2 can be tricky. "The sum of all following terms" seems ambiguous, but is probably best written out as follows:
$$\frac{\frac{1}{r^k} - \frac{1}{r^{k+1}}}{\frac{1}{r^{k+1}}} = \frac{\frac{1}{r^k}}{\sum_{n=k+1}^{\infty} \frac{1}{r^n}}.$$
Some students may need a hint that this can be simplified by multiplying the numerator and denominator of the left-hand side by r^{k+1}, after which the problem could be solved as follows:
$$\frac{r-1}{1} = \frac{\frac{1}{r^k}}{\sum_{n=k+1}^{\infty} \frac{1}{r^n}} \implies \sum_{n=k+1}^{\infty} \frac{1}{r^n} = \frac{1}{r^k} \cdot \frac{1}{r-1} \implies \sum_{n=k+1}^{\infty} \frac{1}{r^n} = \frac{1}{r^{k+1}} \cdot \frac{1}{1-\frac{1}{r}},$$
which thus looks like the sum of a geometric series with first term $1/(r^{k+1})$.

- For Task 10, I'm expecting something like the following:
  (a) $x_H = ar, x_O = ar^2, x_M = ar^3$.
  (b) $GE = \frac{1}{a^2}$, $GH \times GE = \frac{ar-a}{a^2} = \frac{r-1}{a}$.
  (c) $HO = ar^2 - ar$, $HI = \frac{1}{(ar)^2}$, $HI \times HO = \frac{ar^2-ar}{a^2r^2} = \frac{1}{r} \cdot \frac{r-1}{a}$.
  (d) $OM = ar^3 - ar^2$, $HI = \frac{1}{(ar^2)^2}$, $OM \times ON = \frac{ar^3-a^2r}{a^2r^4} = \frac{1}{r^2} \cdot \frac{r-1}{a}$.
  (e) The nth rectangle has area $\frac{r-1}{ar^{n-1}}$.
  (f) $\sum_{n=1}^{\infty} \frac{r-1}{ar^{n-1}} = \frac{r}{a}$, for $|r| > 1$.

Suggestions for Classroom Implementation and Sample Implementation Schedule (based on a 75-minute class period)

This PSP is designed to be implemented in about one (75-minute) class day, with short discussion on the day before implementation (with some assigned homework), and a chance to wrap and discuss the project on the day after implementation. Homework can be done individually, and the bulk of the project should be completed as group work during class.

- **Day 0.** Introduce the project. Depending on whether students have seen Riemann sums, the instructor could either discuss the question of why we usually make all of our "rectangles" of equal width (if yes), or introduce the idea of bounding an infinite area with rectangles (if no).
- **Day 0 Homework:** Assign the beginning of the PSP through Task 3 as homework.
- **Day 1.** Students may benefit from a full-class discussion of Task 2 (which requires strong algebra skills). Students can then work in groups, with the goal of getting through at least part of Task 10, which takes a fair bit of calculation.
- **Day 1 Homework:** Assign the remainder of the PSP as homework.
- **Day 2:** Discuss results either in small groups or as a whole class, possibly asking students to present their solutions to Task 10. This would be an especially good time to discuss the idea that we can bound the area of $f(x) = 1/x^2$ to the right of $x = a$ by r/a^2 for any $|r| > 1$, and to think about what this means in the limiting case as $r \to 1$.

The project itself can also be modified by instructors as desired to better suit their goals for the course. The LaTeX source file for this PSP is available from the author by request.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in teaching standard topics in the calculus sequence. The PSP author name of each is given (together with the general content focus, if this is not explicitly given in the project title). Each of these projects can be completed in 1–2 class days, with the exception of the four projects followed by an asterisk (*) which require 3, 4, 3, and 6 days respectively for full implementation. Classroom-ready versions of these projects can be downloaded from https://digitalcommons.ursinus.edu/triumphs_calculus/.

- *L'Hôpital's Rule*, by Daniel E. Otero

- *The Derivatives of the Sine and Cosine Functions*, by Dominic Klyve
- *Three Hundred Years of Helping Others: Maria Gaetana Agnesi on the Product Rule*, by Kenneth M Monks
- *Fermat's Method for Finding Maxima and Minima*, by Kenneth M Monks
- *Gaussian Guesswork: Elliptic Integrals and Integration by Substitution*, by Janet Heine Barnett
- *Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve*, by Janet Heine Barnett
- *Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean*, by Janet Heine Barnett
- *Investigations Into d'Alembert's Definition of Limit (Calculus version)*, by Dave Ruch (sequence limits)
- *How to Calculate π: Machin's Inverse Tangents*, by Dominic Klyve (infinite series)
- *Euler's Calculation of the Sum of the Reciprocals of Squares*, by Kenneth M Monks (infinite series)
- *Fourier's Proof of the Irrationality of e*, by Kenneth M Monks (infinite series)
- *Jakob Bernoulli Finds Exact Sums of Infinite Series (Calculus Version)*,* by Daniel E. Otero and James A. Sellars
- *Bhāskara's Approximation to and Mādhava's Series for Sine*, by Kenneth M Monks (approximation, power series)
- *Braess' Paradox in City Planning: An Application of Multivariable Optimization*, Kenneth M Monks
- *Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green's Theorem*,* by Abe Edwards
- *The Fermat-Torricelli Point and Cauchy's Method of Gradient Descent*,* by Kenneth M Monks (partial derivatives, multivariable optimization, gradients of surfaces)
- *The Radius of Curvature According to Christiaan Huygens*,* by Jerry Lodder

Recommendations for Further Reading

Without a doubt, the most thorough study of Fermat's work is Michael Sean Mahoney's *The Mathematical Career of Pierre de Fermat (1601–1665)* [Mahoney, 1994]. His discussion of Fermat's "Treatise on Quadrature" begins on page 243 of his book. The curious reader will find a deeper discussion of his work, including an important topic excluded from the PSP: that of *adequality*. The term seems to mean something like "two things are equal in their limits," but the details are vexingly complex to modern readers and contribute little to the understanding of area. After wrestling with how to introduce the idea to students for more time than I would like to admit, I removed these sections entirely.

For a discussion of how Fermat's work can be used in the classroom using modern notation, see

- V. Frederick Rickey. Historical Notes for the Calculus Classroom: Fermat's Integration of Powers, *Convergence* (May 2023). Available at https://www.maa.org/press/periodicals/convergence/historical-notes-for-the-calculus-classroom.

- Amy Shell-Gellasch. Integration à la Fermat, in Amy Shell-Gellasch and Dick Jardine, eds., *Mathematical Time Capsules: Historical Modules for the Mathematics Classroom*, number 77, 111–116, MAA (2011).

The earliest discussion I could find in print of Fermat's integration techniques is the following work by Boyer, which should be read by anyone looking to understand the history of our understanding of Fermat's work.

- Carl B. Boyer. Fermat's Integration of x^n. *National Mathematics Magazine*, 20(1):29–32, 1945.

Acknowledgments

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