Three Hundred Years of Helping Others: Maria Gaetana Agnesi on the Product Rule

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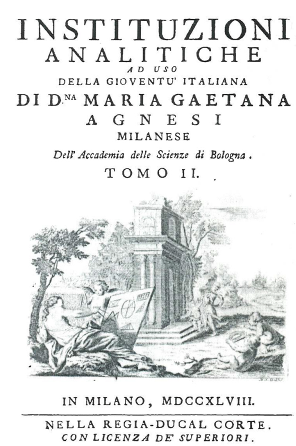
Differential calculus today is typically taught based on the following definition of the derivative of the function $f(x)$:

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

This, in turn, is based on the ϵ–δ definition of the limit. However, it is worth knowing that this is not the only paradigm for exposition of differential calculus. It is actually in some sense quite upside-down: the ϵ–δ definition of a limit was not formally stated until the mid-nineteenth century, in the work of Karl Weierstrass (1815–1897), while the rules of differentiation were worked out centuries earlier. So how were these rules explained before the limit definition was available?

Explanations of differential calculus took different shapes and forms over the years, but surely one of the best is found in the work of the expository genius Maria Gaetana Agnesi (1718–1799).[1] Her two-volume book *Instituzioni Analitiche ad Uso della Gioventù Italiana (Foundations of Analysis for Use of the Italian Youth)* aimed to make accessible to the many knowledge which had previously been held only by the few. Not only did Agnesi's book offer a complete introductory treatment of precalculus, differential calculus, and integral calculus in a single work, it was also one of the first calculus textbooks written in a vernacular language instead of Latin. Her careful systematic approach to presenting its content further added to its readability for those new to the study of calculus (or *analysis*, as it was then called).



Title page from *Instituzioni Analitche ad Uso della Gioventù Italiana, Tomo II.*

However, to leave it at that would be to sorely undersell what a massive accomplishment it was for Agnesi to have done this. To compose such a work today—with plenty of already well-organized sources to draw from—would be a substantial accomplishment, but

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[1]From an early age, Agnesi was quite famous in Europe for the clarity and beauty of her exposition. One anecdote involved two French nobles from Burgundy who traveled to Milan in the hopes of finding works of art from classical antiquity or the High Renaissance—Raphael in particular. When they arrived in Milan, they were disappointed to have found little of what they came for, but they were thrilled to have stumbled upon Maria Agnesi. When they ended up at a social gathering hosted by her father Pietro Agnesi (c. 1692–1752), she spoke on the nature of tides and the origin of spring water, two complicated and controversial topics at the time. The Burgundians were blown away, saying that "She spoke like an angel about these matters" [Mazzotti, 2007, p. 5]. For more details on her life, see [Mazzotti, 2008].

the task is doable for any person trained in mathematics given sufficient time to put into it. But for someone to have done this when calculus was still in its early stages of development—with related results based on different underlying ideas scattered throughout different publications in different languages printed in different countries—would have been incredibly daunting. Although Agnesi's book wasn't the first textbook to be written for the study of calculus, she felt that even the most popular introductory work at the time, the 1707 differential calculus treatise *Analyse des Infinitement Petits* by Guillaume de l'Hôpital (1661–1704), was not accessible to young students.[2] She thus spent five years collecting books and manuscripts with an eye towards writing an introductory text to l'Hôpital's work to make it more accessible to young students. Her teacher Count Carlo Belloni[3] (1701–1747) spoke very directly to how difficult this work was when he told her, "If the Marquis de l'Hôpital took pleasure in these kinds of computations, I am not surprised at all that he died young" [Mazzotti, 2007, p. 54].

The moral of the story: if we want to find out how people were discussing derivatives for centuries before the definition of the limit, and we want not just *an* introduction to that discussion, but a *good* introduction, then we could not do better than consult the writings of the fearless Maria Gaetana Agnesi! Perhaps there is also some insight to be gleaned that was lost when our exposition was rebuilt based on the newer framework of limits.

1 The Reductionist Approach

The standard framework for discussing derivatives takes a reductionist approach, much as a chemist studies a molecule by understanding the atoms it is composed of and how they are bonded. Similarly, the study of differential calculus often begins by learning the derivatives of many "atomic" functions. What is the derivative of x^2? What is the derivative of $\sin x$? And so on. Then, when one meets a more complicated function built of these simpler functions stitched together in some way, one does not have to reinvent the wheel. Instead the idea is to calculate the derivative based on the knowledge of the atomic functions' derivatives and how those atomic functions are "bonded" together.

Task 1 Can you think of another technique you learned in your mathematical journey that took a reductionist approach, in which a mathematical structure was understood by understanding the simpler constituent parts as well as how they were stitched together? Can you think of another technique outside of mathematics that takes a reductionist approach?

Multiplication is a standard way that one binds together our atomic functions to create more complicated functions. Thus, the *product rule* for derivatives is key to having an efficient, operational

[2] This book was essentially a record of l'Hôpital's private lessons and correspondence from his tutor Johann Bernoulli (1667–1748), one of the leading mathematicians of this era. It followed these lessons (for which l'Hôpital paid Bernoulli handsomely) so closely that its publication has left their contemporaries—as well as modern historians of mathematics—asking the question "whose work actually was it?" If you would like to dive directly into this rabbit hole, the English translation of l'Hôpital's book in [Bradley et al., 2015], which also includes source material from Bernoulli, can distract you nearly endlessly from the other more time-sensitive things you should be doing.

[3] Count Belloni made sure that Agnesi studied the geometry and algebra necessary to work through l'Hôpital's book, and later introduced her to Isaac Newton's (1642–1727) *Opticks* and *Principia Mathematica*. He also coached her in disputation techniques and on matters of etiquette. In recognition of his tutoring and guidance, Agnesi dedicated her first mathematical publication to Count Belloni [Mazzotti, 2007, p. 33].

differential calculus; it will let us determine the derivative of a product of functions from knowledge of the derivatives of each factor.

2 Derivatives vs. Differentials: Different, or Same Difference?

As mentioned in the introduction, explanations of differential calculus have taken different shapes and forms over the years. In the earliest days of the calculus, Isaac Newton (1643–1727) and his followers in Great Britain used something called a fluxion, essentially his word for the derivative, when talking about calculus. Meanwhile, on the European continent, Gottfried Leibniz (1646–1716) and his followers used something called a differential, which is discussed below. As you might have guessed, the "differential calculus" of today still uses much of the notation and language that came out of Leibniz's approach to the calculus. In a modern calculus course, however, the majority of attention is usually placed on derivatives. Agnesi's demonstrations focused on the differential, rather than the derivative. As we are about to see, her differential approach is in some ways more elegant and more intuitive!

First, let's consider an important distinction between the *derivative*, a rate of change of one quantity with respect to a change in another,[4] and the *differential*, a small change in a quantity, considered in and of itself.[5] To see how these two ideas can be related to each other, here is an example that may seem familiar from Agnesi's book [Agnesi, 1748, p. 461]:[6]

The differential of x^m, with m being any positive integer, will be $mx^{m-1}dx$.

Setting $y = x^m$, we can write Agnesi's differential statement as

$$dy = mx^{m-1}dx.$$

Dividing both sides of this last equation by dx then creates the following ratio of differentials:

$$\frac{dy}{dx} = mx^{m-1}.$$

This rate of change is known as the derivative of the function $y(x) = x^m$. Similarly, if we started with the derivative fact (usually referred to today as the *power rule*), then we could multiply both sides by dx to obtain the corresponding differential fact.

But how did Agnesi come up with differential statements like the one above in the first place? Let's take a look at her explanation of the product rule to find out.

[4] More specifically, the derivative represents an *instantaneous* rate of change at a particular value of the independent variable.

[5] In [Agnesi, 1748, p. 433], Agnesi herself described a differential as "that infinitesimal portion of a variable quantity." Using her definition, a derivative would then literally be the ratio of two differentials. Today, we tend to instead think of differentials as very, very small finite differences. This is why we still need to take a limit in order to arrive at the instantaneous rate of change (or derivative).

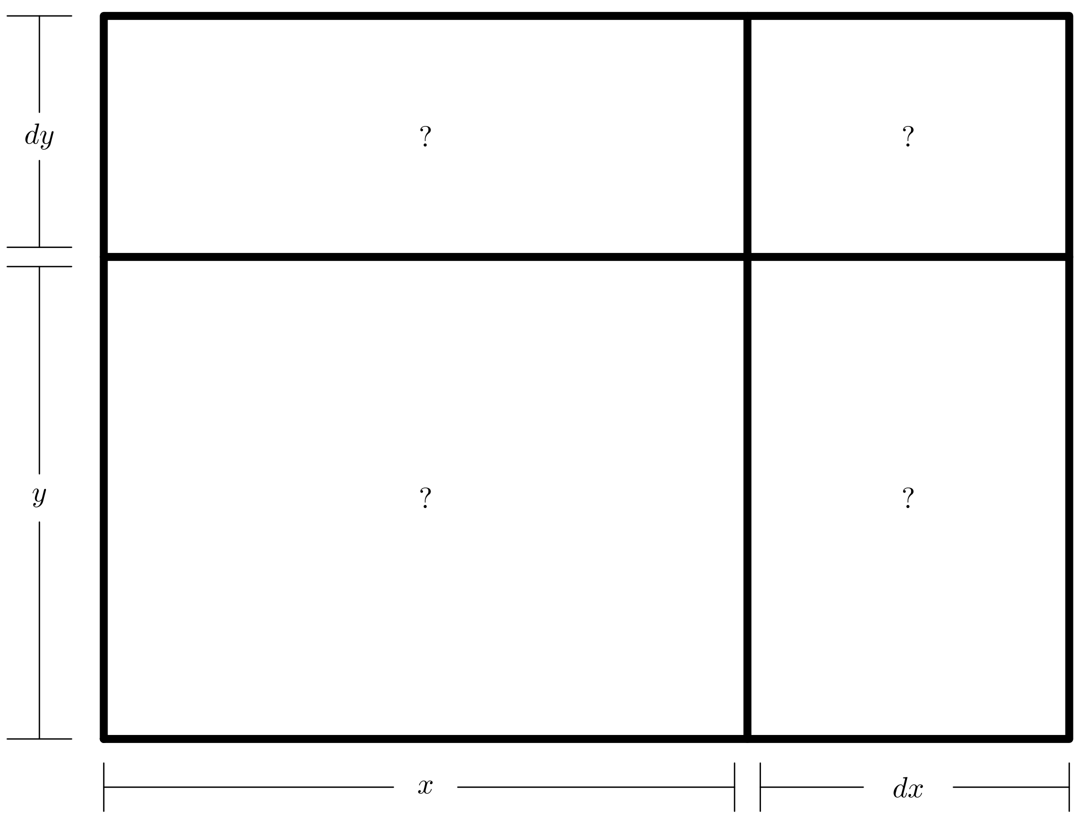
[6] All translations of Agnesi excerpts in this project were prepared by the author, with minor changes made to the source text for readability.

3 Agnesi's Product Rule

We now immerse ourselves in Agnesi's version of the product rule, found in [Agnesi, 1748, p. 458].

If the quantity proposed to be differentiated is the product of several variables, such as xy, whereas x becomes $x+dx$, y becomes $y+dy$, and xy becomes $xy+ydx+xdy+dx\ dy$, which is the product of $x+dx$ with $y+dy$; from this product, then, subtracting the proposed quantity xy, there remains $ydx+xdy+dx\ dy$, but $dx\ dy$ is an infinitely smaller quantity than each of the other two, which are the rectangle of a finite quantity with an infinitesimal, and $dx\ dy$ is the rectangle of two infinitesimals, and therefore infinitely smaller, so this rectangle can frankly be neglected, therefore the differential of xy will be $xdy+ydx$.

Let us illustrate the passage with a diagram, showing dx as a small change in the quantity x and dy as a small change in the quantity y. In particular, our diagram represents products as "rectangles" as Agnesi suggested.



Task 2 In this task, we use the above diagram to make sense of Agnesi's differential rule for products, then use that rule to figure out what the derivative rule for products is.

(a) Let us visualize Agnesi's "quantity proposed" as the area of a rectangle of width x and height y. Label that area in the diagram above, placing the label where one of the question marks is.

(b) Find Agnesi's other quantities, namely xdy, ydx and $dx\ dy$, as areas of rectangles marked with question marks.

(c) When Agnesi said "subtracting the proposed quantity xy, there remains $ydx + xdy + dx\ dy$," what part of the diagram was that part that "remains"? Shade it in your diagram above.

(d) Consider $x = 4$, $y = 3$, and take dx and dy to both be 0.001. (Note that these measurements are not to scale with the diagram above.) Plug all of these numbers in, and see if you agree with her claim that "$dx\ dy$ is an infinitely smaller quantity than each of the other two."

(e) Discarding this "infinitely smaller quantity," we have Agnesi's version of the product rule:

the differential of xy will be $xdy + ydx$.

To translate this into a derivative version, let's first rewrite Agnesi's product rule symbolically using variables that sound more like function names to us:

$$d(fg) = f \cdot dg + g \cdot df.$$

Divide both sides of this by dx, and notice that the left-hand side is now a derivative (i.e., ratio of differentials). As for the right-hand side, if we leave it as just one big expression divided by dx, it isn't incorrect, but we can do better. In particular, rewrite the right-hand side so that it's clear where the derivatives of the two "atomic" functions (f and g) appear in this version of the product rule.

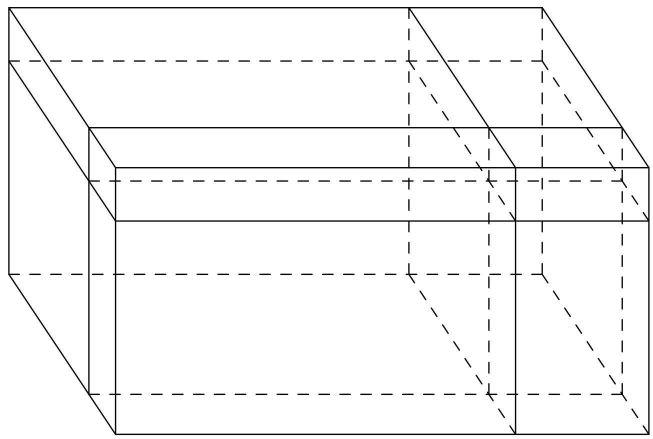
One lovely aspect of Agnesi's treatment of the product rule is that it worked for more than the simple product xy. She proceeded to extend her approach to products of three quantities [Agnesi, 1748, p. 459]!

Let the quantity to be differentiated be xyz; the product of $x + dx$ with $y + dy$ with $z + dz$ is . . .

Task 3 Agnesi showed the answer in her textbook, but let's work this out ourselves!

(a) What is the product of those three quantities she considered, when expanded out? Write it as a sum of eight terms.

(b) See the diagram below, which represents a rectangular prism with seven new smaller rectangular prisms added to it. Label the diagram such that the eight prisms correspond to the eight terms you wrote in part (a). (**Hint.** To get started, revisit the diagram that illustrated the product rule for just two variables. Notice that the same labels could apply to the back face of this more complicated diagram!)



Similar to the process for the product of two quantities, we need to decide which quantities we should keep, and which can be discarded. We consult Agnesi for advice!

... $yzdx + xzdy + xydz + zdx\ dy + ydx\ dz + xdy\ dz + dx\ dy\ dz$; but the first, second, and third terms are each the product of two finite quantities with an infinitesimal; and the fourth, fifth, and sixth are each the product of one finite quantity with two infinitesimals, ... and the last, which is the product of three infinitesimals, therefore neglecting all the terms starting from the fourth, it will be ... .

Task 4 OK, she gave us instructions for what do to here, so let's figure it out.

(a) As before, take $x = 3$, $y = 4$, and $z = 5$, and take $dx = dy = dz = 0.001$. When you plug those numbers in, which terms seem to be "infinitely smaller [quantities]"?

(b) Subtracting the original proposed quantity xyz from the expanded form with eight terms, and discarding those that are "infinitely smaller [quantities]", what is the differential of the product xyz?

Summarizing the results above, and simultaneously writing them using today's prime notation, we can state the product rule of two quantities as

$$(fg)' = f'g + fg'$$

and of three as

$$(fgh)' = f'gh + fg'h + fgh'.$$

Task 5 Calculate the derivative of $x^2 \sin x$ using the rule for the product of two quantities.

Task 6 Calculate the derivative of $x^2 \sin x \cos x$ using the rule for the product of three quantities.

Task 7 Though Agnesi did not specifically write out such a rule for a product of four quantities in general, extrapolate the results above.

(a) What would you expect the derivative of four quantities $f_1 f_2 f_3 f_4$ to be? What about $f_1 f_2 f_3 f_4 f_5$?

(b) Give a verbal description of a process for writing down the derivative of n functions, where n is any natural number.

Task 8 We end by exploring a connection between two of the standard rules of differential calculus, thanks to Agnesi's treatment of the subject.

(a) Find Agnesi's statement of the power rule as quoted in this project. What page number of *Instituzioni* does it appear on?[7]

(b) Find Agnesi's statement of the product rule for multiple factors as quoted in this project. What page number of *Instituzioni* does it appear on?

(c) Based on the page numbers you found, notice which of the above rules appeared first in Agnesi's book! We've already worked through her presentation of the product rule. Because she handled it first, she could then legitimately apply the product rule in order to deduce the rule for x^m—and so can you! Do this now, by explaining how the general product rule that you stated in Task 7(b) gives us the derivative of x^m for an arbitrary natural number m.

4 The "Wishful Thinking" Product Rule

A common mistake that many of us make at least once while we're learning calculus is to "find" the derivative of a product by just taking the product of the derivatives. For example, many calculus students, the author included, at some point have done an invalid operation like this:

$$\frac{d}{dx}\left(x^2 \sin(x)\right) = 2x \cos(x).$$

Especially when we're in a hurry while taking an exam, it can be awfully tempting to just multiply the two derivatives together, which would only work if $(fg)' = f'g'$. This would make calculations cleaner, if only it were true!

Task 9 Use Agnesi's rectangles diagram to explain why it cannot be that $(fg)' = f'g'$.

[7] Note that for each *Instituzioni* excerpt in this project, the corresponding citation includes the page number. There is no need to consult a complete copy of Agnesi's text to complete this task!

5 Epilogue

Agnesi's life was characterized by a passion for helping others. Much of her family life centered around caring for her younger siblings (of which she had twenty!), and her academic work centered around writing for young mathematics students who were unable to access recent developments in calculus (then a new and rapidly-evolving field). She devoted her later years to caring for the sick and the elderly. Although we now typically present calculus as founded on a limit-based framework, Agnesi's way of discussing the subject offers intuition and insight that gets lost in today's formalism. How glad she would be to know that her writing might still be helping students almost three hundred years later!

Task 10 For this task, you will want to consult a source of mathematicians' biographies, perhaps [Koertge, 2008] if you have access to it, or the St Andrews MacTutor Biographies archive found at `https://mathshistory.st-andrews.ac.uk/Biographies/`, or another of your choice.

(a) Maria Agnesi helped so many others; who did she receive help from?

(b) Briefly compare and contrast Agnesi's upbringing, academic career, and impact with that of Laura Bassi (1711–1778).

References

Maria Gaetana Agnesi. *Instituzioni Analitiche ad Uso della Gioventù Italiana*, volume II. Regia-ducal corte, Milan, 1748. Available at `https://archive.org/details/A298184`.

Robert E. Bradley, Salvatore J. Petrilli, and C. Edward Sandifer. *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*. Springer, 2015.

Noretta Koertge, editor. *New Dictionary of Scientific Biography*. Gale, 2008.

Massimo Mazzotti. Maria Gaetana Agnesi: Mathematics and the Making of the Catholic Enlightenment. *Isis*, 92(4):657–683, December 2001.

Massimo Mazzotti. *The World of Maria Gaetana Agnesi, Mathematician of God*. John Hopkins University Press, 2007.

Massimo Mazzotti. Agnesi, Maria Gaetano. In Noretta Koertge, editor, *New Dictionary of Scientific Biography*, volume 1, pages 19–21. Gale, 2008.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) walks a student through Maria Gaetana Agnesi's treatment of the product rule via differentials. Her interpretation of the quantities and their differentials as measurements in a diagram of rectangles provides a visual, intuitive explanation of the rule for students trying to stay afloat in the choppy sea of differential calculus.

This project not only covers the standard product rule for a product of two quantities, but it also shows Agnesi's development of the product rule for three quantities, and has the student hypothesize how these formulas might extend to any number of factors.

Student Prerequisites

This project assumes very little in terms of prerequisites. The only skills assumed are the basics of precalculus: if the student can manipulate sums, differences, and products written in the language of symbolic algebra, they are ready for Agnesi's treatment of the product rule. This was, after all, exactly her goal! She set out to make calculus accessible to the young student with little background, and she succeeded.

PSP Design and Task Commentary

This PSP begins with a high-level introduction regarding what this whole process of derivative rules entails. Task 1 is perfect for an in-class discussion, hopefully showing students how widely-used the reductionist approach is in mathematics, among other fields. Task 2 is the central task in this project; it is where the student dives directly into Agnesi's primary source passage in which she explains the product rule in terms of rectangle areas. Tasks 3–8 are fun extensions of this key idea, and extend the product rule to products of more factors. Task 9 hopes to help students avoid a common error that anyone who has graded a stack of calculus exams has seen far too often!

This PSP allows for a natural place to look at the distinct ideas of "derivative" vs "differential." Students may notice that Agnesi herself sometimes used one word and sometimes used the other, even though her goal was always to find a differential rather than a derivative function. This is a perfectly good place for students to notice how word meanings often change with time; today the derivative always refers to the function, but one can see from these excerpts that it was not so at one point!

Note that the PSP purposefully avoids any discussion of differentiability, as it is in some sense irrelevant to Agnesi's presentation of these ideas. The modern proof of the product rule using the limit definition of the derivative is similarly absent from this PSP; this PSP can be used before, after, or instead of such a treatment of the topic.

Suggestions for Classroom Implementation

This primary source project is likely one of the most accessible of the entire PSP collection. It could be assigned for homework in its entirety to follow a more traditional lesson on the product rule for derivatives, but it is probably more beneficial, and more fun, to work through the introduction and Task 1 at the end of one class (or assign this part for class prep), letting the students attempt

Sections 2–5 in small groups during the next class session. The remaining unfinished work can then be assigned for homework.

A classroom-ready version of this PSP is available at the TRIUMPHS website (find the URL in the Acknowledgements section below). The author is happy to provide LaTeX code for this project. It was created using Overleaf, which makes it convenient to copy and share projects, and can allow instructors to adapt this project in whole or in part as they like for their course.

Sample Implementation Schedule (based on a 50-minute class period)

This PSP can easily be implemented in one 50-minute class period.

- Assign the reading of the introduction and Section 1, and completion of Task 1 for class prep.
- The following class session, begin with 10 minutes for a discussion of the introduction and have some share their responses for Task 1. It is likely that students will have come up with a wide variety of answers!
- Have students work Sections 2–5 in small groups while the instructor/learning assistant helps.
- Students can complete all remaining unfinished tasks for homework.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in teaching standard topics in the calculus sequence. The PSP author name of each is given (together with the general content focus, if this is not explicitly given in the project title). Each of these projects can be completed in 1–2 class days, with the exception of the four projects followed by an asterisk (*) which require 3, 4, 3, and 6 days respectively for full implementation. Classroom-ready versions of these projects can be downloaded from https://digitalcommons.ursinus.edu/triumphs_calculus/.

- *L'Hôpital's Rule*, by Daniel E. Otero
- *The Derivatives of the Sine and Cosine Functions*, by Dominic Klyve
- *Fermat's Method for Finding Maxima and Minima*, by Kenneth M Monks
- *Gaussian Guesswork: Elliptic Integrals and Integration by Substitution*, by Janet Heine Barnett
- *Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve*, by Janet Heine Barnett
- *Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean*, by Janet Heine Barnett
- *Beyond Riemann Sums: Fermat's Method of Integration*, by Dominic Klyve (uses geometric series)
- *Investigations Into d'Alembert's Definition of Limit (Calculus version)*, by Dave Ruch (sequence limits)
- *How to Calculate π: Machin's Inverse Tangents*, by Dominic Klyve (infinite series)
- *Euler's Calculation of the Sum of the Reciprocals of Squares*, by Kenneth M Monks (infinite series)
- *Fourier's Proof of the Irrationality of e*, by Kenneth M Monks (infinite series)

- *Jakob Bernoulli Finds Exact Sums of Infinite Series (Calculus Version)*,* by Daniel E. Otero and James A. Sellars
- *Bhāskara's Approximation to and Mādhava's Series for Sine*, by Kenneth M Monks (approximation, power series)
- *Braess' Paradox in City Planning: An Application of Multivariable Optimization*, Kenneth M Monks
- *Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green's Theorem*,* by Abe Edwards
- *The Fermat-Torricelli Point and Cauchy's Method of Gradient Descent*,* by Kenneth M Monks (partial derivatives, multivariable optimization, gradients of surfaces)
- *The Radius of Curvature According to Christiaan Huygens*,* by Jerry Lodder

Instructors who teach precalculus may also be interested in the following projects based on excerpts from Book 1 of Agnesi's *Instituzioni Analitche ad Uso della Gioventù Italiana*. The first project listed is a 3-day project that subsumes the content of the three shorter projects that follow in the list. Classroom-ready versions of these projects can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_precalc`.

- *Three Hundred Years of Helping Others: Maria Gaetana Agnesi on Precalculus*,* by Kenneth M Monks
- *Three Hundred Years of Helping Others: Maria Gaetana Agnesi on Exponential Notation*,* by Kenneth M Monks
- *Three Hundred Years of Helping Others: Maria Gaetana Agnesi on the Rational Root Theorem*,* by Kenneth M Monks
- *Three Hundred Years of Helping Others: Maria Gaetana Agnesi on Simplifying Radicals*,* by Kenneth M Monks

Recommendations for Further Reading

For students and instructors interested in learning more about Maria Agnesi's life as well as the context in which she did her work, there is the very thorough Mazzotti [2007]. If one desires a more concise description of these things, as well as her mathematics, there is the much shorter article Mazzotti [2001].

Acknowledgments

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For more information about the NSF-funded project TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS), visit

http://blogs.ursinus.edu/triumphs/