The proposed task is to read and interpret the following text – Proposition I.25 – of Piero della Francesca. Following, we will give some suggestions to help in this interpretation.

Proposition I.25

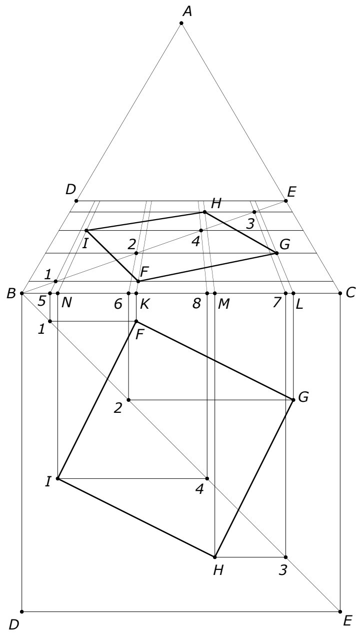
Beginning of text 1

See Bibliographic References

The square BCED (below the line BC) represents the horizontal plane. On this plane is given a square, FGHI. The aim of this proposition is to instruct how to draw its perspective image on the trapezium BCED (above the line BC), that is the image of the square BCED on the painter's canvas .

On the perspective plane, to draw in its place the image of a given square area

Let BCED be the perspective plane and A the observer's eye; let FGHI be the given square in its proper shape and BCED the plane where the square FGHI is given, as it was [said] in the proof; this done, I will draw parallels to BC: first, I will draw a parallel to BC passing through F, that will intersect the diagonal BE at point 1; then, I will draw a parallel to BC passing through G, that will intersect the diagonal BE at point 2; and I will draw a parallel to BC passing through H, that will intersect the



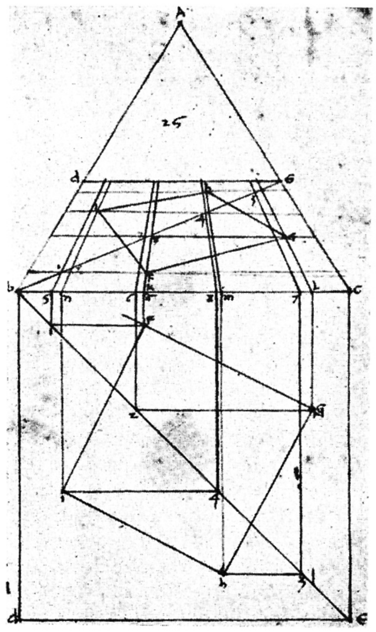


fig. 1

diagonal BE at point 3; and I will draw a parallel to BC passing through I, that will intersect the diagonal BE at point 4; after I will draw a parallel to BD passing through 1, that will intersect BC at point 5; after I will draw a parallel to BD passing through 2, that will intersect BC at point 6; then I will draw a parallel to BD passing through 3, that will intersect BC at point 7; then I will draw a parallel to BD passing through F, that will intersect BC at point 8: then I will draw a parallel to BD passing through G, that will intersect BC at point L, then I will draw a parallel to BD passing through H, that will intersect BC at point M; after that, I will draw a parallel to BD passing through I, that will intersect BC at point N, these points will be used to draw lines on the perspective plane.

First, I will draw the diagonal BE, after I will draw a line from 5 to A, that will intersect BE at point 1; and I will draw a line from 6 to A, that will intersect BE at point 2, I will draw a line from 7 to A, that will intersect BE at point 3, I will draw a line from 8 to A, that will intersect BE at point 4; after I will draw lines through 1, 2, 3 and 4, all parallel to BC and DE; after I will draw a line from K to A, that will intersect the line through 1 at point F; after I will draw a line from L to A, that will intersect the line through 2 at point G; after I will draw a line from M to A, that will intersect the line through 3 at point H; after I will draw a line from N to A, that will intersect the line through 4 at point I; after I will draw the lines FG, GH, HI and IF and the quadrilateral given will be completed. .

End of text 1

Hints for the interpretation

1. In the proposition I.25, Piero give the instructions for the perspective construction of a square *FHGI* given on the horizontal plane α. The plane α is represented by the square *BCED* and the figures in perspective, that is all the lines above the line *BC* of fig. 1, are drawn on the vertical plane π, the painter's canvas. Anyway, the instructions of Piero are always designed as a plane figure.

In the following notes we propose our interpretation of the situation, through drawings in cavalier perspective and some comments. Follow and discuss this interpretation.

See page 2 of Sketchpad document **Piero_eng.gsp**

As usual in Piero della Francesca, different but related points are designated by the same label (for instance the points D, E, F on the final figure of Piero). We will follow here the same convention. The point A on the plane π is the orthogonal projection of the observer's eye (point A space). The plane π is the painter's canvas. Through a central projection from α to π with center A (space), the horizontal square BCED is transformed onto the trapezium BCED on the plane π. (note: the figures 2a and 2b are not included in Piero's book)

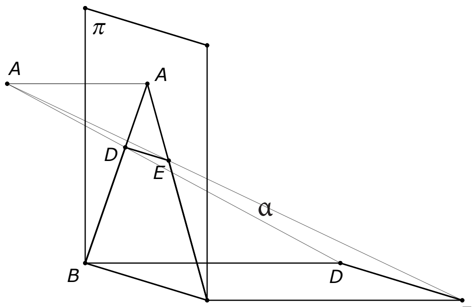


fig. 2a

Through a rabatment, or if you prefer a 90° rotation with axis BC, we are able to make the planes α and π coincident, and in this way to have in the same plane the given figures and their images in perspective. Please note that this procedure:

- *to define a mapping from the square BCED (plane α) onto the trapezium BCED (plane π) ; and*
- *to superimpose the two planes, defining in this way, a bijection between two sets in the same plane;*

was not used in the XVth century.

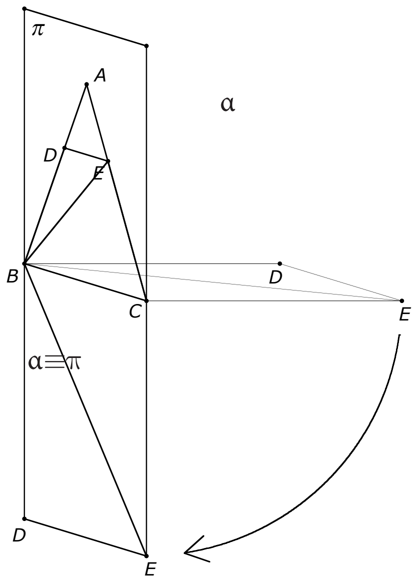


fig. 2b

See page 3 of Sketchpad document **Piero_eng.gsp**

Using this method, we will obtain the plane figure 2c that will be the basis for Piero's construction. The square BCED (below line BC) will represent the plane where the square FHGI is drawn. If the artist is painting the interior of a room, this square could be a figure on the pavement. The aim of Piero is to give clear instructions on how to draw, on the painter's canvas, the perspective image of this square.

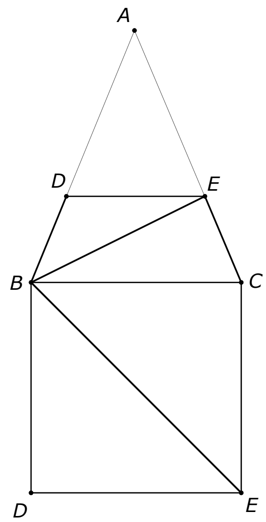


fig. 2c

2. Returning to Pieros’text, we see that after placing the square *FGHI* on the plane α, Piero give instructions to construct the points that, on the perspective plane π, are the images of points *F, G, H* and *I*. Piero repeat for each vertex the construction indicated in fig. 3 (*P* --->*P’*).

For each point *P* in the interior (or on the border) of the square *BCED* we find one point *P’* in the interior (or on the border) of it’s image on the perspective plane. Other propositions deal with other polygons (triangle, octogon, etc.).

The labels of segments *a, b, a’, b’* (that are not included in the Piero’s illustrations) suggest that point *P*, defined by co-ordinates (*a*,*b*) is sent to *P’*, defined by co-ordinates (a’,b’).

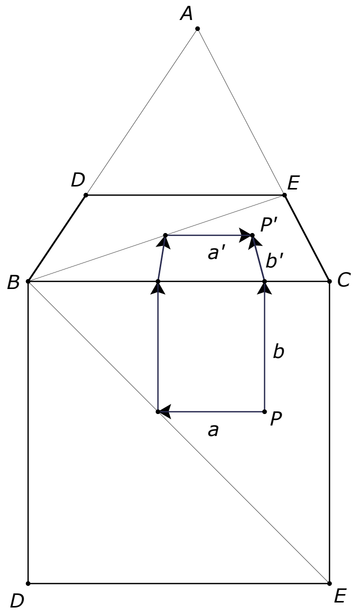


fig. 3

We will try now to find what could have been the perceptions that have lead Piero to this discovery. In the following point, we will look at a previous proposition (I.15).

See page 4 of Sketchpad document **Piero_eng.gsp**

3. 3. Proposition I.15

Given on the horizontal plane α a square decomposed in several small and equal parts, construct the corresponding parts in the square image on the perspective plane.

We will draw Piero’s illustration (fig. 4a) in two steps (fig. 4b and 4c).

In the text of this proposition, Piero takes the figure 2c as a starting point. He divides the square *BCED* in several “equal parts” and show how can be constructed the corresponding dissection of the perspective square:

(i) construction of the lines *FA*, *GA*, *HA*,and *IA*

(ii) constructions of parallel lines to *BC* through the intersections obtained in (i) with the diagonal of the perspective square.

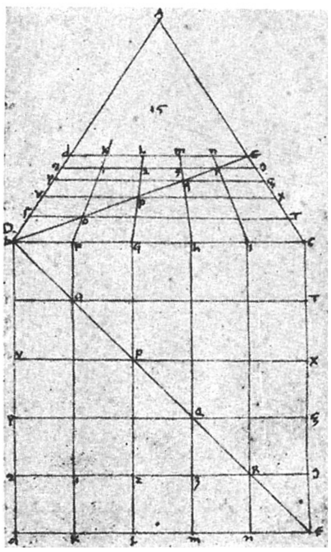


fig. 4a.
The illustration of Piero

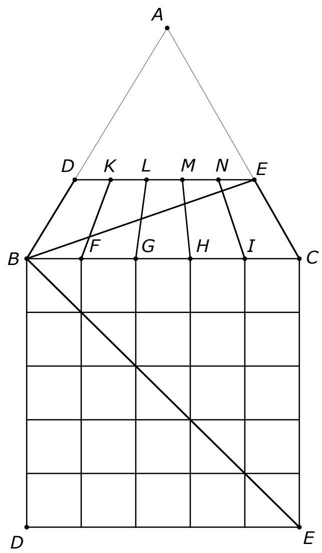


fig. 4b

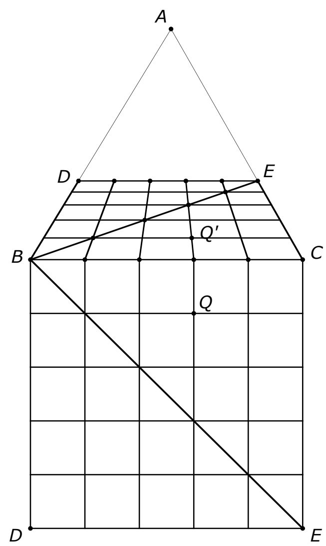


fig. 4c

In this way, the nodes of the two reticulates are corresponding points (as *Q* and *Q’* in the figure 4c). From here to the method indicated in figure 3 it was a small step to Piero, indeed.

Hints for the work with *Sketchpad*

1. We suggest that you try to extend this transformation to the whole plane α, using *The Geometer's Sketchpad*.

To start, please bear in mind that Piero only applies his method to points in the interior (or on the border) of the square *BCDE* that represents the plane α. It seems that the same construction will be valid for every point *P* of α if the supporting lines are substituted for line *BC* and the two diagonals, in order to assure that the intersections needed for the constructions exist for every point *P*. As you will see in your exploration of *Sketchpad*, this is not true.

2. Instructions:

If you have no experience with Sketchpad, you may follow pages 5 et 6 of document **Piero_eng.gsp**

a) Open the page 7 of file Piero_eng.gsp. This is a blank page where you may try some constructions and wher you may use the tools ***P* ---> *P'***, ***P'* ---> *P*** and ***VL1***.

b) Construct an horizontal line *BC* (this is the line that is substituted for the line segment *BC*)

c) Construct two lines *t* and *t'* with a common point on the line *BC* (these are the lines that are substituted for the diagonals)

d) Construct point *A* (it will be the orthogonal projection of the observer's eye on the plane α≡π)

e) Construct point *P* and follow for this point, on this new situation, the instructions given by Piero to construct the image *P'* (use the same labels as in the figure 5)

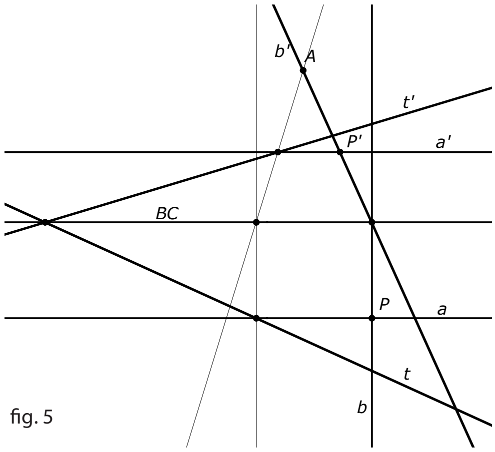


fig. 5

f) Your sketch will be similar to figure 5.

The tool ***P* ---> *P'*** *gives the point P' for a given point P. You may use this tool to verify your construction.*

3. The point *P'* is obtained as the intersection of lines *a'* and *b'*. If you drag point *P*, the position and direction of the lines that are used to construct the point *P'* change, and we could not be sure, without further investigation, that their intersections will always exist...

We proceed with new constructions

a) Select all auxiliary lines for the construction of *P'* (and also the intersections of these lines with *BC*, *t* and *t'*) and use the command "hide" (***Display:hide objects***). (Point *A* must be close to line *t'*, as in fig. 6)

To use tools, press the buton in the toolbox (left side of the screen)

b) Use tool ***VL1*** to show the red line *VL1*.

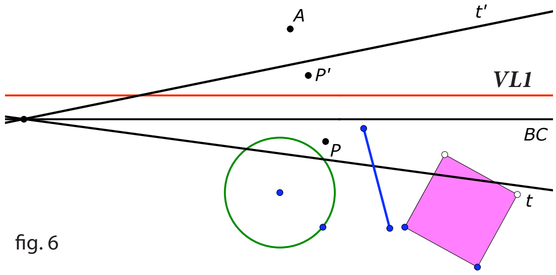


fig. 6

c) Construct some figures: segment, square, circle – below line *BC*.

d) Your sketch must be similar to figure 6.

e) With the procedure ***merge-locus-split*** (*edit:merge*, *construct:locus*, *edit:split)* you will obtain the image of the segment under the transformation *P--->P'*.

f) Drag the line segment until it intersects the line *VL1*.

g) In the same way, construct the images of the square and of the circle under the transformation *P--->P'*. Drag the square and the circle in such a way that they will cross the line *VL1*.

For the meaning of VL1 and VL2, please see page 6 of **Piero_eng.gsp.**

h) Any conjecture on the meaning of line *VL1*?

4. With his method for construction perspective images, Piero della Francesca (1416-1492) defined a projective transformation, more than three centuries before Poncelet (1788-1867).

The proposed task is to read and interpret the following texts, written by Dürer and Dandelin, with the help of the original figures, the side notes and the Sketchpad document **Durer_Dandelin_eng.gsp**.

If you have some practice with this (or another) dynamic geometry program and if you have access to a computer, you may try to follow and model Dürer and Dandelin ideas.

The subtitles have been added.

[Introduction]

Beginning of text 1

See **Bibliographic References**

Greek geometers have shown that we can make three different conic sections that will not produce a curve identical to the one of the base of the cone. In this case, we will cut again the cone in order to have again the shape of a cone. We will not deak here with this case. But the other sections will produce each one a particular curve. These lines, I will teach you how to draw them. Learned Scholars call the first section *elipsis*. This section cuts the cone obliquely and does not amputate its base. This oblique section may begin higher in one side and lower in the other side, so one side of the curve is closer to the base than the other. The second section gives in the drwaing a parallel to the side *AB* of the cone. Learned call this section "parabola". The third section, when we draw it, represents one vertical line, parallel to the line that joins the center of the cone to the vertex *A*. This curve it's called "hyperbola". I don't know how to label these sections with german names, but we would like to give them names allowing us to identify them. The *elipsis*, we will call it "egg line" [*Eierlinie*] because it looks more or less like an egg. The "parabola" will be called "burning line" [*Brennlinie*] because with this curve it's possible to make a mirror that can light up a fire. The "hyperbola", I will call it "fork line" [*Gabellinie*].

See in page 2, Dürer's illustration and a reproduction more readable.

[The *elipsis*]

Now, if I want to draw the egg line–ellipse, first I must draw the cone, after the section and then the plan of the section.

[Cone and section in elevation]

Page 1 of
Dürer_dandelin_eng.gsp

I will do the following: we will call *A* the vertex of the cone and *BCDE* the base. I will draw a vertical line through *A*. The higher point of the section will be *F* and the lower point *C*. This section, I will cut it in twelve parts by eleven points, with numbers starting from *F*.

[Cone in plan view]

Page 2 de
Dürer_dandelin_eng.gsp

Below, I will draw the cone in plan view. So, *A* becomes the centre and *BCDE* a circle. I will draw vertical lines through every point and every number between *F* and *G* that will cut the base and the circle, and I will label the intersection points by the same letters and numbers.

Instead of using a construction by points (11 points in the drawing of Dürer) we will use one variable horizontal plane, a dynamic construction often used with dynamic geometry software. But the idea is the same.

[Section in plan view]

The italic part of Dürer's text underlines his method of construction of the arcs of circle that define the conic section points. As you may see in page 2 of **Dürer_dandelin_eng.gsp** *we replaced the 11 arcs by a green dynamic arc.*

After that, I take a dividers and I place an end on the vertical line through A, at the same height as the point 1 on the line FG. The other end I place it on the line AD. I keep the dividers opening and transfer it to the plan drawn below. I place one end on the center A, the other on the vertical line through 1 and I will draw an arc of circle all around on the side of D to line 1. Again I place the dividers with an end on the cone's vertical line through *A* at the height of point 2 of *FG*, and the other end on the line *AD*. I will transfer the same opening to the plan: I place one end on the center *A*, the other on line 2 and again I will draw an arc of circle in the side of *D* and to line 2. I will repeat this operation for points 3 and 4. After, to deal with point 5, I will use the dividers with one end at line *AB*. I will transport the same opening to the horizontal projection and I will draw an arc of circle with center *A* and from line from vertical line 5 in the side of *D* to line 5. I will do the same for every number and I will transport every element from the cone to the horizontal plane.

[Drawing the section in true size]

Page 3 de **Dürer_dandelin_eng.gsp**

To draw the line *elipsis* in true size, I will do the following: I will draw the length of *GF* in a vertical position, and I will divide it in twelve equal parts by eleven points and I will draw eleven parallel and horizontal lines through all these points. Afterwards I take, in the horizontal plane, the distance between the two intersection points in line 1 and I will place this distance on line 1 on both sides of line *FG*. I will do the same for every number. When all points are determined, I will trace the egg line or *elipsis* from point to point as I have drawn it below.

End of text 1

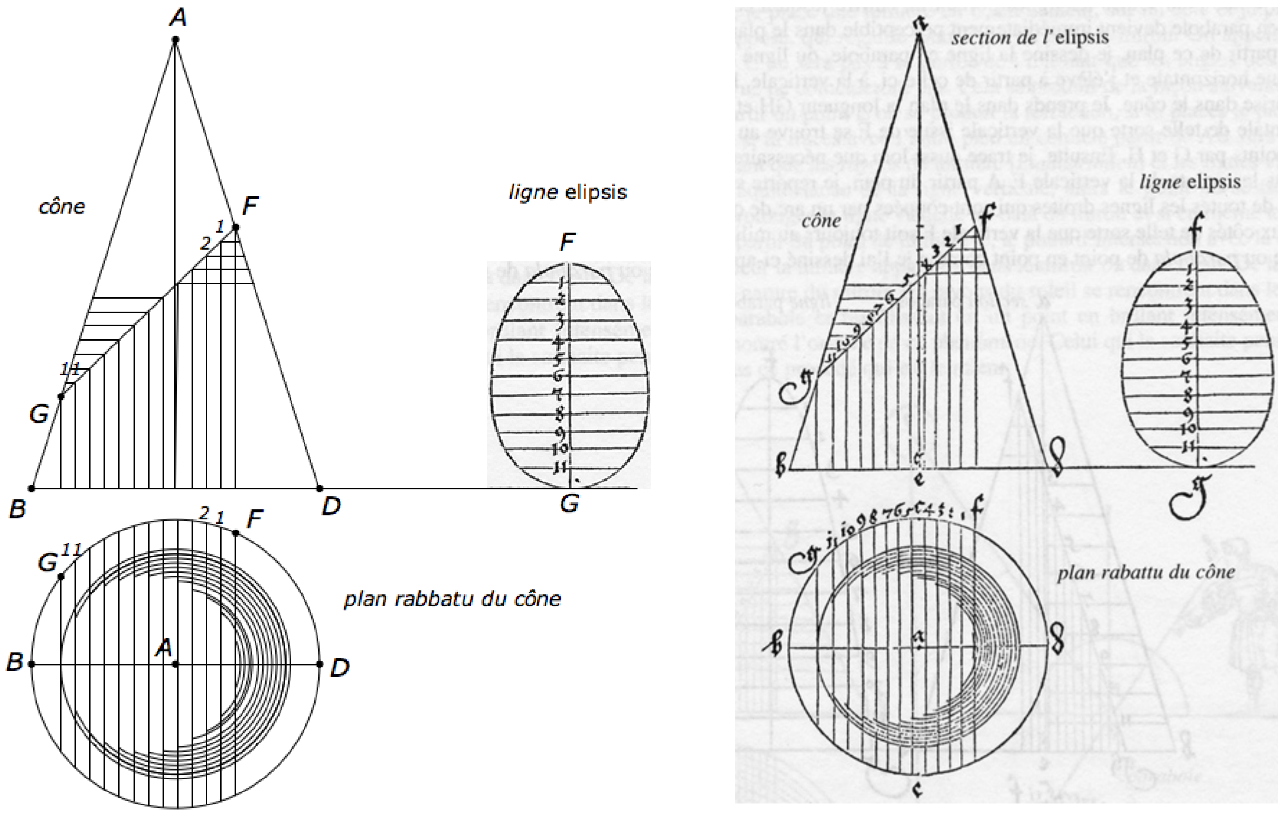


fig. 1

Dürer's instructions to draw an ellipsis are followed by similar texts for the other conic sections. We propose you now to read and interpret one text of Dandelin that will allow us to find the foci of the ellipse constructed by Dürer's method.

[A theorem of Dandelin]

Beginning of text 2
See **Bibliographic References**

Given a right cone and one plane, they will cut each other somewhere in the space, no matter their relative positions; and we can find always two spheres that touch the cone (from the inside) and touch also the secant plane. Then the two contact points of the plane and the spheres are the foci of the conic section.

Draw a plane through the cone axis and orthogonal to the secant plane. Its intersection with the cone, the spheres and the secant plane will be respectively the two lines *AS* and *BS*, the two circles *C* and *c* touching the two lines, and the line *Ff* tangent to the circles *C* and *c* at *F* and *f*, contact points of the spheres and the secant plane.

See fig. 2.

The two spheres will touch the cone in two parallel circles, orthogonal to the plane *ASB*, with traces *aA* and *bB*.

Now, draw through *S* a straight line on the surface of the cone; this line will touch the two spheres at *t* and *T* on the circumferences of the circles *ATB* and *atb*, and the length *Tt* will be equal to the length *Aa*. Also, this line will cut the secant plane at some point *m*, and the projection of *m* on the greater axis of the conic section will be *m'*.

If we draw the lines *mf* and *mT*, they will touch respectively spheres *C* and *c*; but *mt* is also tangent to the sphere *c*, so *mt* and *mf* are equal, and similarly *mT* = *mF*. Thus *mT* + *mt* (or *tT* or *Aa*) is equal to the sum of line segments *mF* and *mf* from *F* and *f* to a pont *m* on the curve; but the point *m* is arbitrary and *Aa* is constant, so therefore this property is true for every point of the section; so the curve is an ellipsis with foci *F* and *f*.

The proof will be analogous for the parabola and the hyperbola, so our general theorem is proved.

End of text 2

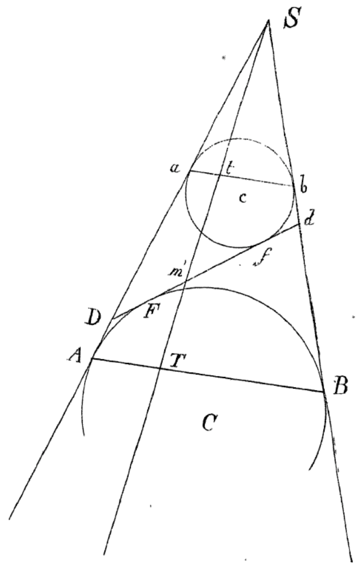


fig. 2

Page 4 of
Dürer_dandelin_eng.gsp

This result of Dandelin will be used to find the foci of the ellipse constructed by Dürer. You may follow the foci construction in the same Sketchpad document.

For the other conic sections, Dürer give similar instructions and similar illustrations (fig. 3).

Page 5 of
Dürer_dandelin_eng.gsp

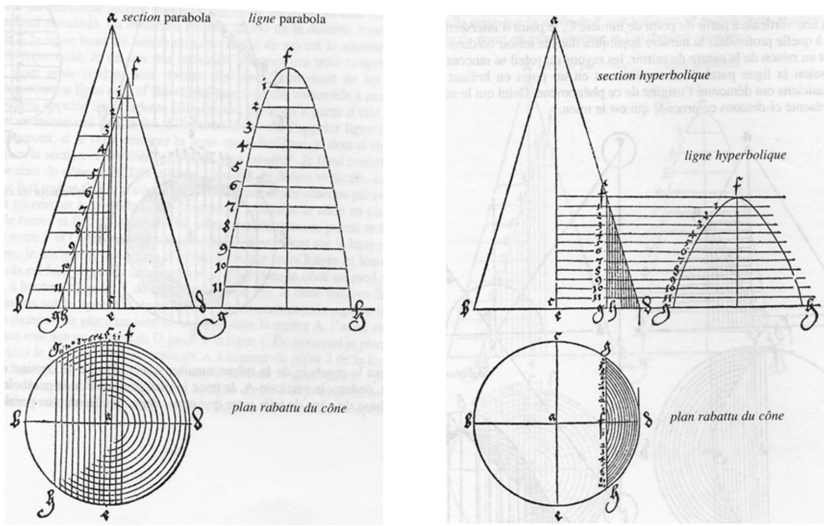


fig. 3

The proposed task is to read and interpret the following texts, written by Roberval and Descartes, with the help of the original figures, the side notes and the Sketchpad document roberv_descartes_eng.gsp.

If you have some practice with this (or another) dynamic geometry program and if you have access to a computer, you may try to follow and model Roberval and Descartes ideas.

[Roberval: principle of invention][1]

Beginning of text 1
See Bibliographic References

Problem I

Fifth proposition

See the French version to read a note written in the beginning of the French original of this text.

To find the tangent to a curve by knowing the motions of the point [which describe it].

We suppose that we are given enough specific properties of the curve to know those motions.

Axiom, or principle of invention

The direction of the motion of a point which describes a curve is the tangent to the curve at each position of the said point.

This principle is easy to understand if you give some attention to it.

General method

By the specific properties of the curve (which you are given), examine the different motions of the point which describe it in the place where you wish to draw the tangent; find the single motion of which these motions are the composition and you will have the tangent to the curve.

The proof is literally in our principle. And as it is very general, and applicable to all the examples we will be dealing with, it is not necessary to repeat it here.

Roberval presents several examples showing the application of his method to the Pascal snail, the conic sections and so on.

[„,]

[Roberval: cycloid construction][2]

Eleventh example, the Roulette or Trochoide of Roberval

In this section, Roberval describes the two uniform motions of point B and give a method to draw the Roulette. Please see fig. 1 (on page 2).

In **Page 2 of Rob_Desc_eng.gsp** *you will find the construction of the same curve in dynamic geometry. In this page we will deal only with the case where the circumference of the circle is equal to the length of line segment* ad.

A circle is given with centre *a* and radius *aB*, and the tangent at the point *B* is extended through point *C*; imagine that circle *aB* rolls along the line *BC*, and the line segment *BC* may be equal, greater or smaller than the circumference of the circle (this is indifferent, and the proof is easy); the point *B* of the circle is driven by two motions, one straight from *B* to *C*, the other circular due to the rolling circle; I say that this point describes the Roulette or the Trochoide; or, if you draw through centre *a* the line segment *ad*, equal and parallel to *BC* and to the same side, we imagine the circle rolling from *B* to *C* without rotating about its axis, and so the centre *a* describes the line *ad* in an uniform motion, and at the same time, the point *B* describes the circumference of its circle, going from *B* to π *Q G B* in an uniform motion, and when the centre reaches *d*, this point reaches *C*, where the line touches the circle, and the two motions, one circular moving the point *B* once on the circumference of its circle, the other straight, dragging the same point towards C, composed as we have said, and being both uniform, drive the Roulette to point *C*.

The two motions being uniform, the point *B* will describe three kinds of Roulettes, as long as the circumference of its circle is equal. greater or smaller than the length of line segment *ad*.

[...]

To draw in a easy way this curve, you will extend line *BC* through *E*; through point *E*, you will draw line segment *EF*, equal and parallel to *aB*; and with centre *F* you

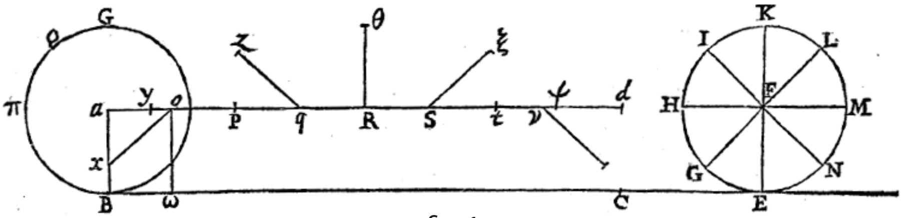


fig. 1

will describe the circle *EGHIKLMN,* equal to the first circle, and you will divide the circumference in any number of equal parts by the points *EGHIKLMN,* and through these points you will draw the lines GF, HF, and so on. After, you will divide the line *ad* in the same number of equal parts, by the points *oPqRStu,* and you will draw *ox* equal and parallel to *FG*, *Py* equal and parallel to *FH*, *qz* equal and parallel to *FI*, and so on; you will obtain points *BxyzθξψC,* and with these points you will describe the Roulette.

The proof of this statement is obvious [...].

[Tangents: method of Roberval]

This knowledge of the Roulette is enough to find the tangents to this curve; it you take a point on the curve, and if you draw two directions, one for the straight motion and one for the circular motion; and if you trace in these directions two line segments whose ratio is the same as the ratio of line segment *BC* to the circumference of the circle of the Roulette, the direction of the composed motion will be the direction of the tangent at that point.

[Tangents: method of Roberval, proof]

In

Page 3 of Rob_Desc_eng.gsp

you will find a dynamic geometry construction of the tangent, by Roberval method.

ABC is the Roulette with base *ADC* and axis *BD*; we want to find the tangent at point *E*. You will describe the circle *BFD* of the Roulette, about axis *BD* or any other diameter perpendicular to the base *ADC*; you will draw through point *E* a line parallel to *AC*, that will cut the circumference of the semicircle (closer to point *E*) at point *F*. Draw the tangent to the semicircle through point *F*, and take a point *H* on the tangent so that the ratio of *AC* to the circumference of the circle is equal to the ratio of *EF* to *FH*; through point *H* draw the line *HE*, this will be the tangent to the Roulette at point *E*.

End of text 1

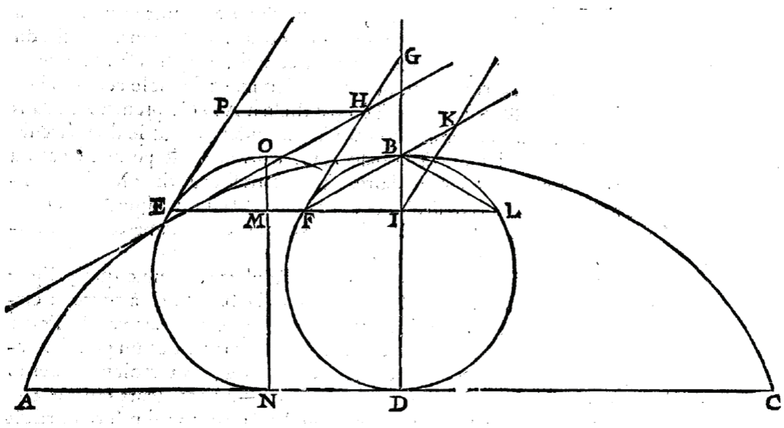


fig. 2

[Tangents: method of Descartes][5]

Beginning of text 2

See **Bibliographic References**

In

Page 4 de Rob_Desc_fr.gsp

you will find the construction of the tangent, by the method of Descartes, in dynamic geometry.

The first of these questions is to find the tangents to curves described by the movement of a roulette. To which I reply that the straight line which passes through the point of the curve where one wishes to find the tangent, and the point of the base which the roulette touches when it describes that point, always cuts that tangent at right angles. It follows that if, for example, one wishes to find the straight line at *B* which touches the curve *ABC*, described on the base *AD* by one of the points of the circumference of the roulette *DNC*, one must draw through this point *B*, the

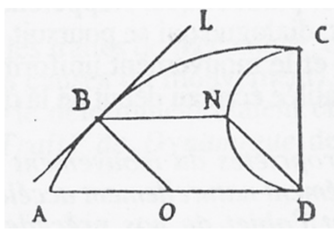


fig. 3

line *BN* parallel to the base *AD*, then draw another line from the point *N*, where this parallel meets the circle, this line being towards the point *D* where the roulette touches the base, and afterwards draw *BO* parallel to *ND*, and finally draw *BL* which is at right angles to *ND*; now this line is the required tangent.

[Tangents: method of Descartes, proof]

For this I shall only offer a proof which is extremely short and extremely simple.
If a rectilinear polygon, of any sort, is made to roll along a straight line, the curve described by one of its points, whatever it might be, will be composed of many portions of circles, and the tangents at all the points of each of these portions of circles, will cut at right angles the lines drawn from these points towards the point at which the polygon touches the base when it describes that portion of a circle. Following from which, taking the circular roulette to be a polygon with an infinite number of sides, it can be clearly seen that it must have this same property, that is that the tangents at each of the points which are on the curve which it describes, must cut at right angles the lines drawn from those points towards the points of the base which they meet [sont touches par elles}, at the same moment when it describes them.

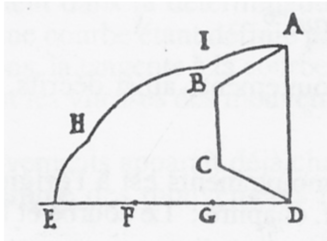


fig. 4

Thus, when the regular hexagon *ABED* is made to roll along the straight line *EFGD*, its point *A* will describe the curve *EHIA*, composed of the arc *EH*, which is described while this hexagon touches the base at the point *F* which is the centre of that arc, the arc *HI* whose centre is *G*, the arc *IA* whose centre is *D* etc., through which centres pass all the lines which meet the tangents at right angles. Now the same will happen with a polygon composed of an hundred thousand million sides, and consequently also for a circle. I could demonstrate this tangent in another way, which is more beautiful in my view and more geometric, but I omit it in order to avoid me the trouble of writing it out because it would be a little longer.

In

Page 5 of Rob_Desc_eng.gsp

you will find a dynamic version of fig. 4 of Descartes.

End of text 2

[Supplement: different kinds of Roulettes]

See

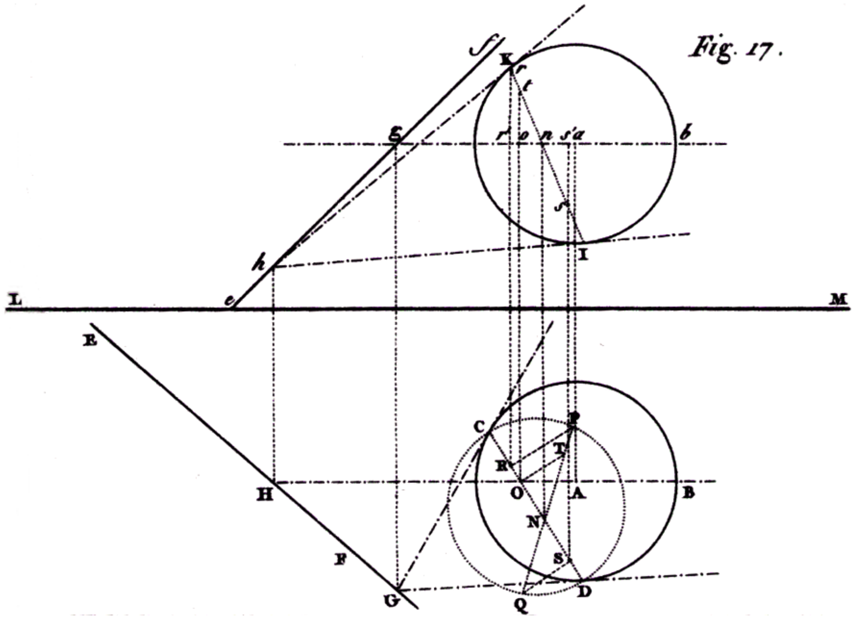
Pages 6 and 7 of Rob_Desc_eng.gsp

The method of Roberval is good for every kind of Roulette (with and without gliding). We have included two pages in the Sketchpad document to illustrate this.

The proposed task is to read the second solution of Gaspard Monge for the question above, and to interpret it with the original figure (Fig. 17), the side notes in this document and the *Sketchpad* file Monge_eng.gsp.

If you have some practice with this (or another) dynamic geometry program and if you have access to a computer, you may try to model Monge's construction. In alternative you can follow a step by step interpretation in the *Sketchpad* file.

See the Fig. 17, from Monge, and a more readable reproduction below.



Open the file ***Monge_eng.gsp****. For a better understanding of the construction you may observe, at any time, a representation in cavalier perspective not only of the objects drawn by Monge, but also others that he did not draw but imagined to conceive the method to construct the solution. For that you must press the buttons in the sketch.*

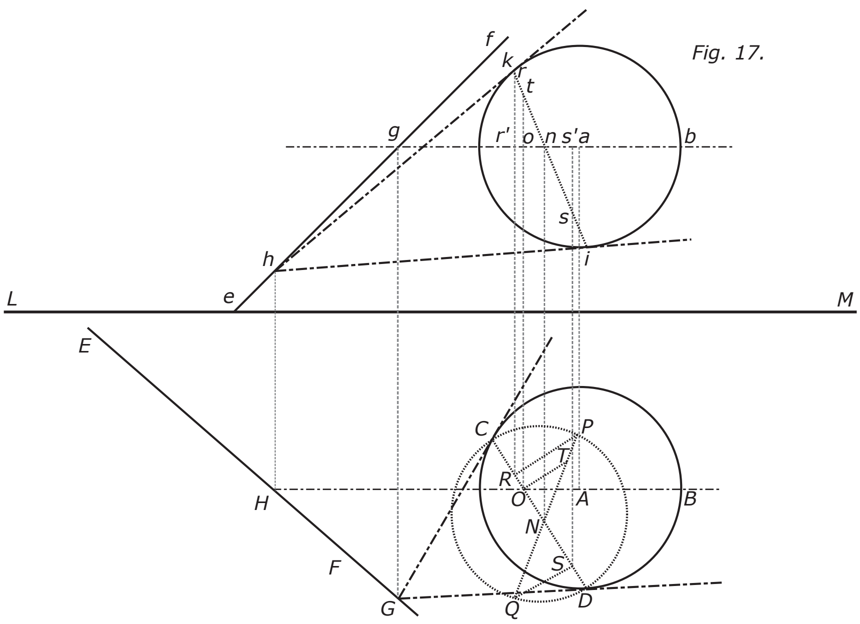
The subtitles have been added.

[Representation of given objects]

Beginning of text 1
See **Bibliographical References**

Let ***A*** and ***a*** (fig. 17) be the two projections of the centre of the sphere, ***AB*** or ***ab*** its radius, ***BCD*** the projection of its horizontal great circle and ***EF***,***ef*** the projections of the given line.

Page 1 of Monge_eng.gsp



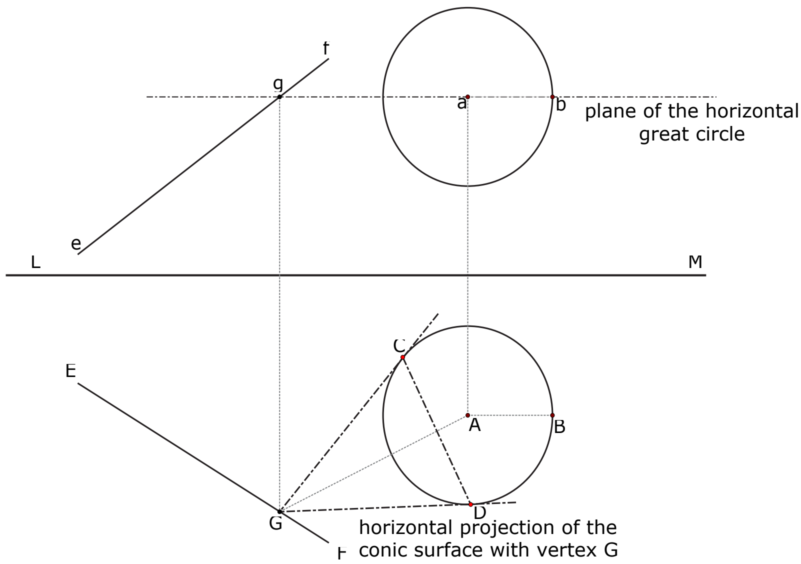
Page 2 of Monge_eng.gsp

Observe the construction of point G and of the horizontal projection of the conic surface that is tangent to the sphere and has vertex G. Observe, also, the perspective and the vertical projection that Monge did not represent because it is not needed to the construction.

[First conic surface]

If we imagine the plane containing the horizontal great circle of the sphere extended until it cuts the given line at a certain point, we will get the vertical projection of this plane if we draw through point ***a*** the indefinite horizontal line ***bag***; the point ***g***, where this horizontal line cuts ***ef***, will be the vertical projection of the intersection of the plane with the given line, and we will have its horizontal projection ***G*** if we project ***g*** onto ***EF***.

If we take this point to be the vertex of a conic surface that envelopes the sphere, so that each generatrix touches the sphere in one point, we will have the projections of the two horizontal generatrices of this conic surface if we draw through point ***G*** two



lines ***GC*** and ***GD***, tangents to the circle ***BCD*** and touching it in two points ***C***, ***D***, that will be easy to find. That conic surface will touch the sphere surface at the circumference of a circle, with ***CD*** as a diameter, whose plane will be orthogonal to the cone axis, and so vertical, and its horizontal projection will be the line ***CD***.

[Explanation]

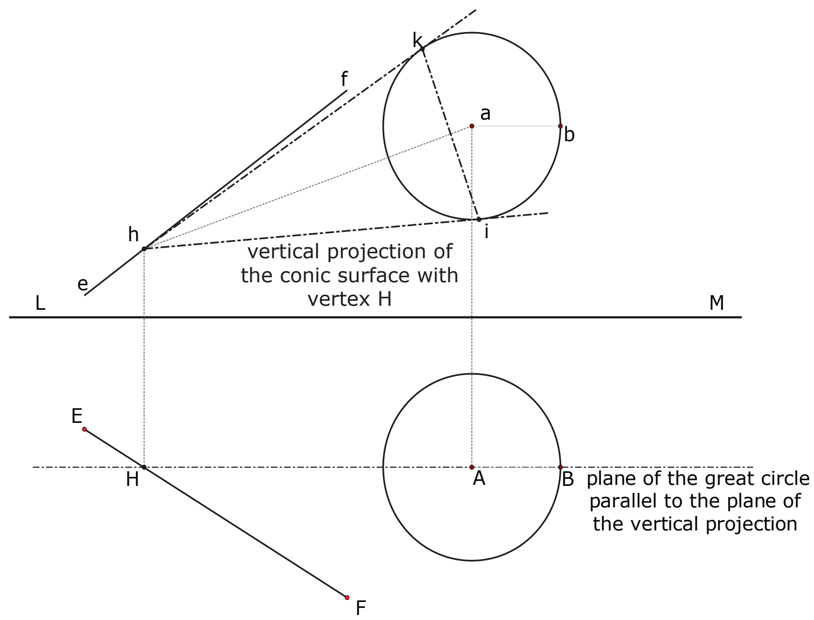
If we imagine two planes containing the given line and tangents to the conic surface, each one will touch it along one of these generatrices, that will be at the same time on the conic surface and on the plane; and because this generatrix touch also the sphere surface in one point of the circumference of the circle projected on ***CD***, it follows that this point is simultaneously on the conic surface, on the tangent plane, on the sphere surface and on the circumference of the circle projected on ***CD***, so it is a common contact point of all these objects. So, (i) the two tangent planes to the conic surface are also tangent planes to the sphere surface, and are the ones whose position we must find, (ii) their contact points with the sphere, being on the circumference of the circle projected on ***CD***, will be projected themselves somewhere on this line, (iii) the line defined by these two contact points, belonging to the plane of the same circle, will be also projected on the indefinite line ***CD***.

Page 3 of Monge_eng.gsp

Observe the construction of point H and the vertical projection of the conic surface with vertex H and tangent to the sphere. Observe, also, the perspective and the vertical projection that Monge did not represent because it is not needed to the construction.

[Second conic surface]

Now we will proceed by doing, in relation to the plane of the great circle parallel to the one of the vertical projection, the same operation we just made with the plane of the horizontal great circle. The horizontal projection of this plane will be the line ***BAH***, indefinitely parallel to ***LM***; the point where this plane meets the given line will be projected horizontally on the intersection ***H*** of lines ***EF***, ***BAH***; and we will get its vertical projection ***h*** by projecting point ***H*** on ***ef***. If we imagine a new conic surface with vertex in this intersection point and enveloping the sphere as the first one, we will have the vertical projections of the two extreme generatrices of this surface if we draw, through the point ***h***, the tangents ***hK***, ***hI*** to the circle ***AKI***, that will touch it in two points ***K***,***I*** that we will find easily. That second conic surface will touch the sur-



face of the sphere in the circumference of a new circle that will have ***KI*** as diameter and which plane, orthogonal to the plane of the vertical projection, will be projected indefinitely on the line ***KI***. The circumference of this circle will contain again the two contact points of the sphere with the two requested planes; so, the vertical projections of these two points will be somewhere on the line ***KI***; so the line defined by these two points will be projected also on the same line ***KI***.

[Finding the two contact points]

Page 4 of Monge_eng.gsp

Press the buttons successively to show the objects referred by Monge.

*The dynamic geometry software enables us to construct and see also the vertical projection of the vertical great circle (**button 4**).*

So, the line defined by the two contact points will have its horizontal projection on ***CD*** and its vertical projection on ***KI***, and will pierce the plane of the horizontal great circle in one point, which the vertical projection is the intersection ***n*** of ***KI*** and ***bag***, and we will find its horizontal projection by projecting ***n*** onto ***CD***.

After this, we may imagine that the plane of the vertical circle turns about its horizontal diameter, to become itself horizontal, dragging by its movement the two contact points of its circumference, and the line defined by them. We will draw the circle in its new position, by describing the circle ***CPDQ*** with diameter ***CD***; and if we draw the new position of the line joining the two contact points, the line will cut the circumference ***CPDQ*** in two points of its horizontal position.

But the position of the point N of the line defined by the two contact points, being in the axis of rotation ***CD***, will not change. So, this line, in its horizontal position, will continue to pass through that point. Also, the horizontal projection of the point where this line pierces the plane of the great circle parallel to the vertical projection is the intersection ***O*** of the lines ***CD***, ***BAH***, and its vertical projection is the projection ***t*** of ***O*** onto ***KI***; I say that this point, in its rotation about the axis ***CD***, will describe a vertical quarter of circle perpendicular to ***CD***, and the radius of this circle is the vertical ***to***; so, if we draw through point ***O*** a line perpendicular to ***CD***, and if on this line we mark ***to*** from ***O*** to ***T***, the point ***T*** will be one point of the two contact points line, in its horizontal position. So, if we draw a line through the two points ***N*** and ***T***, the two intersection points ***P*** and ***Q*** with the circumference ***CPDQ*** will be the two contact points on the vertical plane after rotation.

Button 5

To find the horizontal projections of the same points in their natural position, we must imagine that the circle ***CPDQ*** returns to its original position, turning about the same axis ***CD***. Through this movement, the two points ***P*** and ***Q*** will describe quarters of circle in vertical planes, orthogonal to ***CD***, and their horizontal projections

Button 6

Button 7

will be the perpendicular lines ***PR*** and ***QS***. So, the horizontal projections of the two contact points will be on the lines ***PR*** and ***QS***, and as we have seen that they must be also on the line ***CD***, they are the intersection points ***R*** and ***S***.

For the vertical projections ***r*** and ***s*** of these same points, we will project ***R*** and ***S*** onto ***KI***; or, what is the same, we will mark on the vertical lines ***Rr*** and ***Ss***, from the horizontal line bag, ***r'r*** equal to ***PR*** and ***s's*** equal to ***QS***.

Button 8

As we have now the horizontal and vertical projections of the two contact points, we will find the traces of the two tangent planes as in the first solution.

End of text 1

[Construction of the traces of the two tangent planes]

Page 5 of Monge_eng.gsp

One page was added in the file *Sketchpad*, to finish the resolution of the problem.

I. Piero della Francesca
On the perspective plane to construct [the image] of a given square

Piero della Francesca (c. 1413-1492)

Text 1/French

Extracted from the book

Piero della Francesca. *De la Perspective en Peinture*. Translated from the latin text by Jean-Pierre Neraudau. Paris: In Media Res, 1998.

Pages 78-80.

Text 1/English

Free translation of text1/French.

II. Albrech Dürer and Germinal P. Dandelin
Conic sections by double projection and Dandelin spheres

Albrecht Dürer (1471-1528)

Text 1/French

Extracted from the book

Albrecht Dürer. *Instruction sur la Manière de Mesurer*. Translated from the German by Jeannine Bardy and Michael Van Peene. Paris: Flammarion, 1995.

Pages 54-55.

Text 1/English

Free translation of Text 1/French.

Germinal Pierre Dandelin (1794-1847)

Text 2/French

Extracted from

Dandelin, G. Mémoire sur l'Hyperboloïde de Révolution et sur les Hexagones de Pascal et de M. Brianchon. In *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles*, vol. 3, 1826.

Göttinger Digitalisierungs-Zentrum

Pages 3-4.

Text 2/English

Free translation of text 2/French.

III. Gilles P. de Roberval and René Descartes
The tangent to the cycloid

Gilles Personne de Roberval (1602-1675)

Text 1/French

Extracted from

Roberval, Gilles Personne de. *Divers ouvrages de M. de Roberval*. 1693. In Gallica, site of the Bibliothèque Nationale de France, address: http://gallica.bnf.fr/ark:/12148/bpt6k862896

Pages 80 and 105-107.

Text 1/English

Free translation of text 1/French.

René Descartes (1596-1650)

Text 2/French

Extract from a letter to Père Mersenne, cited in (page 130)

Commission Inter-IREM, *Histoires de Problèmes, Histoire des Mathématiques*. IREM de Lyon, 1992.

Text 2/English

Extract from a letter to Père Mersenne, cited in (page 144)

Commission Inter-IREM. *History of Mathematics, History of Problems*. Paris: Éditions Ellipses, 1997.

IV. Gaspard Monge

To find a plane tangent to the surface of a given sphere and containing a given line

Gaspard Monge (1746-1818)

Text 1/French

Extracted from

Monge, Gaspard. *Géométrie Descriptive*. Paris: Éditions Jacques Gabay, 1989

Pages 47-49.

Text 1/English

Free translation of text 1/French.