

ON THE CONTROVERSY  
BETWEEN MESSRS. LEIBNIZ AND BERNOULLI  
CONCERNING LOGARITHMS  
OF NEGATIVE AND IMAGINARY NUMBERS

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However firmly established the theory of logarithms may be, in that its results appear to be proved as rigorously as those of Geometry, nevertheless Mathematicians are still strongly divided concerning the nature of the logarithms of negative and imaginary numbers; and while this controversy does not appear to be widely discussed, the reason apparently is that no one wishes to cast doubt on the certainty of all that has been achieved in pure Mathematics, by bringing before the eyes of the whole world the difficulties and even the contradictions to which the opinions of Mathematicians concerning the logarithms of negative and imaginary numbers are subject. For, even though their views may be very different on issues concerning applied Mathematics, in which the different ways of conceiving objects and of referring them to precise ideas can give rise to real controversies, it has always been claimed that the pure parts of Mathematics are completely free from any item of dispute, and that one will find there nothing of which we are not in a position to demonstrate either the truth or the falsity.

Since the theory of logarithms belongs unquestionably to pure Mathematics, one will be surprised to hear that up to now it has been subject to controversies so perplexing that, whichever side one takes, one tumbles into contradictions which seem impossible to remove. However, if truth must always prevail, there is no doubt that all the contradictions, no matter how obvious they seem to be, can be only apparent, and that the means cannot be lacking for establishing the truth, even though we may not know where those means are to be found.

This controversy concerning the logarithms of negative and imaginary numbers has been discussed with a great deal of vigor in the literary exchange between M. LEIBNIZ and M. JEAN BERNOULLI. These two great Mathematicians, to whom we are largely indebted for the Analysis of the infinite, were so divided concerning this matter that they were unable to come to an agreement about it, even though both had no object but the truth, and neither would have maintained his views obstinately. But each found in the opinions of the other such contradictions that it would have been an undue accommodation if one had changed his view in the other's favor. For it must be remarked that the contradictions with which these two Illustrious men reproached one another were quite real, and not in the least of the type which only appear contradictory to the opposite party, infatuated with his own opinion.

In order thus to put this remarkable controversy in its proper context, I will here set out separately the positions of M. BERNOULLI and of M. LEIBNIZ; I will then adjoin all the arguments which each makes use of to support his position, and finally I will detail the objections which can be made, whether against the corresponding arguments or against the opinions themselves, and I will make clear in all their force, all the contradictions to which both the one and the other of these views are subject, so that we will be in a better position to judge how difficult it may be to discover the truth and to secure it against all these objections, for which these two illustrious gentlemen have toiled in vain.

POSITION OF M. BERNOULLI

*M. BERNOULLI holds that the logarithms of negative numbers are the same as those of positive numbers, in other words that the logarithm of the negative number  $-a$  is equal to the logarithm of the positive number  $+a$ . Thus, the position of M. BERNOULLI implies that  $l-a = l+a$ .*

M. LEIBNIZ gave occasion to this declaration of M. BERNOULLI when he argued, in Letter CXC of the *Exchange*, that the ratio of  $+1$  to  $-1$ , or of  $-1$  to  $+1$ , was imaginary, because the logarithm of the measure

of that ratio, that is to say the logarithm of  $-1$ , which is the exponent of that ratio, was imaginary. To this, M. BERNOULLI declared, in Letter CXCI, that he was not of this opinion, and that he believed in fact that the logarithms of negative numbers were not only real, but also equal to the logarithms of the same numbers, taken positively. M. BERNOULLI supported his position with the following proofs.

## PROOF 1.

In order to prove that  $l-x = l+x$ , whatever number we denote by  $x$ , he appeals to differentials; and because the differential of  $l-x$  is  $\frac{-dx}{-x}$  or  $\frac{dx}{x}$ , which is the same as that of  $l+x$ , he concludes that the quantities themselves,  $l-x$  and  $l+x$ , whose differentials are equal, must themselves be equal, and hence  $l-x = l+x$ .

## PROOF 2.

This proof is derived from the nature of the logarithmic curve. To make this easier to grasp, let  $VBM$  (Fig. 1) be a Logarithmic described on the axis  $OAP$ , which is at the same time its asymptote. We know

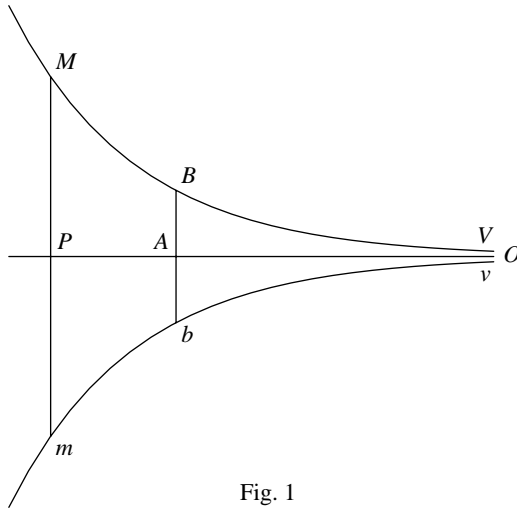


Fig. 1

that the subtangent of that Logarithmic is constant; let it be  $= 1$ ; and let the fixed ordinate  $AB$  also be  $= 1$ . This being so, if we call the arbitrary abscissa  $AP = x$ , taken from the fixed point  $A$ , and the corresponding ordinate  $PM = y$ , then we know that  $x$  represents the logarithm of  $y$ , that is  $x = ly$ . Thus, taking differentials, we will obtain for that logarithmic curve the differential equation  $dx = \frac{dy}{y}$ , or  $y dx = dy$ . As that equation remains the same when we substitute  $-y$  in place of  $y$ , M. BERNOULLI concludes that the curve  $VBM$  is accompanied, in virtue of the law of continuity, by the branch  $vbm$ , which is equal and similar to it, but lying on the other side of the axis  $OP$ , so that that axis is also a diameter of the complete curve. And consequently, since the same abscissa  $AP$  corresponds equally to the two ordinates  $PM$  and  $Pm$ , of which the one is the negative of the other, so that, since  $PM = y$ , we have  $Pm = -y$ , it follows that  $x$  is as much the logarithm of  $-y$  as of  $+y$ , and consequently  $l-y = l+y$ .

## PROOF 3.

Since it all amounts to proving that the Logarithmic is composed of two equal branches, lying on either side of the asymptote  $OP$ , M. BERNOULLI adduces also another reason, namely, that if we consider the curves given by the more general equation  $dx = \frac{dy}{y^n}$ , it is agreed that all these curves, when the exponent  $n$  is an odd number, have two branches such that the axis on which the abscissas  $x$  are taken is a diameter. Consequently that property holds also if  $n = 1$ ; but in that case we will have the Logarithmic of the preceding article; whence it follows that the logarithm of  $PM = +y$  and the logarithm of  $Pm = -y$  are both the same  $= AP = x$ .

## PROOF 4.

Since it is certain from the nature of logarithms that the logarithm of an arbitrary power  $p^n$  is equal to the logarithm of the root  $p$  multiplied by the exponent  $n$ , or that  $lp^n = nlp$ , it follows that if we take for  $p$  a negative number  $-a$ , we will have  $l(-a)^n = n l(-a)$ . Let  $n = 2$ , and this will become  $l(-a)^2 = 2l(-a)$ . But since  $(-a)^2 = a^2$ , we will have  $l(-a)^2 = la^2 = 2la$ ; whence it follows that  $2l(-a) = 2la$ , and consequently  $l-a = l+a$ . This can be seen more briefly in the following way: Because  $(-a)^2 = (+a)^2$ , it follows that  $l(-a)^2 = l(+a)^2$ , so  $2l-a = 2l+a$ , and consequently  $l-a = l+a$ .

All the other reasons which could be alleged to prove this view reduce easily to one of the four which I have just described. I now go on, therefore, to list the objections which can be made against this position and the reasons on which it is based.

## OBJECTION 1.

M. LEIBNIZ raised the objection to the first proof, that the rule for differentiating the logarithm of a variable quantity  $x$ , by dividing the differential of  $x$  by the same quantity  $x$ , does not hold unless  $x$  stands for

a positive quantity, so that one errs in setting the differential of  $l-x$  equal to  $\frac{-dx}{-x}$ , or to  $\frac{dx}{x}$ . Now it must be admitted that this objection is not only extremely feeble, not being supported by any valid reason, but that it would completely overturn the differential calculus of logarithms. For, inasmuch as that calculus deals with variable quantities, i.e., with quantities considered in general, if it were not true generally that  $d.lx = \frac{dx}{x}$ , whatever quantity one assigns to  $x$ , whether positive or negative, or even imaginary, then one would never be able to make use of that rule, the truth of the differential calculus being based on the generality of the rules which comprise it. Now M. LEIBNIZ had no need to make use of that objection in order to maintain his own position, since he could have attacked M. BERNOULLI's proof with a much stronger objection, namely the following.

#### OBJECTION 2.

As M. BERNOULLI wishes to prove by the equality of differentials that  $l-x = l+x$ , observe that we could prove by the same reasoning that  $l2x = lx$ ; for the differential of  $l2x$  is  $\frac{2dx}{2x} = \frac{dx}{x}$ , the same as that of  $lx$ . Thus, if the reasoning of M. BERNOULLI were correct, it would follow not only that  $l-x = l+x$ , but also that  $l2x = lx$ , and in general  $lnx = lx$ , whatever number is denoted by  $n$ ; a consequence which M. BERNOULLI himself would never accept. Now we know that when the differentials of two variable quantities are equal, it follows only that these variable quantities differ from one another by a constant quantity; and we cannot in general conclude that they must be equal. Thus, although the differential of  $x+a$  is  $=dx$ , the same as that of  $x$ , it would be a false conclusion if we were to deduce that  $x+a = x$ . For that reason, it is therefore clear that, since the differential of  $l-x$  and of  $l+x$  is the same  $\frac{dx}{x}$ , the quantities  $l-x$  and  $l+x$  differ from one another only by a constant quantity, which in any case is evident from the fact that  $l-x = l-1 + lx$ . And hence we understand easily that since  $lnx = lx + ln$ , the differential of  $lnx$  must be equal to the differential of  $lx$ . It is true that M. BERNOULLI supposes that  $l-1 = 0$ , just as  $l1 = 0$ , so that we would have  $l-x = lx + l-1 = lx$ . But since that is precisely what M. BERNOULLI hopes to prove by this argument, it is clear that that supposition cannot be admitted.

#### OBJECTION 3.

One can make the same objection to M. BERNOULLI's second argument, when he tries to prove by the differential equation of the Logarithmic  $y dx = dy$ , that that curve has two similar branches lying on either side of the axis. For, not only does that equation remain the same if one substitutes  $-y$  in place of  $y$ , but also if one puts  $2y$ , or in general  $ny$  for  $y$ ; whence it would follow that that curve had an infinity of branches, and that the abscissa  $x$  were the logarithm in common, not only of  $y$  and of  $-y$ , but also of  $2y$ , and in general of  $ny$ , for any number  $n$ . Thus, for the same reason which allows us to deny the infinity of branches of the Logarithmic, we will deny also the existence of the two branches which M. BERNOULLI wishes to establish.

#### OBJECTION 4.

This objection is also directed against the two branches of the logarithmic curve. For, even though one can surely deduce the existence of a diameter of a curve, when its equation in the coordinates  $x$  and  $y$  is such that it remains unaltered if one substitutes  $-y$  in place of  $y$ , nevertheless that criterion is not valid unless the equation for the curve is algebraic, or expressed in finite terms. For we know that a differential equation is much more general than the finite equation from which it has been derived, and that it includes an infinity of curves which are not contained in the finite equation. Thus the equation of the parabola  $yy = ax$  has the differential  $2y dy = a dx$ ; but that same differential equation belongs equally to the general equation  $yy = ax \pm ab$ , which includes at once an infinity of parabolas. It is the same for the differential equation of the Logarithmic  $y dx = dy$ , which belongs equally well to the finite equation  $x = lny$  as to  $x = ly$ , which however was the only one originally under discussion. Thus it follows that we cannot determine the form of a curve by considering only its differential equation.

#### OBJECTION 5.

This one deals with the third proof, which is undoubtedly much stronger. For, if all the curves comprised in that general equation  $dx = \frac{dy}{y^n}$ , where  $n$  denotes an odd number, are endowed with a diameter, the same property must hold if  $n = 1$ , which is the case for the Logarithmic. But, since that property is not evident, except by considering the integrals of the equation  $dx = \frac{dy}{y^n}$ , which one can always carry out algebraically, except for the case  $n = 1$ , in the same way that one makes an exception in that case, when the question concerns the integrability of the equation  $dx = \frac{dy}{y^n}$ , we will have also the right to make the same exception

when it has to do with the existence of a diameter. Thus, if one cannot prove by any other reason that the Logarithmic has a diameter, this argument derived from the general equation  $dx = \frac{dy}{y^n}$  will not be convincing. In order to show more clearly the insufficiency of the argument, I will give an example of a case, even among algebraic curves, where a general equation comprises curves each endowed with a diameter, and where nevertheless it is necessary to make an exception in a particular case.

Take the equation

$$y = \sqrt{ax} + \sqrt[4]{a^3(b+x)}.$$

We can certainly conclude that the curves expressed by this equation have a diameter, since when we rationalize the equation  $y = \sqrt{ax} + \sqrt[4]{a^3(b+x)}$ , we obtain an equation of the eighth degree in which all the exponents of  $y$  are even numbers. Nevertheless, however sure that conclusion may appear, it is necessary to make an exception in case  $b = 0$ ; for in that case, when we remove the radical signs, the equation  $y = \sqrt{ax} + \sqrt[4]{a^3x}$  rises only to the fourth degree, giving

$$y^4 - 2axy - 4aaxy + aaxx - a^3x = 0,$$

which, because of the term  $4aaxy$ , lacks a diameter. From all this, it follows that the argument of M. BERNOULLI is not rigorous enough to prove his position.

#### OBJECTION 6.

I pass to M. BERNOULLI's fourth proof, which is undoubtedly the strongest; for we cannot call into question any of the points on which it is based, without overturning the most well-established principles of Analysis and of the theory of logarithms. For we could not deny that  $(-a)^2 = (+a)^2$ , hence there is no doubt that their logarithms are equal, i.e.,  $l(-a)^2 = l(+a)^2$ . Furthermore, it is equally certain that we have in general  $lp^2 = 2lp$ , whence  $l(-a)^2 = 2l-a$  and  $l(+a)^2 = 2l+a$ ; and consequently, it will certainly follow that  $2l-a = 2l+a$ . The halves of these two quantities will therefore also incontestably be equal to one another and, consequently,  $l-a = l+a$ , just as M. BERNOULLI maintains.

But if this reasoning is correct, we can deduce from it other consequences which no one, least of all M. BERNOULLI, would accept; for we can prove in the same way that the logarithms of imaginary quantities would also be real, just like those of negative numbers. For, it is undoubtedly the case that  $(a\sqrt{-1})^4 = a^4$ , hence it will follow also that  $l(a\sqrt{-1})^4 = la^4$ , and further  $4l(a\sqrt{-1}) = 4la$ , so that  $l(a\sqrt{-1}) = la$ . Furthermore, since  $\left(\frac{-1+\sqrt{-3}}{2}a\right)^3 = a^3$ , it will follow that  $l\left(\frac{-1+\sqrt{-3}}{2}a\right)^3 = la^3$ , and hence  $3l\frac{-1+\sqrt{-3}}{2}a = 3la$ , whence  $l\frac{-1+\sqrt{-3}}{2}a = la$ , which can hardly be accepted without overturning the whole theory of logarithms.

We would therefore have, according to the system of M. BERNOULLI, not only  $l-1 = l1 = 0$ , but also  $l\sqrt{-1} = 0$ ,  $l-\sqrt{-1} = 0$ , and  $l\frac{-1+\sqrt{-3}}{2} = 0$ . Now, M. BERNOULLI having so brilliantly reduced the quadrature of the circle to logarithms of imaginary numbers, if the logarithm of  $\sqrt{-1}$  were  $= 0$ , his beautiful discovery that the radius is to the fourth part of the circumference as  $\sqrt{-1}$  is to  $l\sqrt{-1}$  would be completely false. Thus, letting the ratio of the diameter to the circumference  $= 1 : \pi$ , it will follow that  $\frac{1}{2}\pi = \frac{l\sqrt{-1}}{\sqrt{-1}}$ , and hence  $l\sqrt{-1} = \frac{1}{2}\pi\sqrt{-1}$ , which would be absurd if  $l\sqrt{-1} = 0$ . It is therefore not true that  $l\sqrt{-1} = 0$ , whence it must be concluded that, however sound the 4<sup>th</sup> proof appears to be, it must be treated with caution, since it would imply  $l\sqrt{-1} = 0$  as well as  $l-1 = 0$ . Consequently, we cannot admit that the position of M. BERNOULLI has been sufficiently proved.

It is indeed quite astonishing that, whether we adopt or reject M. BERNOULLI's position, we encounter either way insurmountable difficulties, and even contradictions. For, if we maintain that  $l-a = l+a$  or  $l-1 = l+1 = 0$ , we are obliged to admit also that  $l\sqrt{-1} = 0$ , since  $l\sqrt{-1} = \frac{1}{2}l-1$ . Now it would not only be absurd to claim that the logarithms of imaginary quantities were not imaginary, but it would also be wrong that  $l\sqrt{-1} = \frac{1}{2}\pi\sqrt{-1}$ , which nonetheless is rigorously proved. Thus, in adopting the position of M. BERNOULLI, we fall into contradiction with very firmly established truths.

Grant that M. BERNOULLI's position is false, and that it is not the case that  $l-1 = 0$ ; for that is what M. BERNOULLI's position reduces to; then we will be obliged to question some of the operations on which the reasoning of the 4<sup>th</sup> proof is based; this we will not be able to do without again falling into contradiction with other demonstrated truths. To make this clearer, let  $l-1 = \omega$ , and if it is not the case that  $\omega = 0$ , its double  $2\omega$  will also not  $= 0$ ; now  $2\omega$  is the logarithm of the square of  $-1$ ; since that is  $= +1$ , the logarithm

of  $+1$  would no longer  $= 0$ , which is a new contradiction. Further,  $-x$  is both  $= -1 \cdot x$  and  $= \frac{x}{-1}$ , whence  $l-x = lx + l-1 = lx - l-1$ ; from this it would follow that  $l-1 = -l-1$ , even though it was not the case that  $l-1 = 0$ ; but it is a contradiction to say that  $+a = -a$  but not  $a = 0$ .

So whether we say the one thing or the other, whether M. BERNOULLI's position is true or false, either way we are plunged into the greatest difficulties, having to deal with manifest contradictions. However, it is absolutely necessary that this position be either true or false, and there does not appear to be any other possibility. How then shall we get out of this mess and preserve the truth against these great contradictions? I pass to the examination of the position of M. LEIBNIZ.

#### POSITION OF M. LEIBNIZ

*M. LEIBNIZ holds that the logarithms of all negative numbers, and even more so those of imaginary numbers, are imaginary; thus, since  $l-a = la + l-1$ , he holds that  $l-1$  is an imaginary quantity.*

I have already remarked that M. LEIBNIZ held that the ratio of  $+1$  to  $-1$  or of  $-1$  to  $+1$  is imaginary, since the logarithm of that ratio or  $l-1$  is imaginary. We see, of course, that all the objections which were made against the system of M. BERNOULLI serve to strengthen this position, and that the reasons advanced to support M. BERNOULLI's position must be contrary to those of M. LEIBNIZ. Nevertheless, one can bring forward particular proofs to confirm M. LEIBNIZ's position, which will be the subject of my examination which follows.

#### PROOF 1.

Having observed that the logarithm of the number  $1+x$  is equal to the sum of this series

$$l(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \text{etc.},$$

from which we see to begin with that if  $x = 0$ , it follows that  $l1 = 0$ , now to obtain the logarithm of  $-1$  we must set  $x = -2$ , whence we obtain

$$l-1 = -2 - \frac{1}{2} \cdot 4 - \frac{1}{3} \cdot 8 - \frac{1}{4} \cdot 16 - \frac{1}{5} \cdot 32 - \frac{1}{6} \cdot 64 - \text{etc.}$$

Now, there is no doubt that the sum of this divergent series could not be  $= 0$ ; thus, it is certain that  $l-1$  is not  $= 0$ . The logarithm of  $-1$  will thus be imaginary, since it is also clear that it could not be real, i.e., positive or negative.

#### PROOF 2.

Let  $y = lx$ , and letting  $e$  be the number whose logarithm is  $= 1$ , the approximate value of which is, as we know,  $e = 2,718281828459$ , since we have  $yle = 1lx$ , we find that  $x = e^y$ . Thus the logarithm of the number  $x$ , being the exponent of a power of  $e$  which is equal to the number  $x$ , it is clear that no real exponent of a power of  $e$  could produce a negative number, and therefore, in order for  $e^y$  to become  $= -1$ , neither  $y = 0$  nor any other real number taken for  $y$  could satisfy that condition. And in general, taking for  $x$  a negative number  $-a$ , whose logarithm we suppose to be  $= y$ , the equation  $e^y = -a$  will always be impossible, so that the value of  $y$  will be imaginary.

#### PROOF 3.

Since in general the value of  $e^y$  is expressed by this infinite series

$$e^y = 1 + \frac{y}{1} + \frac{y^2}{1 \cdot 2} + \frac{y^3}{1 \cdot 2 \cdot 3} + \frac{y^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{y^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \text{etc.},$$

which is always convergent, however large the number which is substituted for  $y$ , so that the objections drawn from the nature of divergent sequences, as in the first proof, do not arise here, hence the logarithm of the number  $x$  being set  $= y$ , we will have

$$x = 1 + \frac{y}{1} + \frac{y^2}{1 \cdot 2} + \frac{y^3}{1 \cdot 2 \cdot 3} + \frac{y^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.},$$

and therefore, if  $y$  denotes the logarithm of  $-1$ , or if we let  $x = -1$ , we will have the equality

$$-1 = 1 + \frac{y}{1} + \frac{y^2}{1 \cdot 2} + \frac{y^3}{1 \cdot 2 \cdot 3} + \frac{y^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.},$$

which, as is immediately clear, can't be satisfied by the value  $y = 0$ , since it would follow that  $-1 = +1$ . Consequently, it is certain that the logarithm of  $-1$  is not  $= 0$ .

I will rest content with having presented these three proofs, because the other arguments which can be used to confirm M. LEIBNIZ's position are already contained in the objections which were made against the system of M. BERNOULLI. Nevertheless, the three arguments which I have expounded are subject to the following objections.

#### OBJECTION 1.

Contrary to the first proof, we will say to begin with that the continual increase of the terms, which are all negative, of the sequence

$$-2 - \frac{1}{2} \cdot 4 - \frac{1}{3} \cdot 8 - \frac{1}{4} \cdot 16 - \frac{1}{5} \cdot 32 - \frac{1}{6} \cdot 64 - \text{etc.}$$

is not a sure sign that the sum of that sequence could not be  $= 0$ . For if the geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - \text{etc.}$$

in the case  $x = -2$  gives

$$-1 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \text{etc.}$$

and in the case  $x = -3$  gives

$$-\frac{1}{2} = 1 + 3 + 9 + 27 + 81 + 243 + \text{etc.},$$

why then, we will say, could it not be possible that the sum of a series whose terms increase, always having the same sign, could be  $= 0$ ? To give an example, we have only to add to the last series term by term the series

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \text{etc.}$$

and we will indeed get

$$0 = 2 + 2 + 10 + 26 + 82 + 242 + 730 + \text{etc.}$$

Thus, if the sum of that series is  $= 0$ , what absurdity will there be in maintaining that it could also be the case that

$$0 = -2 - \frac{1}{2} \cdot 4 - \frac{1}{3} \cdot 8 - \frac{1}{4} \cdot 16 - \frac{1}{5} \cdot 32 - \frac{1}{6} \cdot 64 - \text{etc.},$$

and therefore the first argument is not convincing.

#### OBJECTION 2.

The second argument could also be used to prove the opposite view. For since we have  $x = e^y$ , where  $y$  is the logarithm of the number  $x$ , hence whenever  $y$  is a fraction having an even number for denominator, it must be admitted that in that case the value of  $e^y$ , and hence also of  $x$ , is negative as well as positive. Thus, if  $\frac{m}{2n}$  is a logarithm, the corresponding number  $x$  being  $e^{m:2n} = \sqrt{e^{m:n}}$  will be both positive and negative; so that in this case, both  $x$  and  $-x$  will have the same logarithm  $\frac{m}{2n}$ . Thus, since logarithms are not rational numbers, and consequently are equivalent to fractions whose numerators and denominators are infinitely large, we can always regard the denominators as even numbers; it follows that the same logarithm which belongs to the positive number  $+x$  will also belong to the negative number  $-x$ .

## OBJECTION 3.

The third proof is undoubtedly the strongest, since it seems to exclude absolutely the negative numbers from among those which correspond to real logarithms. For it is clear that, whatever real number we substitute for  $y$ , the value of the series

$$x = 1 + \frac{y}{1} + \frac{y^2}{1 \cdot 2} + \frac{y^3}{1 \cdot 2 \cdot 3} + \frac{y^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

could never become negative, so that no real logarithm could correspond to a negative number. However, as that series is only valid to the extent that it comes from the finite formula  $e^y$ , the preceding objections also apply here. For, if  $e^y$  could give a negative number, is it of great significance whether the series to which it is equal also gives one or not? In order to recognize this, we have only to consider a radical formula, such as  $\frac{1}{\sqrt{1-x}}$ , which is both  $\frac{+1}{\sqrt{1-x}}$  and  $\frac{-1}{\sqrt{1-x}}$ , whereas the series to which it is equal

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \text{etc.}$$

gives only the positive value, whatever value be substituted for  $x$ .

M. LEIBNIZ would not have failed to respond to these objections; and since the first does not prove the view contrary to his, but only renders the first proof doubtful, he would not have scrupled to abandon that first proof, and to base his stance principally on the others. For, basically, the second objection does not demolish his position, which reduces merely to show that  $l-1$  is not  $= 0$ ; but the second objection has no force against that, showing merely that if  $e^y$  has to be  $= -1$ , the exponent  $y$  could not be any fraction of the form  $\frac{m}{2n}$ , for which the radical sign could produce a negative value. For we check easily that, whether we substitute for  $y$  a positive number greater than 0, or an arbitrary negative number for  $y$ ,<sup>1)</sup> the value of the power  $e^y$  never becomes  $= -1$ . Thus, if  $y$  is not imaginary,  $e^y = -1$  would have to occur in the case  $y = 0$ . But in that case, all ambiguity about signs, which might have arisen because of the radical signs, vanishes, and it is indubitable that  $e^0 = +1$ . And if we wished to say that we could regard 0 as  $\frac{0}{2}$ , and  $e^0$  as  $\sqrt{e^0} = \sqrt{1}$ , whose value could also be  $= -1$ , that would be a very weak objection, since by the same argument we could prove that  $-a = +a$ ; for, setting  $a = a^{\frac{2}{2}} = \sqrt{a^2}$ , we could deduce equally well  $a = -a$  as  $a = +a$ . In order to forestall such false consequences, we need only remark that such an expression  $a^{\frac{m}{2n}}$  only has two values, one positive and the other negative, when the fraction  $\frac{m}{2n}$  is reduced to lowest terms and the denominator still remains an even number. Thus, since the value of the powers,  $a^1, a^2, a^3, a^4$ , etc. is not ambiguous, neither will  $a^0$  be ambiguous. It is thus always the case that  $a^0 = 1$ , which suffices to invalidate the second objection; and the third has no force unless the second holds.

It would thus appear that M. LEIBNIZ's view is better-founded, since it is not contrary to the discovery of M. BERNOULLI, that is

$$l\sqrt{-1} = \frac{1}{2}\pi\sqrt{-1},$$

since M. LEIBNIZ holds that the logarithm of  $-1$ , and even more so that of  $\sqrt{-1}$ , is imaginary. But, in adopting M. LEIBNIZ's position, we plunge into the aforementioned difficulties and contradictions. For, if  $l-1$  were imaginary, its double, i.e. the logarithm of  $(-1)^2 = +1$ , would also be imaginary, which doesn't accord with the first principle of the theory of logarithms, in virtue of which we suppose that  $l+1 = 0$ .

Thus, whichever way we turn, whether we embrace the view of M. BERNOULLI or that of M. LEIBNIZ, we encounter everywhere such great obstacles to maintaining our position that we are unable to defend it against contradictions. Nevertheless, it appears that if one of the two views is false, the other must necessarily be true, and that there is no third choice. This is therefore an extremely important problem, to establish the theory of logarithms in such a way that it is no longer subject to any contradiction.

But, having carefully weighed the contradictions which we find on either side, we will be led to believe that such a reconciliation is utterly impossible; and there will be no lack of enemies of Mathematics to draw very troublesome consequences concerning the certitude of that science. For, while the Pyrrhonists have attacked all the sciences, it is easy to see that none of the objections which they have brought against any

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<sup>1)</sup> EULER repeats 'pour  $y$ '. *Tr.*

science are even close to approaching, in solidity, the objections which I have been describing against the theory of logarithms.

However, I will show, so clearly that not the least doubt will remain, that that theory is firmly established, and that all the aforementioned difficulties derive their origin from a single mistaken idea; so that, having rectified that idea, all the difficulties and contradictions, however overwhelming they may have seemed, will immediately vanish; and then the whole theory of logarithms will stand so firmly that we will be in a position to resolve easily all the objections which have previously appeared insoluble. Without this development, which however has been unknown to Mathematicians up to now, I don't know from what point of view we could envisage the theory of logarithms: on the one hand, we would have to say that it is true and as solidly established as any other branch of Analysis; but on the other hand, we would be unable to deny that that same theory is subject to contradictions which it is impossible to remove. We would therefore be obliged to admit that Mathematics, and even Analysis, involves mysteries incomprehensible to our intellects. Hence, if these mysteries have been the result simply of one idea which was not completely correct, we can draw the very important consequence that it is extremely dangerous to make judgments about things of which we have only an imperfect idea: but it is wellnigh certain that, outside of Mathematics, the number of distinct and complete ideas is very small.

#### SOLUTION OF THE PRECEDING DIFFICULTIES

It must first be stated that if the idea which Messrs. LEIBNIZ and BERNOULLI have attached to the term 'logarithm', and which all Mathematicians have had up to now, were perfectly correct, it would be absolutely impossible to rescue the theory of logarithms from the contradictions which I have been propounding. Now, the idea of logarithms having been derived from an origin of which we have a perfect understanding, how is it possible that it could be defective? When we say that the logarithm of a given number is the exponent of the power of a certain number taken arbitrarily, which becomes equal to the given number, it appears that nothing is lacking to the correctness of that idea. And that is perfectly true; but we generally put that idea together with a condition which does not suit it at all: that is, we ordinarily suppose, almost without noticing it, that to each number there corresponds only one logarithm; now, with only a little consideration, we will find that all the difficulties and contradictions by which the theory of logarithms appears to be embarrassed, persist only to the extent that we suppose that to each number there corresponds only one logarithm. Thus, in order to remove all the difficulties and contradictions, I say that, just by virtue of the given definition, there corresponds to each number an infinity of logarithms; which I will prove in the following theorem.

#### THEOREM

*There is always an infinity of logarithms which belong equally to each given number; that is, if  $y$  denotes the logarithm of the number  $x$ , I say that  $y$  contains an infinity of different values.*

#### PROOF

I will restrict myself here to hyperbolic logarithms, since we know that the logarithms of all other types have a constant ratio to these; thus, when the hyperbolic logarithm of the number  $x$  is taken  $= y$ , the tabular logarithm of that same number will be  $= 0,4342944819 \cdots y$ .

Now, the foundation of hyperbolic logarithms is that, if  $\omega$  signifies an infinitely small number, the logarithm of the number  $1 + \omega$  will be  $= \omega$ , or that  $l(1 + \omega) = \omega$ . From this it follows that  $l(1 + \omega)^2 = 2\omega$ ,  $l(1 + \omega)^3 = 3\omega$ , and in general

$$l(1 + \omega)^n = n\omega.$$

But, since  $\omega$  is an infinitely small number, it is clear that the number  $(1 + \omega)^n$  could not become equal to any given number  $x$ , unless the exponent  $n$  is an infinite number. Therefore let  $n$  be an infinitely large number and set

$$x = (1 + \omega)^n,$$

and the logarithm of  $x$ , which has been taken  $= y$ , will be  $y = n\omega$ . Thus, to express  $y$  in terms of  $x$ , as the first formula gives  $1 + \omega = x^{\frac{1}{n}}$  and  $\omega = x^{\frac{1}{n}} - 1$ , that value substituted for  $\omega$  in the other formula will produce

$$y = nx^{\frac{1}{n}} - n = lx.$$



Whence it is clear that the value of the formula  $nx^{\frac{1}{n}} - n$  will more closely approach the logarithm of  $x$ , the larger the number  $n$  is taken, and that if we substitute for  $n$  an infinite number, this formula will give the true value of the logarithm of  $x$ . Now, since it is certain that  $x^{\frac{1}{2}}$  has two different values,  $x^{\frac{1}{3}}$  three,  $x^{\frac{1}{4}}$  four, and so forth, it will be equally certain that  $x^{\frac{1}{n}}$  must have an infinity of different values, since  $n$  is an infinite number. Consequently, that infinity of different values of  $x^{\frac{1}{n}}$  will also produce an infinity of different values for  $lx$ , so that the number  $x$  must have an infinity of logarithms. Q.E.D.

From this, it follows that the logarithm of  $+1$  is not only  $= 0$ , but that there is also an infinity of other quantities, each of which is equally the logarithm of  $+1$ . However, we see easily that all these other logarithms, apart from the first,  $0$ , will be imaginary quantities; so that in calculation we are entitled to regard only  $0$  as the logarithm of  $+1$ , just as when it is a question of the cube root of  $1$ , we only use  $1$ , even though the imaginary quantities  $\frac{-1+\sqrt{-3}}{2}$  and  $\frac{-1-\sqrt{-3}}{2}$  are equally cube roots of  $1$ . But when we wish to compare the logarithm of  $1$  with the logarithms of  $-1$ , or of  $\sqrt{-1}$ , which are all, as I will show in what follows, imaginary, it is necessary to consider the logarithm of  $1$  in its full extent; and then all the difficulties and contradictions recounted above will disappear of themselves. For, let  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ , etc. be the imaginary logarithms of unity, which correspond to it just as well as  $0$  does, and we will see easily that it can be the case that  $2l-1 = l+1$ , even though all the logarithms of  $-1$  are imaginary; for, in order to satisfy the equation  $2l-1 = l+1$ , it suffices that the double of each of the logarithms of  $-1$  occur among the imaginary logarithms of  $+1$ . Similarly, since  $4l\sqrt{-1} = l+1$ , each logarithm of  $\sqrt{-1}$  multiplied by  $4$  must be found in the series  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ , etc. Thus, the equalities  $2l-1 = l+1$  and  $4l\sqrt{-1} = l+1$  can be maintained, without our being obliged to hold that either  $l-1 = 0$  or  $l\sqrt{-1} = 0$ , as M. BERNOULLI has claimed. But this will all become completely clear when I actually determine all the logarithms of any given number, which will be the subject of the following problems.

### PROBLEM 1

*To determine all the logarithms which correspond to any given positive number  $+a$ .*

### SOLUTION

Since  $a$  is a positive number, it will certainly have a real logarithm, which can be found by rules which are well-known. Therefore let  $A$  be the real logarithm of the number  $a$ , and since  $a = 1 \cdot a$ , we will have  $la = l1 + A$ : thus, if each logarithm of unity is added to  $A$ , it will produce a logarithm of the given number  $a$ ; and in order to find all its logarithms, we need only look for all the logarithms of unity  $+1$ . Thus, letting  $y$  denote an arbitrary logarithm of  $+1$ , the values of  $y$  must be obtained from the equation derived in the theorem, taking  $x = 1$ , so that we will have the equation

$$y = n1^{\frac{1}{n}} - n,$$

which transforms into

$$1 + \frac{y}{n} = 1^{\frac{1}{n}},$$

and removing the fractional exponent we will have

$$\left(1 + \frac{y}{n}\right)^n = 1,$$

where  $n$  denotes an infinite number. This equation now being rational, so to speak, each of its roots will give a suitable value for  $y$ , that is to say a logarithm of  $+1$ . Now, to find all the roots of this equation, we know that they come from the factors of the formula  $(1 + \frac{y}{n})^n - 1$ , if we set each factor  $= 0$ . But, in general, it has been proved that a typical factor of such a formula  $p^n - q^n$  is

$$p^2 - 2pq \cos \frac{2\lambda\pi}{n} + q^2,$$

where  $\lambda$  denotes an arbitrary whole number and  $\pi$  the angle  $180^\circ$ , or half the circumference of a circle whose radius is  $= 1$ ; so that on giving to  $\lambda$  successively all the possible values  $0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ , etc., we

obtain finally all the factors of the formula  $p^n - q^n$ . And thus, all the roots of the equation  $p^n - q^n = 0$  will be comprised in this general expression

$$p = q \left( \cos \frac{2\lambda\pi}{n} \pm \sqrt{-1} \cdot \sin \frac{2\lambda\pi}{n} \right),$$

which comes from setting

$$p^2 - 2pq \cos \frac{2\lambda\pi}{n} + qq = 0.$$

Thus, all the roots of our derived equation  $(1 + \frac{y}{n})^n - 1 = 0$ , putting  $p = 1 + \frac{y}{n}$  and  $q = 1$ , will be contained in this general expression

$$1 + \frac{y}{n} = \cos \frac{2\lambda\pi}{n} \pm \sqrt{-1} \cdot \sin \frac{2\lambda\pi}{n}.$$

But, since  $n$  denotes an infinite number, the arc  $\frac{2\lambda\pi}{n}$  will be infinitely small; thus we will have

$$\cos \frac{2\lambda\pi}{n} = 1 \quad \text{and} \quad \sin \frac{2\lambda\pi}{n} = \frac{2\lambda\pi}{n},$$

whence it follows that

$$1 + \frac{y}{n} = 1 \pm \frac{2\lambda\pi}{n} \sqrt{-1},$$

and hence

$$y = \pm 2\lambda\pi \sqrt{-1}.$$

We have now only to substitute for  $\lambda$  successively all the definite values which it contains, that is 0, 1, 2, 3, 4, 5, 6, 7, etc. to infinity; and all the logarithms of unity, or all the possible values of  $l1$  will be

$$0, \pm 2\pi\sqrt{-1}, \pm 4\pi\sqrt{-1}, \pm 6\pi\sqrt{-1}, \pm 8\pi\sqrt{-1}, \text{ etc.}$$

Thus, all the logarithms of the given number  $a$ , of which we know already the real logarithm  $A$ , will be

$$A, A \pm 2\pi\sqrt{-1}, A \pm 4\pi\sqrt{-1}, A \pm 6\pi\sqrt{-1}, A \pm 8\pi\sqrt{-1}, \text{ etc.}$$

Q.E.I.

From this, it is clear that each positive number has only one real logarithm, and that all its infinitely many other logarithms are imaginary. However, all the imaginary logarithms enjoy the same property as the real, and we could make use of them equally well in calculations by following the same rules. For, let  $A, B, C, D$ , etc. be the real logarithms of the positive numbers  $a, b, c, d$ , etc., so that in general

$$la = A \pm 2\lambda\pi\sqrt{-1}, lb = B \pm 2\mu\pi\sqrt{-1}, lc = C \pm 2\nu\pi\sqrt{-1}, \text{ etc.}$$

Now let  $c = ab$ , and we know that we will have  $C = A + B$ ; but, taking the logarithms in general, we will see also that the sum of the logarithms of the factors  $a, b$  is always equal to the logarithm of the product  $ab = c$ . For we will have

$$la + lb = A + B \pm 2\zeta\pi\sqrt{-1}$$

taking for  $\zeta$  an arbitrary whole number which could result from adding the terms  $\pm 2\lambda\pi\sqrt{-1}$  and  $\pm 2\mu\pi\sqrt{-1}$ . Now it is clear that putting  $A + B = C$ , that expression for  $la + lb$  agrees perfectly with  $lc = C \pm 2\nu\pi\sqrt{-1}$ . The same agreement will be found also in division, raising to powers, and extraction of roots, in which we make use of logarithms. But in connection with the extraction of roots, as the number of roots is always equal to the exponent of the radical sign, this way of expressing logarithms generally has this advantage over the ordinary method, that it shows us all the roots; whereas by the ordinary method we find in each case only one root, that is, the real one, which furthermore is positive; all of which we will see more clearly, when I have determined all the logarithms of numbers which are negative or imaginary.

## PROBLEM 2

To determine all the logarithms which correspond to an arbitrary negative number  $-a$ .

## SOLUTION

Since  $-a = -1 \cdot a$ , we will have  $l-a = la + l-1$ , and, taking for  $la$  the real logarithm of  $a$ , we will have all the logarithms of the negative number  $-a$  if we get all the logarithms of  $-1$ . But we have seen that, putting  $y$  for the logarithm of the number  $x$  in general, we have  $y = nx^{\frac{1}{n}} - n$ , whence it follows that  $1 + \frac{y}{n} = x^{\frac{1}{n}}$  and hence  $(1 + \frac{y}{n})^n - x = 0$ . Thus,  $y$  will express all the logarithms of  $-1$  if we set  $x = -1$ , so that all the logarithms of  $-1$  will be the roots of the equation

$$\left(1 + \frac{y}{n}\right)^n + 1 = 0,$$

taking the number  $n$  to be infinitely large.

Now we know that all the roots of the general equation  $p^n + q^n = 0$  are found by solving the formula

$$p^2 - 2pq \cos \frac{(2\lambda - 1)\pi}{n} + q^2 = 0,$$

taking for  $\lambda$  successively all whole numbers, both positive and negative, and hence we will have

$$p = q \left( \cos \frac{(2\lambda - 1)\pi}{n} \pm \sqrt{-1} \cdot \sin \frac{(2\lambda - 1)\pi}{n} \right).$$

Thus, the roots of the given equation  $(1 + \frac{y}{n})^n + 1 = 0$  will all be comprised in the general formula

$$1 + \frac{y}{n} = \cos \frac{(2\lambda - 1)\pi}{n} \pm \sqrt{-1} \cdot \sin \frac{(2\lambda - 1)\pi}{n},$$

and since  $n = \infty$ , this becomes

$$y = \pm(2\lambda - 1)\pi\sqrt{-1}.$$

Consequently, taking for  $\lambda$  successively all the values which belong to it, all the logarithms of  $-1$  will be

$$\pm\pi\sqrt{-1}, \pm3\pi\sqrt{-1}, \pm5\pi\sqrt{-1}, \pm7\pi\sqrt{-1}, \pm9\pi\sqrt{-1}, \text{ etc.}$$

Thus, if we let  $l+a = A$ , that is,  $A$  denotes the real logarithm of the positive number  $+a$ , all the logarithms of the negative number  $-a$  will be:

$$A \pm \pi\sqrt{-1}, A \pm 3\pi\sqrt{-1}, A \pm 5\pi\sqrt{-1}, A \pm 7\pi\sqrt{-1}, \text{ etc.}$$

of which the number is infinite. Q.E.I.

From this, it is clear that all the logarithms of an arbitrary negative number are imaginary, and that there is no negative number having any of its logarithms real. M. LEIBNIZ was thus correct to maintain that the logarithms of negative numbers were imaginary. However, since the positive numbers also have an infinity of imaginary logarithms, all of M. BERNOULLI's objections against that position lose their force. For, although  $l-1 = \pm(2\lambda - 1)\pi\sqrt{-1}$ , the logarithm of its square will be  $l(-1)^2 = \pm 2(2\lambda - 1)\pi\sqrt{-1}$ , an expression which occurs among the logarithms of  $+1$ , so that it remains true that  $2l-1 = l+1$ , even though none of the logarithms of  $-1$  occur among the logarithms of  $+1$ . Let  $A$  be the real logarithm of the positive number  $+a$ , and let  $p$  represent in general all the even numbers and  $q$  all the odd numbers, and since in general

$$l+1 = \pm p\pi\sqrt{-1} \quad \text{and} \quad l-1 = \pm q\pi\sqrt{-1},$$

and

$$l+a = A \pm p\pi\sqrt{-1} \quad \text{and} \quad l-a = A \pm q\pi\sqrt{-1},$$

it follows that

$$l(-a)^2 = 2l-a = 2A \pm 2q\pi\sqrt{-1}.$$

Now,  $2q$  being  $= p$  and  $2A$  the real logarithm of  $a^2$ , we see that  $2A \pm p\pi\sqrt{-1}$  is the general formula for the logarithms of  $a^2$ ; thus  $l(-a)^2 = la^2$  or rather  $2l-a = 2l+a$ , even though it is not the case that  $l-a = l+a$ ; which would without doubt be contradictory, if the numbers  $+a$  and  $-a$  had only one logarithm; for in that case we would have to conclude that  $l-a = l+a$ , if we had  $2l-a = 2l+a$ . But, once we agree that both  $-a$  and  $+a$  have an infinity of logarithms, that consequence, however necessary it may formerly have been, is no longer valid, since in order that  $2l-a = 2l+a$ , it suffices that the doubles of all the logarithms of  $-a$  occur among the logarithms of  $+aa$ . That can well happen, as we see, without any of the logarithms of  $-a$  being equal to any of the logarithms of  $+a$ .

It must however be admitted that all the values of  $2l-a$  are different from the values of  $2l+a$ , inasmuch as

$$2l+a = 2A \pm 2p\pi\sqrt{-1} \quad \text{and} \quad 2l-a = 2A \pm 2q\pi\sqrt{-1},$$

where  $2p$  represents an arbitrary evenly even number, and  $2q$  an arbitrary oddly even number. But it must be remarked that the logarithms of  $+a^2$ , which are those of a positive number whose real logarithm is  $= 2A$ , are comprised in the general formula  $la^2 = 2A \pm p\pi\sqrt{-1}$ , where  $p$  represents an arbitrary even number including possibly zero. That being so, it is clear that all the values of  $2l-a$ , together with those of  $2l+a$ , are included among those of  $la^2$ . Thus, although we could say that  $2l-a = la^2$  and  $2l+a = la^2$ , taking the sign  $=$  to indicate that the values of  $2l-a$  or of  $2l+a$  occur among the values of  $la^2$ , we couldn't say, in truth, that  $2l-a = 2l+a$ . Nevertheless, since in the formulas  $l+a = A \pm p\pi\sqrt{-1}$  and  $l-a = A \pm q\pi\sqrt{-1}$  the numbers  $p$  and  $q$  are indeterminate, nothing obliges us, when we double the logarithms, to take for  $p$  and  $q$  the same numbers. Thus, in order to carry out the multiplications in their full extent, if  $p, p', p'', p'''$ , etc., stand for arbitrary even numbers, equal or unequal, and  $q, q', q'', q'''$ , etc. for odd numbers, equal or unequal to one another, these duplications will be done in the following way:

$$\begin{array}{r} l+a = A \pm p\pi\sqrt{-1} \\ l+a = A \pm p'\pi\sqrt{-1} \\ \hline 2l+a = 2A \pm (p+p')\pi\sqrt{-1}, \end{array} \quad \begin{array}{r} l-a = A \pm q\pi\sqrt{-1} \\ l-a = A \pm q'\pi\sqrt{-1} \\ \hline 2l-a = 2A \pm (q+q')\pi\sqrt{-1}. \end{array}$$

Now here, as  $p+p'$  represents the sum of two arbitrary even numbers and  $q+q'$  the sum of two arbitrary odd numbers, both  $p+p'$  and  $q+q'$  will represent an arbitrary even number; and thus, we will have  $p+p' = q+q'$ , so that  $2l-a = 2l+a$ . Consequently, in this sense, we will be able to say that  $2l-a = 2l+a$ , even though it is not the case that  $l-a = l+a$ . In the same way, we will have

$$\begin{aligned} 3l+a &= 3A \pm (p+p'+p'')\pi\sqrt{-1} = 3A \pm p\pi\sqrt{-1} = l+a^3, \\ 3l-a &= 3A \pm (q+q'+q'')\pi\sqrt{-1} = 3A \pm q\pi\sqrt{-1} = l-a^3, \end{aligned}$$

since  $p+p'+p''$  produces all the even numbers and consequently agrees with  $p$ ; in the same way,  $q+q'+q''$  produces all the odd numbers and agrees with  $q$ . Now, since  $q+q'+q''+q'''$  produces all even numbers, this expression will be equivalent to  $p$ ; hence the quadruples will be

$$\begin{aligned} 4l+a &= 4A \pm (p+p'+p''+p''')\pi\sqrt{-1} = 4A \pm p\pi\sqrt{-1} = l+a^4, \\ 4l-a &= 4A \pm (q+q'+q''+q''')\pi\sqrt{-1} = 4A \pm p\pi\sqrt{-1} = l-a^4. \end{aligned}$$

Thus, this way of finding the logarithms of the powers both of  $+a$  and of  $-a$  agrees perfectly well with the known principles of powers and of logarithms, and all the objections recounted above no longer have any grip on demonstrated truths. The same agreement will be seen also in the case of logarithms of imaginary quantities, which I go on to develop in the following problem.

### PROBLEM 3

*To find all the logarithms of an arbitrary imaginary quantity.*

## SOLUTION

It has been proved that every imaginary quantity, however complicated it may be, can always be reduced to the form  $a + b\sqrt{-1}$ , where  $a$  and  $b$  are real quantities. I now set

$$\sqrt{(aa + bb)} = c$$

and  $\frac{a}{\sqrt{(aa+bb)}}$  and  $\frac{b}{\sqrt{(aa+bb)}}$  will be the cosine and the sine of a certain angle which can easily be found from tables. Let this angle therefore =  $\varphi$ , which represents at the same time the quantity of the arc of the circle which is its measure, the total sine being taken = 1. We will thus have

$$a = c \cos \varphi \quad \text{and} \quad b = c \sin \varphi,$$

and the imaginary formula, of which we seek to find all the logarithms, will be

$$a + b\sqrt{-1} = c(\cos \varphi + \sqrt{-1} \cdot \sin \varphi)$$

or, since  $c$  is a positive number, let  $C$  be its real logarithm, and we will have

$$l(a + b\sqrt{-1}) = C + l(\cos \varphi + \sqrt{-1} \cdot \sin \varphi).$$

Thus the problem is to find all the logarithms of the imaginary quantity  $\cos \varphi + \sqrt{-1} \cdot \sin \varphi$ ; calling this  $x$ , its logarithms will be the values of  $y$  found from the equation

$$\left(1 + \frac{y}{n}\right)^n - x = 0,$$

where  $n$  represents an infinite number. But in order to be able to compare this equation with the general form  $p^n - q^n = 0$ , I remark that

$$x = \cos \varphi + \sqrt{-1} \cdot \sin \varphi = \left(1 + \frac{\varphi\sqrt{-1}}{n}\right)^n,$$

the truth of which has been sufficiently proved elsewhere. For we know that

$$\cos \varphi = 1 - \frac{\varphi^2}{1 \cdot 2} + \frac{\varphi^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{\varphi^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}$$

and

$$\sin \varphi = \varphi - \frac{\varphi^3}{1 \cdot 2 \cdot 3} + \frac{\varphi^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \text{etc.}$$

Now, since  $n$  is an infinite number, we will have

$$\left(1 + \frac{\varphi\sqrt{-1}}{n}\right)^n = 1 + \frac{\varphi\sqrt{-1}}{1} - \frac{\varphi^2}{1 \cdot 2} + \frac{\varphi^3\sqrt{-1}}{1 \cdot 2 \cdot 3} + \frac{\varphi^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{\varphi^5\sqrt{-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \text{etc.},$$

whence it is clear that

$$\left(1 + \frac{\varphi\sqrt{-1}}{n}\right)^n = \cos \varphi + \sqrt{-1} \cdot \sin \varphi.$$

We will thus have<sup>1)</sup>

$$p = 1 + \frac{y}{n} \quad \text{and} \quad q = \frac{1 + \varphi\sqrt{-1}}{n}$$

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<sup>1)</sup> The second equation should read  $q = 1 + \frac{\varphi\sqrt{-1}}{n}$ . *Tr.*

in the equation to be solved  $p^n - q^n = 0$ . But, as we have already seen that each of the roots of this equation is contained in the general formula

$$p = q \left( \cos \frac{2\lambda\pi}{n} \pm \sqrt{-1} \cdot \sin \frac{2\lambda\pi}{n} \right),$$

taking for  $\lambda$  all whole numbers, whether positive or negative, we will have in our case

$$1 + \frac{y}{n} = \left( 1 + \frac{\varphi\sqrt{-1}}{n} \right) \left( \cos \frac{2\lambda\pi}{n} \pm \sqrt{-1} \cdot \sin \frac{2\lambda\pi}{n} \right)$$

and since, the number  $n$  being infinite,

$$\cos \frac{2\lambda\pi}{n} = 1 \quad \text{and} \quad \sin \frac{2\lambda\pi}{n} = \frac{2\lambda\pi}{n},$$

we will have

$$1 + \frac{y}{n} = \left( 1 + \frac{\varphi\sqrt{-1}}{n} \right) \left( 1 \pm \frac{2\lambda\pi}{n} \sqrt{-1} \right),$$

which gives

$$y = \varphi\sqrt{-1} \pm 2\lambda\pi\sqrt{-1},$$

whence all the logarithms of the formula  $\cos \varphi + \sqrt{-1} \cdot \sin \varphi$  will be

$$\varphi\sqrt{-1}, (\varphi \pm 2\pi)\sqrt{-1}, (\varphi \pm 4\pi)\sqrt{-1}, (\varphi \pm 6\pi)\sqrt{-1}, \text{ etc.}$$

and the logarithms of the imaginary formula  $a + b\sqrt{-1}$ , if we set

$$c = \sqrt{(aa + bb)} \quad \text{and} \quad \text{tang } \varphi = \frac{b}{a}, \quad \text{or} \quad \cos \varphi = \frac{a}{c} \quad \text{and} \quad \sin \varphi = \frac{b}{c}$$

and in addition

$$lc = C,$$

will be

$$C + \varphi\sqrt{-1}, C + (\varphi \pm 2\pi)\sqrt{-1}, C + (\varphi \pm 4\pi)\sqrt{-1}, C + (\varphi \pm 6\pi)\sqrt{-1}, \text{ etc.}$$

Q.E.I.

From this, it is clear that all the logarithms of an imaginary quantity are also imaginary; for when either  $\varphi = 0$  or  $\varphi = \pm 2\lambda\pi$ , which are the cases in which any of the logarithms could become real, this could occur only when  $\text{tang } \varphi = \frac{b}{a} = 0$ ; it would follow that  $b = 0$ , and the quantity  $a + b\sqrt{-1}$  would cease to be imaginary. Thus, if we take  $p$  to signify every even number, whether positive or negative, all the logarithms of the imaginary quantity  $a + b\sqrt{-1}$  will be included in the general formula

$$C + (\varphi + p\pi)\sqrt{-1},$$

where  $C$  is the real logarithm of the positive quantity  $\sqrt{(aa + bb)} = c$ , and the arc or the angle  $\varphi$  is taken so that  $\sin \varphi = \frac{b}{c}$  and  $\cos \varphi = \frac{a}{c}$ . Now, since there is an infinity of angles which have the same sine  $\frac{b}{c}$  and cosine  $\frac{a}{c}$ , all of which are comprised in the formula  $\varphi + p\pi$ , we could omit the term  $p\pi$ , and say that the logarithm of  $a + b\sqrt{-1}$  is in general  $= C + \varphi\sqrt{-1}$ ; since that angle  $\varphi$  already contains all the angles. However, if we take for  $\varphi$  the smallest positive angle which corresponds to the sine  $\frac{b}{c}$  and the cosine  $\frac{a}{c}$ , the general formula for the logarithms of  $a + b\sqrt{-1}$  will be  $= C + (\varphi + p\pi)\sqrt{-1}$ .

If the angle  $\varphi$  is such that it has a commensurable ratio with  $\pi$  or the circumference of a circle, that will always be an indication that a certain power of the imaginary quantity  $a + b\sqrt{-1}$  becomes real. For let  $\varphi = \frac{\mu}{\nu}\pi$ , and since  $l(a + b\sqrt{-1}) = C + (\frac{\mu}{\nu}\pi + p\pi)\sqrt{-1}$ , we will have

$$l(a + b\sqrt{-1})^\nu = \nu C + (\mu + \nu p)\pi\sqrt{-1},$$

whence we see that if  $\mu + \nu p$  is an even number or just  $\mu$  even, the power  $(a + b\sqrt{-1})^\nu$  will be a positive real number, and in fact  $= c^\nu = (\sqrt{(aa + bb)})^\nu$ ; but if  $\mu + \nu p$  or just  $\mu$  is an odd number, the power  $(a + b\sqrt{-1})^\nu$  will be a negative number  $= -c^\nu$ .

Up to this point, we might perhaps have thought that we could have given to  $\pi$  whatever value we liked, since there does not appear to be anything, either in relation to the logarithms of positive numbers  $l+a = A \pm p\pi\sqrt{-1}$ , or to those of negative numbers  $l-a = A \pm q\pi\sqrt{-1}$ , from which we would be able to understand why the letter  $\pi$  should represent the semi-circumference of a circle of radius  $= 1$ , rather than any other number. But now, when it has to do with the logarithms of imaginary numbers, the reason becomes evident; since it is necessary to compare the angle  $\varphi$  with  $\pi$ , so that if we were to give to  $\pi$  any value other than that of two right angles, the formulas would become false, and would no longer be in accord with those which we have obtained for positive and negative numbers.

In order to be able to see this more clearly, suppose that  $c = 1$  and  $C = 0$ , so that we have the formula  $\cos \varphi + \sqrt{-1} \cdot \sin \varphi$ , all the logarithms of which will be included in the general formula

$$l(\cos \varphi + \sqrt{-1} \cdot \sin \varphi) = (\varphi + p\pi)\sqrt{-1}$$

where  $p$  represents an arbitrary even whole number, whether positive, negative, or even zero.

From this, we will obtain to begin with the previous formulas for the logarithms of positive or negative real numbers. For, let  $\varphi = 0$ , and inasmuch as  $\cos \varphi = 1$  and  $\sin \varphi = 0$ , we will have  $l+1 = p\pi\sqrt{-1}$ , or more explicitly

$$l+1 = 0, \pm 2\pi\sqrt{-1}, \pm 4\pi\sqrt{-1}, \pm 6\pi\sqrt{-1}, \pm 8\pi\sqrt{-1}, \text{ etc.},$$

but, setting  $\varphi = \pi = 180^\circ$ , and since  $\cos \varphi = -1$  and  $\sin \varphi = 0$ , we will have

$$l-1 = (1 + p)\pi\sqrt{-1} = q\pi\sqrt{-1},$$

taking  $q$  to stand for an arbitrary odd number. We will thus have

$$l-1 = \pm\pi\sqrt{-1}, \pm 3\pi\sqrt{-1}, \pm 5\pi\sqrt{-1}, \pm 7\pi\sqrt{-1}, \text{ etc.}$$

Let us now work out also the simplest cases of imaginary numbers, and let

1.  $\varphi = 90^\circ = \frac{1}{2}\pi$ , and since  $\cos \varphi = 0$  and  $\sin \varphi = +1$ , we will have

$$l+\sqrt{-1} = \left(\frac{1}{2} + p\right)\pi\sqrt{-1};$$

whence all the logarithms of  $+\sqrt{-1}$  will be

$$\begin{aligned} &+\frac{1}{2}\pi\sqrt{-1}, +\frac{5}{2}\pi\sqrt{-1}, +\frac{9}{2}\pi\sqrt{-1}, +\frac{13}{2}\pi\sqrt{-1}, +\frac{17}{2}\pi\sqrt{-1}, \text{ etc.} \\ &-\frac{3}{2}\pi\sqrt{-1}, -\frac{7}{2}\pi\sqrt{-1}, -\frac{11}{2}\pi\sqrt{-1}, -\frac{15}{2}\pi\sqrt{-1}, -\frac{19}{2}\pi\sqrt{-1}, \text{ etc.} \end{aligned}$$

If we here add two arbitrary values together to get the logarithm of  $l(+\sqrt{-1})^2$ , that is, of  $l-1$ , we will obtain  $\pm\pi\sqrt{-1}$ , or  $\pm 3\pi\sqrt{-1}$ , or  $\pm 5\pi\sqrt{-1}$ , etc., which are all the logarithms of  $-1$ .

2. Let  $\varphi = 270^\circ = \frac{3}{2}\pi$ , and because  $\cos \varphi = 0$  and  $\sin \varphi = -1$ , we will have

$$l-\sqrt{-1} = \left(-\frac{1}{2} + p\right)\pi\sqrt{-1};$$

thus all the logarithms of  $-\sqrt{-1}$  will be contained in the following expressions

$$\begin{aligned} &+\frac{3}{2}\pi\sqrt{-1}, +\frac{7}{2}\pi\sqrt{-1}, +\frac{11}{2}\pi\sqrt{-1}, +\frac{15}{2}\pi\sqrt{-1}, +\frac{19}{2}\pi\sqrt{-1}, \text{ etc.} \\ &-\frac{1}{2}\pi\sqrt{-1}, -\frac{5}{2}\pi\sqrt{-1}, -\frac{9}{2}\pi\sqrt{-1}, -\frac{13}{2}\pi\sqrt{-1}, -\frac{17}{2}\pi\sqrt{-1}, \text{ etc.}, \end{aligned}$$

where it is clear, as before, that two arbitrary values added together give  $q\pi\sqrt{-1}$ , taking  $q$  to be an arbitrary odd number, which is the logarithm of  $-1$  or of  $(-\sqrt{-1})^2$ . Furthermore, if we add an arbitrary logarithm of  $-\sqrt{-1}$  to an arbitrary logarithm of  $+\sqrt{-1}$  to get a logarithm of the product  $(+\sqrt{-1}) \cdot (-\sqrt{-1})$ , which is

= +1, we do indeed just get the logarithms of +1. And it is clear, similarly, that  $l(+\sqrt{-1}) - l(-\sqrt{-1}) = l-1$  or  $l(-\sqrt{-1}) - l(+\sqrt{-1}) = l-1$ , just as is required by the nature of those expressions.

3. Let  $\varphi = 60^\circ = \frac{1}{3}\pi$  or  $\cos \varphi = \frac{1}{2}$  and  $\sin \varphi = \frac{\sqrt{3}}{2}$ ; we will find that

$$l \frac{1+\sqrt{-3}}{2} = \left(\frac{1}{3} + p\right) \pi \sqrt{-1},$$

so that all the logarithms of this imaginary expression  $\frac{1+\sqrt{-3}}{2}$  will be

$$\begin{aligned} &+\frac{1}{3}\pi\sqrt{-1}, +\frac{7}{3}\pi\sqrt{-1}, +\frac{13}{3}\pi\sqrt{-1}, +\frac{19}{3}\pi\sqrt{-1}, +\frac{25}{3}\pi\sqrt{-1}, \text{ etc.} \\ &-\frac{5}{3}\pi\sqrt{-1}, -\frac{11}{3}\pi\sqrt{-1}, -\frac{17}{3}\pi\sqrt{-1}, -\frac{23}{3}\pi\sqrt{-1}, -\frac{29}{3}\pi\sqrt{-1}, \text{ etc.}, \end{aligned}$$

where it is clear that any three of these logarithms added together will produce  $q\pi\sqrt{-1}$  or one of the logarithms of -1, since

$$\left(\frac{1+\sqrt{-3}}{2}\right)^3 = -1.$$

4. Let  $\varphi = 120^\circ = \frac{2}{3}\pi$  or  $\cos \varphi = -\frac{1}{2}$  and  $\sin \varphi = \frac{\sqrt{3}}{2}$ , and we will get

$$l \frac{-1+\sqrt{-3}}{2} = \left(\frac{2}{3} + p\right) \pi \sqrt{-1}.$$

Thus all the logarithms of the imaginary formula  $\frac{-1+\sqrt{-3}}{2}$  will be

$$\begin{aligned} &+\frac{2}{3}\pi\sqrt{-1}, +\frac{8}{3}\pi\sqrt{-1}, +\frac{14}{3}\pi\sqrt{-1}, +\frac{20}{3}\pi\sqrt{-1}, +\frac{26}{3}\pi\sqrt{-1}, \text{ etc.} \\ &-\frac{4}{3}\pi\sqrt{-1}, -\frac{10}{3}\pi\sqrt{-1}, -\frac{16}{3}\pi\sqrt{-1}, -\frac{22}{3}\pi\sqrt{-1}, -\frac{28}{3}\pi\sqrt{-1}, \text{ etc.}, \end{aligned}$$

and since

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^3 = +1,$$

we will see that we do get the logarithms of +1 by adding together any three of these logarithms.

5. Let  $\varphi = 240^\circ = \frac{4}{3}\pi$  or  $\cos \varphi = -\frac{1}{2}$  and  $\sin \varphi = \frac{-\sqrt{3}}{2}$ , and we will have

$$l \frac{-1-\sqrt{-3}}{2} = \left(\frac{4}{3} + p\right) \pi \sqrt{-1},$$

so that all the logarithms of this formula  $\frac{-1-\sqrt{-3}}{2}$  will be

$$\begin{aligned} &+\frac{4}{3}\pi\sqrt{-1}, +\frac{10}{3}\pi\sqrt{-1}, +\frac{16}{3}\pi\sqrt{-1}, +\frac{22}{3}\pi\sqrt{-1}, +\frac{28}{3}\pi\sqrt{-1}, \text{ etc.} \\ &-\frac{2}{3}\pi\sqrt{-1}, -\frac{8}{3}\pi\sqrt{-1}, -\frac{14}{3}\pi\sqrt{-1}, -\frac{20}{3}\pi\sqrt{-1}, -\frac{26}{3}\pi\sqrt{-1}, \text{ etc.}, \end{aligned}$$

whence we will get as before, adding any three of these logarithms together, any of the logarithms of +1, since

$$\left(\frac{-1-\sqrt{-3}}{2}\right)^3 = +1.$$

Similarly, any two of these logarithms added together will produce a logarithm of  $\frac{-1+\sqrt{-3}}{2}$ ; because

$$\left(\frac{-1-\sqrt{-3}}{2}\right)^2 = \frac{-1+\sqrt{-3}}{2}.$$

And since conversely

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2 = \frac{-1-\sqrt{-3}}{2},$$

we will also see that the sum of any two logarithms of  $\frac{-1+\sqrt{-3}}{2}$  produces a logarithm of  $\frac{-1-\sqrt{-3}}{2}$ .



6. Let  $\varphi = 300^\circ = \frac{5}{3}\pi$  or  $\cos \varphi = \frac{1}{2}$  and  $\sin \varphi = \frac{-\sqrt{3}}{2}$ , and we will have

$$l^{\frac{1-\sqrt{-3}}{2}} = \left(\frac{5}{3} + p\right) \pi \sqrt{-1}.$$

Consequently, the logarithms of the formula  $\frac{+1-\sqrt{-3}}{2}$  will be

$$\begin{aligned} &+\frac{5}{3}\pi\sqrt{-1}, +\frac{11}{3}\pi\sqrt{-1}, +\frac{17}{3}\pi\sqrt{-1}, +\frac{23}{3}\pi\sqrt{-1}, +\frac{29}{3}\pi\sqrt{-1}, \text{ etc.} \\ &-\frac{1}{3}\pi\sqrt{-1}, -\frac{7}{3}\pi\sqrt{-1}, -\frac{13}{3}\pi\sqrt{-1}, -\frac{19}{3}\pi\sqrt{-1}, -\frac{25}{3}\pi\sqrt{-1}, \text{ etc.}, \end{aligned}$$

whence it is evident that any three of these logarithms being added together give a logarithm of  $-1$ , in conformity with

$$\left(\frac{1-\sqrt{-3}}{2}\right)^3 = -1.$$

And in general, we will always find that all the operations which we perform with logarithms are in perfect agreement with the corresponding operations performed with the numbers which belong to them, so that we will not encounter the slightest discrepancy between the operations with logarithms and the corresponding operations with numbers.

7. Let  $\varphi = 45^\circ = \frac{1}{4}\pi$  or  $\cos \varphi = \frac{1}{\sqrt{2}}$  and  $\sin \varphi = \frac{1}{\sqrt{2}}$ , and we will have

$$l^{\frac{+1+\sqrt{-1}}{\sqrt{2}}} = \left(\frac{1}{4} + p\right) \pi \sqrt{-1}.$$

Thus, all the logarithms of the imaginary expression  $\frac{+1+\sqrt{-1}}{\sqrt{2}}$  will be

$$\begin{aligned} &+\frac{1}{4}\pi\sqrt{-1}, +\frac{9}{4}\pi\sqrt{-1}, +\frac{17}{4}\pi\sqrt{-1}, +\frac{25}{4}\pi\sqrt{-1}, +\frac{33}{4}\pi\sqrt{-1}, \text{ etc.} \\ &-\frac{7}{4}\pi\sqrt{-1}, -\frac{15}{4}\pi\sqrt{-1}, -\frac{23}{4}\pi\sqrt{-1}, -\frac{31}{4}\pi\sqrt{-1}, -\frac{39}{4}\pi\sqrt{-1}, \text{ etc.} \end{aligned}$$

8. Let  $\varphi = 135^\circ = \frac{3}{4}\pi$  or  $\cos \varphi = \frac{-1}{\sqrt{2}}$  and  $\sin \varphi = +\frac{1}{\sqrt{2}}$ , and we will have

$$l^{\frac{-1+\sqrt{-1}}{\sqrt{2}}} = \left(\frac{3}{4} + p\right) \pi \sqrt{-1}.$$

And consequently, all the logarithms of the formula  $\frac{-1+\sqrt{-1}}{\sqrt{2}}$  will be

$$\begin{aligned} &+\frac{3}{4}\pi\sqrt{-1}, +\frac{11}{4}\pi\sqrt{-1}, +\frac{19}{4}\pi\sqrt{-1}, +\frac{27}{4}\pi\sqrt{-1}, +\frac{35}{4}\pi\sqrt{-1}, \text{ etc.} \\ &-\frac{5}{4}\pi\sqrt{-1}, -\frac{13}{4}\pi\sqrt{-1}, -\frac{21}{4}\pi\sqrt{-1}, -\frac{29}{4}\pi\sqrt{-1}, -\frac{37}{4}\pi\sqrt{-1}, \text{ etc.} \end{aligned}$$

Each the these logarithms added to any one of the preceding logarithms of  $\frac{+1+\sqrt{-1}}{\sqrt{2}}$  produces a logarithm of the form  $q\pi\sqrt{-1}$ , or a logarithm of  $-1$ , just as it should, since

$$\frac{+1+\sqrt{-1}}{\sqrt{2}} \cdot \frac{-1+\sqrt{-1}}{\sqrt{2}} = -1.$$

9. Let  $\varphi = 225^\circ = \frac{5}{4}\pi$  or  $\cos \varphi = \frac{-1}{\sqrt{2}}$  and  $\sin \varphi = \frac{-1}{\sqrt{2}}$ , and we will have

$$l^{\frac{-1-\sqrt{-1}}{\sqrt{2}}} = \left(\frac{5}{4} + p\right) \pi \sqrt{-1}.$$

Thus, all the logarithms of the formula  $\frac{-1-\sqrt{-1}}{\sqrt{2}}$  will be

$$\begin{aligned} &+\frac{5}{4}\pi\sqrt{-1}, +\frac{13}{4}\pi\sqrt{-1}, +\frac{21}{4}\pi\sqrt{-1}, +\frac{29}{4}\pi\sqrt{-1}, \text{ etc.} \\ &-\frac{3}{4}\pi\sqrt{-1}, -\frac{11}{4}\pi\sqrt{-1}, -\frac{19}{4}\pi\sqrt{-1}, -\frac{27}{4}\pi\sqrt{-1}, \text{ etc.}, \end{aligned}$$

which are the negatives of the preceding ones; this is also in perfect agreement with the analytic operations, since

$$\frac{-1-\sqrt{-1}}{\sqrt{2}} = 1 : \frac{-1+\sqrt{-1}}{\sqrt{2}}$$

and hence

$$l\frac{-1-\sqrt{-1}}{\sqrt{2}} = -l\frac{-1+\sqrt{-1}}{\sqrt{2}}.$$

10. Let  $\varphi = 315^\circ = \frac{7}{4}\pi$  or  $\cos \varphi = +\frac{1}{\sqrt{2}}$  and  $\sin \varphi = -\frac{1}{\sqrt{2}}$ , whence we will have

$$l\frac{+1-\sqrt{-1}}{\sqrt{2}} = \left(\frac{7}{4} + p\right) \pi \sqrt{-1}.$$

Consequently, all the logarithms of the formula  $\frac{+1-\sqrt{-1}}{\sqrt{2}}$  will be

$$\begin{aligned} & +\frac{7}{4}\pi\sqrt{-1}, +\frac{15}{4}\pi\sqrt{-1}, +\frac{23}{4}\pi\sqrt{-1}, +\frac{31}{4}\pi\sqrt{-1}, \text{ etc.} \\ & -\frac{1}{4}\pi\sqrt{-1}, -\frac{9}{4}\pi\sqrt{-1}, -\frac{17}{4}\pi\sqrt{-1}, -\frac{25}{4}\pi\sqrt{-1}, \text{ etc.} \end{aligned}$$

All the logarithms of the last four articles have the property that each multiplied by 4 produces a logarithm of  $-1$ , which is in conformity with the truth, since the squared-squares of the four formulas

$$\frac{+1+\sqrt{-1}}{\sqrt{2}}, \frac{-1+\sqrt{-1}}{\sqrt{2}}, \frac{-1-\sqrt{-1}}{\sqrt{2}}, \frac{+1-\sqrt{-1}}{\sqrt{2}}$$

produce the number  $-1$ .

These examples suffice to make clear that the idea of logarithms which I have tried to establish is the true one, and that it is in perfect agreement with all the operations which are included in the theory of logarithms, so that we no longer encounter any difficulty, and all the contradictions to which that theory appeared to be subject have entirely disappeared. Consequently, the great controversy in which Messrs. LEIBNIZ and BERNOULLI formerly took part is now perfectly decided, so that neither the one nor the other would find the slightest pretext for withholding their agreement.

The beautiful discovery of M. BERNOULLI, which reduces the quadrature of the circle to imaginary logarithms, also appears not only in perfect agreement with this theory, but is a necessary consequence of it, and is even extended by it to an infinitely greater scope, since we see that the logarithms of all numbers, so far as they are imaginary, depend on the quadrature of the circle. Thus, since the logarithms of  $+1$  are  $\pm p\pi\sqrt{-1}$ ,  $\frac{l+1}{\sqrt{-1}}$  will always be a real quantity, but which includes an infinity of values, because of the infinity of the logarithms of  $+1$ . In consequence, if we take the ratio of the diameter to the circumference to be  $= 1 : \pi$ , all the values of the expression  $\frac{l+1}{\sqrt{-1}}$  will be the following:

$$0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \pm 8\pi, \pm 10\pi, \text{ etc.}$$

Similarly, the logarithms of  $-1$  divided by  $\sqrt{-1}$  will furnish the following real quantities which also belong to the quadrature of the circle. For the values of  $\frac{l-1}{\sqrt{-1}}$  are

$$\pm\pi, \pm 3\pi, \pm 5\pi, \pm 7\pi, \pm 9\pi, \text{ etc.}$$

In the same way, we see that the values of the following expressions will be:

The values of	$\frac{l(+\sqrt{-1})}{\sqrt{-1}}$	will be these to infinity $+\frac{1}{2}\pi, +\frac{5}{2}\pi, +\frac{9}{2}\pi, +\frac{13}{2}\pi, +\frac{17}{2}\pi, \text{ etc.}$ $-\frac{3}{2}\pi, -\frac{7}{2}\pi, -\frac{11}{2}\pi, -\frac{15}{2}\pi, -\frac{19}{2}\pi, \text{ etc.}$
	$\frac{l(-\sqrt{-1})}{\sqrt{-1}}$	$+\frac{3}{2}\pi, +\frac{7}{2}\pi, +\frac{11}{2}\pi, +\frac{15}{2}\pi, +\frac{19}{2}\pi, \text{ etc.}$ $-\frac{1}{2}\pi, -\frac{5}{2}\pi, -\frac{9}{2}\pi, -\frac{13}{2}\pi, -\frac{17}{2}\pi, \text{ etc.}$

and we will obtain in the same way from the other examples developed above similar real expressions which will all involve the quadrature of the circle.

I have already made clear the excellent agreement between logarithms and extraction of roots, having shown that the doubles of the logarithms of both  $-1$  and  $+1$  are contained among the logarithms of  $+1$ , since

$$1 = (+1)^2 = (-1)^2;$$

similarly, since

$$1 = (+1)^3 = \left(\frac{-1+\sqrt{-3}}{2}\right)^3 = \left(\frac{-1-\sqrt{-3}}{2}\right)^3,$$

we will see that the triples of the logarithms of  $+1$ , of  $\frac{-1+\sqrt{-3}}{2}$ , and of  $\frac{-1-\sqrt{-3}}{2}$  occur among the logarithms of  $+1$ . But I remark here in addition, inasmuch as  $1$  has only two square roots  $+1$  and  $-1$ , thus if we arrange the doubles of all the logarithms of both  $+1$  and  $-1$  in a sequence, we will obtain the complete series of all the logarithms of  $+1$ ; thus

$$\begin{aligned} 2l+1 \text{ is } & 0, \pm 4\pi\sqrt{-1}, \pm 8\pi\sqrt{-1}, \pm 12\pi\sqrt{-1}, \text{ etc.} \\ 2l-1 \text{ is } & \pm 2\pi\sqrt{-1}, \pm 6\pi\sqrt{-1}, \pm 10\pi\sqrt{-1}, \text{ etc.} \end{aligned}$$

In the same way, the three cube roots of  $+1$  being

$$+1, \frac{-1+\sqrt{-3}}{2} \text{ and } \frac{-1-\sqrt{-3}}{2},$$

if we arrange the triples of all the logarithms of these three roots in a sequence, the result will be the complete sequence of logarithms of  $+1$ , since

$$\begin{array}{l} 3l+1 \text{ gives } 0, \pm 6\pi\sqrt{-1}, \pm 12\pi\sqrt{-1}, \pm 18\pi\sqrt{-1}, \text{ etc.} \\ 3l\frac{-1+\sqrt{-3}}{2} \quad \left\{ \begin{array}{l} +2\pi\sqrt{-1}, +8\pi\sqrt{-1}, +14\pi\sqrt{-1}, \text{ etc.} \\ -4\pi\sqrt{-1}, -10\pi\sqrt{-1}, -16\pi\sqrt{-1}, \text{ etc.} \end{array} \right. \\ 3l\frac{-1-\sqrt{-3}}{2} \quad \left\{ \begin{array}{l} +4\pi\sqrt{-1}, +10\pi\sqrt{-1}, +16\pi\sqrt{-1}, \text{ etc.} \\ -2\pi\sqrt{-1}, -8\pi\sqrt{-1}, -14\pi\sqrt{-1}, \text{ etc.} \end{array} \right. \end{array}$$

Each logarithm of  $+1$  occurs in these three series, and none occurs more than once; which indicates that unity has only three cube roots, and that all three together are required to exhaust the nature of unity.

It is the same with all the other roots of unity, and inasmuch as the biquadratic roots of  $+1$  are

$$+1, -1, +\sqrt{-1} \text{ and } -\sqrt{-1},$$

we will see that the quadruples of the logarithms of each of these roots give only the fourth part of the logarithms of  $+1$ . Now, all these quadruples of all the fourth roots together produce the whole sequence of logarithms of  $+1$ . It is also remarkable that all the logarithms of any given root are different from the logarithms of every other root of the same number. However, although the two logarithms  $l+1$  and  $l-1$  are different from one another, it is nevertheless the case that  $2l+1 = l+1$  and  $2l-1 = l+1$ , even though we do not have  $2l+1 = 2l-1$ . In the same way, the three logarithms

$$l+1, l\frac{-1+\sqrt{-3}}{2} \text{ and } l\frac{-1-\sqrt{-3}}{2}$$

are different from one another; however, despite that inequality, we have

$$3l+1 = l+1, 3l\frac{-1+\sqrt{-3}}{2} = l+1, \text{ and } 3l\frac{-1-\sqrt{-3}}{2} = l+1.$$

We see thus that it is essential to the nature of logarithms that each number have an infinity of logarithms, and that all these logarithms be different not only from one another, but also from all the logarithms of every other number. It is the same with logarithms as with angles or arcs of a circle; for, since to each sine or cosine there corresponds an infinity of different arcs, so to each number there belongs an infinity of

different logarithms. But it is necessary here to call attention to a great difference, which is that all the arcs corresponding to the same sine or cosine are real, whereas all the logarithms of the same number are imaginary, with the exception of one, when the given number is positive; for all the logarithms of numbers which are either negative or imaginary are without exception imaginary. Now, since only one sine or cosine belongs to a given arc, so also to a given logarithm there corresponds only one number; so that, when we require the number which corresponds to a given logarithm, this is a problem which admits only one solution.

#### PROBLEM 4

*An arbitrary logarithm being given, to find the number which corresponds to it.*

#### SOLUTION

Suppose first that the given logarithm is a real quantity  $= f$ ; then we know that, denoting the number  $= e$  whose real logarithm is  $= 1$ , the number which corresponds to the logarithm  $f$  will be  $= e^f$ .

Secondly, let the given logarithm be  $= g\sqrt{-1}$  or simply imaginary, and let  $x$  be the number which corresponds to it. Since  $g$  is a real number, let it be compared with  $\pi$ , say  $g = m\pi$ , and it is clear that if  $m$  is an even or odd whole number, the number  $x$  will be  $+1$  or  $-1$ . But in every other case whatever, the number  $x$  will be imaginary, and in order to find it we have only to take a circular arc  $= g$ , the radius being  $= 1$ , and having found its sine and cosine the desired number will be

$$x = \cos g + \sqrt{-1} \cdot \sin g.$$

In the third place, let the given logarithm be an arbitrary imaginary quantity  $= f + g\sqrt{-1}$ , since we know that every imaginary quantity can be reduced to this form  $f + g\sqrt{-1}$ , where  $f$  and  $g$  are real numbers. This being so, it is clear that the desired number  $x$  will be the product of the two numbers of which the logarithm of one is  $f$  and of the other  $g\sqrt{-1}$ . Consequently, the number which corresponds to the logarithm  $f + g\sqrt{-1}$  will be

$$= e^f (\cos g + \sqrt{-1} \cdot \sin g).$$

Q.E.I.

We see thus that the number which corresponds to the given logarithm  $f + g\sqrt{-1}$  will be real when  $\sin g = 0$ , that is, when  $g = m\pi$ , the coefficient  $m$  being an arbitrary whole number, whether positive or negative. In this case, we see also that if  $m$  is an even number, inasmuch as  $\cos g = +1$ , the desired number will be positive, but if  $m$  is an odd number, since  $\cos g = -1$ , the desired number will be negative  $= -e^f$ . In all the other cases, where  $m$ , that is  $\frac{g}{\pi}$ , is a fraction, or even irrational, the number which corresponds to the logarithm  $f + g\sqrt{-1}$  will infallibly be imaginary.

By means of this rule, we can also make use of logarithms in calculation with imaginary numbers. As an example, let us find the value of the expression

$$\left(\frac{-1 + \sqrt{-3}}{2}\right)^4 \left(\frac{+1 + \sqrt{-1}}{\sqrt{2}}\right)^3 \left(\frac{-1 - \sqrt{-3}}{2}\right)^2 \sqrt{-1} = A.$$

In order to do this, we have only to take an arbitrary logarithm of each factor, and to carry out the operations according to the generally received rules, as follows:

$$\begin{aligned} l \frac{-1 + \sqrt{-3}}{2} &= \frac{2}{3}\pi\sqrt{-1}, & \text{thus} & & 4l \frac{-1 + \sqrt{-3}}{2} &= \frac{8}{3}\pi\sqrt{-1}, \\ l \frac{+1 + \sqrt{-1}}{\sqrt{2}} &= \frac{1}{4}\pi\sqrt{-1}, & \dots & & 3l \frac{+1 + \sqrt{-1}}{\sqrt{2}} &= \frac{3}{4}\pi\sqrt{-1}, \\ l \frac{-1 - \sqrt{-3}}{2} &= \frac{4}{3}\pi\sqrt{-1}, & \dots & & 2l \frac{-1 - \sqrt{-3}}{2} &= \frac{8}{3}\pi\sqrt{-1}, \end{aligned}$$

and finally

$$l\sqrt{-1} = \frac{1}{2}\pi\sqrt{-1}.$$

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Thus, the sum is  $lA = \frac{79}{12}\pi\sqrt{-1}.$

Consequently, the required product will be

$$A = \cos \frac{79}{12}\pi + \sqrt{-1} \cdot \sin \frac{79}{12}\pi$$

or in fact

$$A = \cos \frac{7}{12}\pi + \sqrt{-1} \cdot \sin \frac{7}{12}\pi.$$

I remark also that the given logarithm being  $= f + g\sqrt{-1}$ , the corresponding number according to the usual rule is found to be  $= e^{f+g\sqrt{-1}}$ . Now, this expression is completely equivalent to the one which we have found. For we already know that  $e^{g\sqrt{-1}} = \cos g + \sqrt{-1} \cdot \sin g$  and consequently

$$e^{f+g\sqrt{-1}} = e^f \cdot e^{g\sqrt{-1}} = e^f(\cos g + \sqrt{-1} \cdot \sin g),$$

but this last expression is more convenient than the first, in which the imaginaries enter the exponent.