Queen Dido Activity
Teacher Notes

Level: This activity is designed for students in middle school through high school.

Objective: Students will compare the areas of shapes with the same perimeter. They should conclude that, among shapes formed using a straight "coastline" as one edge and a string of fixed length for the remaining edge(s), the semicircle has the largest area.

Materials: Copy the Student Page to distribute to students. Provide each pair of students with a 12" string, a ruler, and 1/4-inch graph paper.

When to Use: Use this activity in Pre-algebra, Algebra I, or Geometry classes after introducing formulas for areas of polygons. For other activities involving areas of isoperimetric shapes, see the **Housing Activity** and the **Geometry of Beehives Activity.**

How to Use: Students should compute areas of several different shapes, working individually or in pairs. You might suggest shapes for the students to try, such as triangles, rectangles, circles, pentagons, hexagons, or irregular shapes. You might direct students to calculate areas using a specific method, or you might let them experiment and come up with methods of their own. For each shape they try, students should provide a sketch and record the total area. Students could use any or all of the following methods:

1. Use a formula to find the area of the region.

2. Divide the region into smaller shapes whose areas are known and sum the areas.

3. Use grid paper and count the squares to approximate the area as closely as possible.

Solution: Among shapes formed using a straight coastline as one edge and a string or rope of fixed length for the remaining edge(s), the semicircle has largest area. Therefore, Dido should have laid her cowhide strips in the shape of a semicircle. A semicircle formed from a 12" string has radius $r = \frac{12}{\pi}$ (since $\pi r = 12$) and area $A = \frac{1}{2}\pi r^2 = \frac{\pi}{2}\left(\frac{12}{\pi}\right)^2 = \frac{72}{\pi} \approx 22.9183$ square inches.

Background Information: The Roman poet Virgil (70-19 BCE) wrote his famous epic poem, the *Aeneid,* during the last ten years of his life. He modeled the *Aeneid* on the *Iliad* and especially the *Odyssey,* the well-known epic poems of the Greek poet, Homer (*c.* 750 BCE). In the *Iliad,* Homer tells of the Trojan War, fought in Troy between the Greeks and the hometown Trojans; in the *Odyssey,* he recounts the adventures of Odysseus, a hero of the Trojan War on the winning Greek side, during his circuitous ten-year journey home to Ithaca. In the *Aeneid,* Virgil's protagonist, Aeneas, also a hero of the Trojan War despite having been on the losing Trojan side, flees Troy after the city is destroyed with a band of loyal followers. As they wander about the Mediterranean, he and his men have a series of hair-raising adventures until, finally, Aeneas is able to fulfill his destiny by settling in Italy where his descendants will found Rome.

Queen Dido Activity
Student Page

According to the Roman poet Virgil's epic poem, the *Aeneid*, Princess Dido, the daughter of the king of the ancient Phoenician city of Tyre, fled Tyre after her brother, Pygmalion, murdered her husband. She ended up in what is now Tunisia on the Mediterranean coast of Africa, where she agreed to pay a certain sum of money for as much land as she "could enclose with one bull's hide" (Fitzgerald, *Aeneid*, Book I, 16). Dido then took a bull's hide, cut it up into long, thin strips, tied the strips together end-to-end, and set out to enclose the largest amount of land possible. She chose land along the sea, so that she could use the shoreline as one edge of her enclosure. She still needed to decide in what shape to lay the bull's hide in order to enclose the largest area possible (Kline, 134-135; Perl, 72-73).

Your assignment: Using the edge of your desk as the coastline and a 12" piece of string as the bull's hide, form different shapes and compute their areas. Sketch each shape you create and be sure to record its area. Which shape has the largest area? In what shape should Dido have laid the hide in order to enclose the largest area possible?

Ancient City of Carthage (from: http://sunsite.auc.dk/cgfa/turner/p-turner19.htm)

According to the *Aeneid*, the land Dido purchased became the great city of Carthage and Dido herself became its queen. Unfortunately, Queen Dido did not live to perform many more mathematical feats. Not too many years after she founded Carthage, the mythical Trojan hero Aeneas blew into town. Dido fell in love with Aeneas and begged him to stay. When he refused, Dido threw herself on a sword Aeneas had left behind, committing suicide (Fitzgerald, *Aeneid,* Book IV, 119-121). Again according to the *Aeneid,* Aeneas went on to fulfill his destiny by settling in what is now Italy, where his descendants would found the city of Rome. The historical city of Carthage flourished from the ninth century BCE until 146 BCE, when it was destroyed by the Romans.

Geometry of Beehives Activity
Teacher Notes

Level: This activity is designed for students in middle school through high school.

Objective: Students will compare the areas of shapes with the same perimeter. They should conclude that, of an equilateral triangle, a square, and a regular hexagon with fixed perimeter, the regular hexagon has the largest area, and that, among all shapes with fixed perimeter, the circle has the greatest area.

Materials: Copy the Student Page to distribute to students. Depending on the methods you intend for students to use, you might provide each pair of students with cutouts of regular polygons and with a 12" string, a ruler, and 1/4-inch graph paper.

When to Use: Use this activity in Pre-algebra, Algebra I, or Geometry classes after introducing formulas for areas of polygons.

How to Use: Students should complete the activity individually or in pairs. Students could complete Problem 1 by experimenting with pattern blocks or cutouts of regular polygons, and/or by using the fact that the interior angles of a regular *n*-gon sum to $180(n-2)$ degrees, hence are each of measure $\frac{180(n-2)}{n} = 180 - \frac{360}{n}$ degrees. In Problems 2 and 3, students could use any or all of the following methods for finding areas:

a. Use a formula to find the area of the region.

b. Divide the region into smaller shapes whose areas are known and sum the areas.

c. Use grid paper and count the squares to approximate the area as closely as possible. You might direct students to calculate areas using a specific method, or you might let them experiment and come up with methods of their own.

Background Information: In his *Mathematical Collection*, the Greek mathematician Pappus of Alexandria (*c.* 320 CE) collected, clarified, and often extended the geometrical results of his predecessors Pythagoras, Euclid, and Archimedes. He generalized the Pythagorean Theorem to parallelograms (generalization of squares) constructed on the sides of an arbitrary triangle (generalization of a right triangle). He also established that the volume of the solid of revolution generated by revolving a plane region about an axis is equal to the product of the area of the plane region and the distance traveled by the center of gravity of the plane region (Burton 160, 237).

Although the question of how the area of a rectangle was related to its perimeter was studied in ancient civilizations, early peoples generally seem to have believed that the area of a plane figure was directly related to its perimeter. That is, they believed that plane figures having the same perimeter had the same area and that the area increased as the perimeter increased.

Land of greater perimeter was thought to have greater area, resulting in some notorious swindles. The size of an army was determined by the perimeter of its camp and the size of an island by the time it took to sail around it, resulting in some serious miscalculations (Burton, 64).

The work of Pappus, then, on areas of plane figures having the same perimeter and on volumes of solids having the same surface area, fit into a long tradition of intellectual inquiry and perhaps also straightened out some misconceptions. For other activities involving areas of isoperimetric shapes, see the **Housing Activity** and the **Queen Dido Activity.**

Solutions:

1. Pappus assumes that there should be no gaps or openings between cells and that each cell should have the same shape. The equilateral triangle, square, and regular hexagon are the only regular polygons that tessellate the plane by themselves. This is because the equilateral triangle, square, and regular hexagon are the only regular polygons for which the interior angle–60°, 90°, and 120°, respectively–divides 360° evenly.

2. The equilateral triangle of perimeter 12” has side length 4” and, by the Pythagorean Theorem, an altitude of $2\sqrt{3}$ inches. Its area, then, is $\frac{1}{2}\cdot 4\cdot 2\sqrt{3} = 4\sqrt{3} \approx 6.9282$ square inches. The square of perimeter 12” has side length 3” and area $3^2 = 9$ square inches. The regular hexagon of perimeter 12” has side length 2” and, by the Pythagorean Theorem, an apothem of length $\sqrt{3}$ inches. Since the area A of a regular polygon with apothem a and perimeter P is given by the formula $A = \frac{1}{2}aP$, then the area of the regular hexagon of perimeter $P = 12$ (and apothem $a = \sqrt{3}$) is $A = \frac{1}{2}\sqrt{3}\cdot 12 = 6\sqrt{3} \approx 10.3923$ square inches. Hence, the regular hexagon of perimeter 12” has larger area than do the equilateral triangle and the square of perimeter 12”.

3. To see that the circle of circumference 12” has an even larger area, use the formula $C = 2\pi r$ for circumference of a circle in terms of radius r to determine that the circle with circumference $C = 12$ has radius $r = \frac{12}{2\pi} = \frac{6}{\pi}$ and area $A = \pi r^2 = \pi\left(\frac{6}{\pi}\right)^2 = \frac{36}{\pi} \approx 11.4592$ square inches. Bees probably should not build hives made of circular cells because circles do not tessellate the plane.

Geometry of Beehives Activity
Student Page

The Greek geometer Pappus of Alexandria (*c.* 320 CE) studied the relationship between perimeter and area. He published his results in the fifth book of his collection of eight mathematical works called, appropriately, the *Mathematical Collection*. In the following passage from Book Five of the *Collection*, he discusses the geometry of beehives and, more generally, of isoperimetric shapes, or shapes having the same perimeter.

> Bees, then, know just this fact which is of service to themselves, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material used in constructing the different figures. We, however, claiming as we do a greater share in wisdom than bees, will investigate a problem of still wider extent, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest plane figure of all those which have a perimeter equal to that of the polygons is the circle.
> (Heath, *A History of Greek Mathematics*, vol. 2, 390)

1. Pappus implies that the only possible shapes for the cells in a beehive are equilateral triangles, squares, and regular hexagons. Why should the possibilities for the shape of each of the cells in a beehive be limited to the equilateral triangle, the square, and the regular hexagon?

2. Pappus claims that, if an equilateral triangle, a square, and a regular hexagon have the same perimeter, the hexagon has the largest area. Verify his claim by computing the area of an equilateral triangle, a square, and a regular hexagon of perimeter 12”.

3. Pappus claims that, among all plane figures having the same perimeter, the circle has the largest area. Compute the area of a circle of circumference 12”. Does the circle have larger area than the regular hexagon of perimeter 12” from Problem 2? Should bees build hives made of circular cells?

In addition to comparing the areas of plane figures with equal perimeters, Pappus also compared the volumes of solids with equal surface areas. He proved that the sphere has greater volume than any solid with equal surface area and that, for two regular solids with equal surface area, the one with the greater number of faces has the greater volume.