Research on the Precession of the Equinoxes and on the Nutation of the Earth's Axis*

Leonhard Euler†

Lemma 1

1. *Supposing the earth AEBF (fig. 1) to be spherical and composed of a homogenous substance, if the mass of the earth is denoted by M and its radius $CA = CE = a$, the moment of inertia of the earth about an arbitrary axis, which passes through its center, will be* $= \frac{2}{5}Maa$.

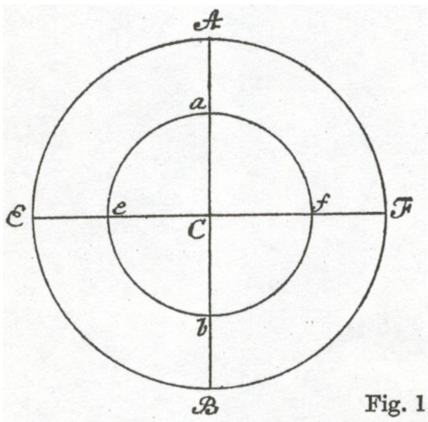

Fig. 1

*Leonhard Euler, *Recherches sur la précession des equinoxes et sur la nutation de l'axe de la terr*, in *Opera Omnia*, vol. II.30, p. 92-123, originally in *Mémoires de l'académie des sciences de Berlin* **5** (1749), 1751, p. 289-325. This article is numbered E171 in Eneström's index of Euler's work.

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Corollary

2. Although the earth may not be spherical, since its figure differs from that of a sphere ever so slightly, we readily understand that its moment of inertia can be nonetheless expressed as $\frac{2}{5}Maa$. For this expression will not change significantly, whether we let a be its semi-axis or the radius of its equator.

Remark

3. Here we should recall that the moment of inertia of an arbitrary body with respect to a given axis about which it revolves is that which results from multiplying each particle of the body by the square of its distance to the axis, and summing all these elementary products. Consequently this sum will give that which we are calling the moment of inertia of the body around this axis.

Lemma 2

4. *If the earth, being considered as spherical, has about its center an equally spherical core $aebf$, the density of which is to the crust as $1+\nu$ is to 1; then letting the mass of the entire earth $= M$, the radius $CA = a$, and the radius of the core $Ca = \alpha$, the moment of inertia with respect to an arbitrary axis, which passes through its center, will be*

$$= \frac{2}{5}M\frac{a^5 + \nu\alpha^5}{a^3 + \nu\alpha^3}.$$

Corollary 1

5. If the core is more dense than the rest of the earth, the value of ν will be positive, but if the core is less dense than the crust, ν will be a negative number. Now, if the earth were entirely hollow inside or if the space $aebf$ were empty, we would have $\nu = -1$, and in this case the moment of inertia would become

$$= \frac{2}{5}M\frac{a^5 - \alpha^5}{a^3 - \alpha^3}.$$

Corollary 2

6. We will agree still that a small aberration of the spherical figure does not significantly change the expression of the moment of inertia, so long as the

figure of the core is more or less spherical. At least it is quite certain that in the application of these two propositions to the precession of the equinoxes, a small difference would be seen to be of little consequence, since we will be obliged to content ourselves with approximations.

Hypothesis

7. The earth, having once received a rotational movement around an axis, which agrees with its axis on the figure or only differs from it slightly, will always conserve this uniform movement, and its axis of rotation will always remain the same and will be directed toward the same points of the sky, unless the earth should be subjected to external forces which might cause some change either in the speed of rotational movement or in the position of the axis of rotation.

Remark

8. In order for a body to be able to turn freely around an axis, it is not sufficient that this axis passes through its center of gravity, it must also be the case that all the centrifugal forces maintain themselves in equilibrium, and when this last condition does not hold, it is impossible for the axis of rotation to remain the same. Instead, it will continually be changed by the centrifugal forces of the body. It depends therefore solely on the distribution of the mass of the earth, whether or not the axis around which it turns is endowed with this property. For if all the centrifugal forces do not mutually cancel each other, the poles of the earth will undergo changes, even if there would be no external forces. It is therefore only pure supposition that I am making when I say that the axis of the earth, around which it turns, is not subjected to such changes, but it seems that this supposition is confirmed by the phenomena, and it will be entirely so when I have shown that there are actually no other changes in the poles except for those which are caused by the forces of the sun and the moon.

Corollary

9. From this it follows that any force, the direction of which passes through the center of gravity of the earth changes neither its rotational movement nor the position of its axis. Therefore we will only make use of those forces acting

on the earth, whose direction does not pass through the center of gravity, when explaining the changes we observe in its poles.

Problem 1

10. *All the parts of the earth being attracted toward a fixed point* 0*, of which the forces are proportional to an arbitrary power of the distances, to find the total force to which the earth will be subjected, supposing the earth to be spherical and composed of a homogenous substance.*

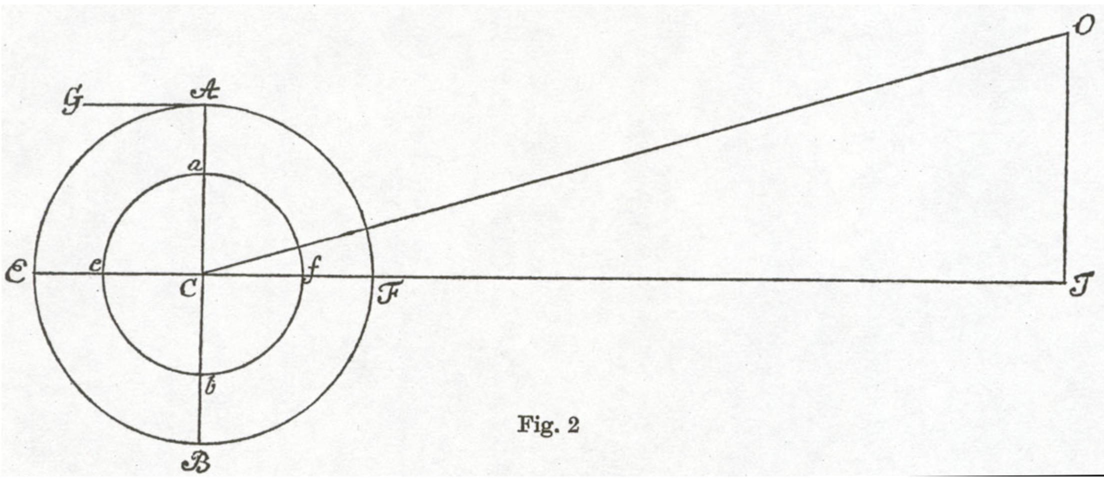

Fig. 2

Solution

Let the semi-axis of the earth $CA = CB = a$, the radius of the equator $CE = CF = e$, and from the attracting point O let us drop to the plane of the equator the perpendicular OT, and having drawn the straight lines CT, CO, let $CT = f$, $TO = g$, $CO = h = \sqrt{ff + gg}$. Let the attracting force from the point O at an arbitrary distance z be $\frac{k^n}{z^n}$. That is, let this force be reciprocally proportional to the power z^n of the distance. Moreover, let the mass of the earth $= M$. Given this we find by the Theory of Forces that the total force, to which the earth is drawn toward the point O, results from two forces.

1. The force along CO, which is

$$= \frac{Mk^n}{h^n}\left(1 - \frac{(n+1)aa - 4(n+1)ee}{10hh} + \frac{(n+1)(n+3)(aagg+eeff)}{10h^4}\right)$$

2. The force along TO, which is

$$= \frac{n+1}{5} M \frac{k^n}{h^n} \cdot \frac{g(ee-aa)}{h^2}.$$

We might well reduce these two forces into a single one, which would pass through the point O and a point of the axis a little below the center C; but for our intentions, it would be more convenient to make use of these two forces along CO and TO.
Q.E.D.

Corollary 1

11. As the direction of the force along CO passes through the center of the earth, it contributes neither to the rotational movement of the earth nor to the to the changing of its poles. And, if the earth were subject to this force alone, neither its rotational movement nor the position of its poles would undergo change.

Corollary 2

12. Therefore, it is only necessary to consider the force along TO, if we wish to investigate the changes that the attraction from point O can produce in the rotational movement of the earth and in the position of the poles. And to this end we easily understand that these effects result only from the moment of this force.

Corollary 3

13. Now the moment of this force along TO is in the direction of a diameter of the equator which is perpendicular to the plane ABO, which is determined by the axis of the earth AB and the point O, and the moment of this force will be

$$= \frac{n+1}{5} \cdot M \cdot \frac{k^n}{h^n} \cdot \frac{g(ee-aa)}{h^3} \cdot CT.$$

This moment will therefore be

$$= \frac{n+1}{5} \cdot M \cdot \frac{k^n}{h^n} \cdot \frac{fg(ee-aa)}{h^3}.$$

Corollary 4

14. We could therefore conceive of a force equivalent to AG applied perpendicularly to the axis at A, which tends to move point A away from point O; this direction AG being [likewise situated] in the plane ABO. Therefore the moment of the force will be:

$$AG \cdot AC = \frac{n+1}{5} \cdot M \cdot \frac{k^n}{h^n} \cdot \frac{fg(ee-aa)}{h^3}.$$

And this moment will produce the same effect on the rotational movement of the earth as the force which attracts it toward point O.

Corollary 5

15. If we call the angle $ACO = \varphi$, which will denote the apparent distance from point O to the pole of the earth A, seen from the center of the earth, we will have $\frac{f}{h} = \sin\varphi$ and $\frac{g}{h} = \cos\varphi$. Therefore the moment in question will be

$$AG \cdot AC = \frac{n+1}{5} \cdot M \cdot \frac{k^n}{h^{n+1}}(ee-aa)\sin\varphi\cos\varphi.$$

Corollary 6

16. It is clear that the moment would vanish, if the earth were spherical, since then we would have $e = a$. But if the diameter of the equator EF exceeds the axis of the earth AB, or if $e > a$, then this moment is positive, such as it is presented in the figure. But if the diameter of the equator were smaller than that of the earth's axis, this moment would become negative and would consequently produce a contrary effect.

Corollary 7

17. This same moment will still become negative, although $e > a$, if the angle ACO is obtuse or larger than a right angle. For if $\varphi > 90°$, its cosine or $\cos\varphi$ will become negative, and therefore also the moment. From this we

would also note that this moment will vanish when the angle ACO is right, and that this same moment will be maximized when the angle $ACO = \varphi$ is half a right angle or a right angle and a half.

Problem 2

18. *The earth being supposed to have a spheroidical core about its center, if all its parts are attracted towards a center of force O, which attracts according to an arbitrary power of the distance, to find the moment of force to which the earth will be subjected.*

Solution

As before let the demi-axis of the earth be $CA = CB = a$, the radius of its equator $CE = CF = e$. For the core, let the demi-axis be $Ca = Cb = \alpha$, and the radius of its equator $Ce = Cf = \epsilon$. Then let the density of the core [be] to that of the crust as $1 + \nu$ is to 1. Then having dropped from the point O onto the plane of the equator the perpendicular OT, let $CT = f$, $TO = g$, $CO = h = \sqrt{ff + gg}$. Now at the distance $= z$ let the attractive force of the point O be $\frac{k^n}{z^n}$. Given this, and denoting the mass of the whole earth [by] M, the force by which the earth is attracted towards point O will produce the same effect on its rotational movement, as if one applied to the axis in the plane ABO a certain force AG, the moment of which would be

$$= \frac{(n+1)Mk^n fg}{5h^{n+3}} \cdot \frac{aee(ee - aa) + \nu\alpha\epsilon\epsilon(\epsilon\epsilon - \alpha\alpha)}{aee + \nu\alpha\epsilon\epsilon}.$$

Or rather denoting the angle $ACO = \varphi$, this moment will be

$$AC \cdot AG = \frac{(n+1)Mk^n \sin\varphi\cos\varphi}{5h^{n+1}} \cdot \frac{aee(ee - aa) + \nu\alpha\epsilon\epsilon(\epsilon\epsilon - \alpha\alpha)}{aee + \nu\alpha\epsilon\epsilon}.$$

Q.E.D.

Corollary 1

19. If the aberration of the spherical figure, both of the earth and its core, is extremely small, as one may surely suppose, this moment will be almost exactly

$$AC \cdot AG = \frac{(n+1)Mk^n \sin\varphi\cos\varphi}{5h^{n+1}} \cdot \frac{a^3(ee - aa) + \nu\alpha^3(\epsilon\epsilon - \alpha\alpha)}{a^3 + \nu\alpha^3}.$$

Corollary 2

20. If the core were exactly spherical, this moment would be

$$AC \cdot AG = \frac{(n+1)Mk^n \sin\varphi\cos\varphi}{5h^{n+1}} \cdot \frac{a^3(ee-aa)}{a^3+\nu\alpha^3}.$$

Now if the whole earth were composed of a uniform substance, this moment would be

$$= \frac{(n+1)Mk^n \sin\varphi\cos\varphi}{5h^{n+1}}(ee-aa).$$

Therefore the moment for an earth endowed with a spherical core will be smaller than that for an homogenous earth, if the core is denser than the crust. However if the core were less dense, the first moment would exceed the other.

Corollary 3

21. If the point O were attracted exactly by virtue of the inverse square of the distance, that is if $n=2$, then the moment we seek would be:

$$AC \cdot AG = \frac{(n+1)Mkk \sin\varphi\cos\varphi}{5h^3} \cdot \frac{a^3(ee-aa)+\nu\alpha^3(\epsilon\epsilon-\alpha\alpha)}{a^3+\nu\alpha^3}.$$

and this would be so in the case of the force of the sun on the earth.

Corollary 4

22. If the force from the point O were directly proportional to the distance, we would have $n=-1$, and so the moment would vanish entirely. This becomes quite clear from the outset, if one but reflects on the nature of this force.

Corollary 5

23. If the force were composed of two parts, or if it were expressed in the form $\frac{k^n}{z^n} \pm \frac{i^m}{z^m}$, we easily see that the moment we seek would be expressed in this manner:

$$\frac{M\sin\varphi\cos\varphi}{5h}\left(\frac{(n+1)k^n}{h^n} \pm \frac{(n+1)i^m}{h^m}\right)\frac{a^3(ee-aa)+\nu\alpha^3(\epsilon\epsilon-\alpha\alpha)}{a^3+\nu\alpha^3}.$$

Corollary 6

24. Therefore if the force of the moon were expressed at the distance z in the form $\frac{k^2}{h^2} - \delta$, where δ denotes a constant quantity, as several phenomena appear to confirm, as a result of $n = 2$, $m = 0$ and $\frac{i^m}{h^m} = \delta$, the moment in question would be, supposing the moon to be at O

$$AG \cdot AC = \frac{M \sin\varphi \cos\varphi}{5h} \left(\frac{3kk}{hh} \pm \delta \right) \frac{a^3(ee - aa) + \nu\alpha^3(\epsilon\epsilon - \alpha\alpha)}{a^3 + \nu\alpha^3}.$$

Corollary 7

25. This moment would therefore be smaller than it would be if the force of the moon were entirely due to the inverse square of the distance. It is surely appropriate to take into account the constant δ in the calculations, which I will undertake on the variation of the poles of the earth, in order to see if this new hypothesis is confirmed or not.

Problem 3

26. *The earth, while it turns about the axis CA, being subjected to a force AG applied to the end point A of this axis, of which the moment $AG \cdot AC$ is known, to find the instantaneous change that will be caused by this force in the axis of rotation.*

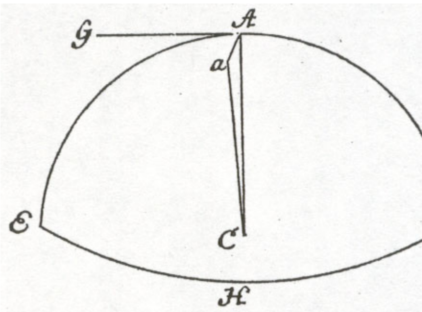

Fig. 3

Solution

Let C be the center of the earth, and CA the axis of the earth extended to the heavens, around which the earth is presently turning in the direction

EHF; of which the speed is such that in an infinitely small time $= dt$, it describes around its axis an angle $= ds$. Let us denote as before the mass of the earth $= M$, and its demi-axis $= a$. In this instant therefore A will be the pole of the earth projected into the heavens, around which the earth would continue to turn uniformly, if it were not subjected to any external force. But as it is subjected to the force AG, of which the moment $AG \cdot AC$ is $= S$, this force, since it passes through the axis of the earth will change nothing in the speed of rotation, but it would require the earth to turn around another axis, which will be such that after an infinitely small time $= dt$ the pole of the earth will no longer correspond to the point A in the heavens, but to another point a situated on the meridian, which is perpendicular to the meridian AE, which corresponds to the direction of the force AG, in the direction of the rotational movement. And by the principles of the Mechanics, that I will explain elsewhere, we find that this change, or the angle ACa, will be expressed as follows:

$$ACa = \frac{Sdt^2}{2ds} : \frac{2}{5}Maa = \frac{5Sdt^2}{4Maads},$$

where $\frac{2}{5}Maa$ denotes the moment of inertia of the earth, supposing it to be homogenous. Now if the earth had a core around its center, of which the radius $= \alpha$, and the density to that of the crust were as $1 + \nu : 1$, then in place of $\frac{2}{5}Maa$ we would need to write

$$\frac{2}{5}M \cdot \frac{a^5 + \nu\alpha^5}{a^3 + \nu\alpha^3}.$$

Hence the [instantaneous] change of the pole would be:

$$ACa = \frac{Sdt^2}{2ds} : \frac{2}{5}M\frac{a^5 + \nu\alpha^5}{a^3 + \nu\alpha^3} = \frac{5Sdt^2(a^3 + \nu\alpha^3)}{4Mds(a^5 + \nu\alpha^5)}.$$

Q.E.D.

Corollary 1

27. The effect of such a force AG will thus consist in how the poles about which the earth turns correspond over time to other points in the heavens. And it is also clear that the line Ca will pass through different points of the earth from the line CA, in such a way that the location of the poles on the

earth will also change continually. But they will always differ so slightly from the extremities of the earth's axis that the difference, being only about $1\frac{1}{3}$ third parts, is totally imperceptible.

Corollary 2

28. If we conceive of a center of forces O, toward which the earth is attracted, in the meridian AF opposite to AE, we have seen that from this force results a moment such that $AG{\cdot}AC = S$. Therefore the effect of this force will be the same as that which we have just determined in the solution of this problem.

Corollary 3

29. Thus if we suppose the distance from this center of forces to the earth $CO = h$, the angle ACO or the arc $AO = \varphi$, the force to the distance z as $\frac{k^n}{z^n}$, the moment $AG \cdot AC$ will be

$$S = \frac{(n+1)Mk^n}{5h^{n+1}} \sin\varphi\cos\varphi(ee - aa),$$

supposing the earth homogenous (see §15). Consequently the change of the pole caused by the this force will be

$$ACa = \frac{(n+1)k^n}{4h^{n+1}} \cdot \frac{dt^2}{a^2 ds}(ee - aa)\sin\varphi\cos\varphi.$$

Corollary 4

30. If the earth is not homogenous, but it has a core around its center, of which the demi-axis $= \alpha$, the demi-diameter of its equator $= \epsilon$, and the density to that of the crust is $1+\nu$ to 1, we have seen that it will be:

$$S = \frac{(n+1)Mk^n \sin\varphi\cos\varphi}{5h^{n+1}} \cdot \frac{a^3(ee - aa) + \nu\alpha^3(\epsilon\epsilon - \alpha\alpha)}{a^3 + \nu\alpha^3}$$

from which the change in the pole will be

$$ACa = \frac{(n+1)k^n}{4h^{n+1}} \cdot \frac{a^3(ee - aa) + \nu\alpha^3(\epsilon\epsilon - \alpha\alpha)}{a^5 + \nu\alpha^5} \cdot \frac{dt^2}{ds}\sin\varphi\cos\varphi.$$

Remark

31. To give the expression of time in terms of better known quantities, let us consider the mean motion of the earth. Let the average distance of the earth to the sun $= b$, the attractive force of the sun to the earth $= \frac{cc}{bb}$, and that during an element of time dt the earth travels in its mean motion through an angle with respect to the sun $= dv$. Given this, by the same principles of Mechanics we find $bdv^2 = \frac{ccdt}{2bb}$, from which we have $dt^2 = \frac{2b^3}{cc}dv^2$. Therefore, we need only write this value $\frac{2b^3}{cc}dv^2$ in place of dt^2 to express the time in terms of the mean motion of the sun. Then while the sun advances by its mean motion through the angle dv, and the earth through its diurnal motion through the angle ds, the change of the pole in the case of the preceding corollary will be:

$$ACa = \frac{(n+1)k^n}{2h^{n+1}} \cdot \frac{b^3}{cc} \cdot \frac{a^3(ee-aa)+\nu\alpha^3(\epsilon\epsilon-\alpha\alpha)}{a^5+\nu\alpha^5} \cdot \frac{dv^2}{ds} \sin\varphi\cos\varphi.$$

Now ds is to dv as the diurnal motion of the earth is to its annual motion. Therefore since the earth makes each revolution about its axis or 360° in 23h, 56', 4", and since in this time the average motion of the sun is 58', 58", we have $ds = 366\frac{25}{81}dv$. And therefore the change found in the pole will be:

$$ACa = \frac{1}{732\frac{50}{81}} \frac{(n+1)k^n}{h^{n+1}} \cdot \frac{b^3}{cc} \cdot \frac{a^3(ee-aa)+\nu\alpha^3(\epsilon\epsilon-\alpha\alpha)}{a^5+\nu\alpha^5} dv \sin\varphi\cos\varphi.$$

Problem 4

32. To find the elementary change of the pole of the earth in the heavens, given that it is caused as much by the force of the sun as by that of the moon.

Solution

Let a remain the demi-axis of the earth, e the radius of its equator, and let the earth be homogenous with the exception of a spheroidical core about its center, of which the demi-axis is $= \alpha$, the radius of its equator $= \epsilon$, and the density to that of the crust as $1+\nu$ is to 1. For simplicity let

$$\frac{a^3(ee-aa)+\nu\alpha^3(\epsilon\epsilon-\alpha\alpha)}{a^r+\nu\alpha^5} = N$$

and let the north pole of the earth at the given instant correspond to the point A in the heavens. Now let the sun be at O, and the arc of the meridian $AO = \varphi$; the distance of the sun to the earth $= b$, and its force on the earth $= \frac{cc}{bb}$. Given this we will have $h = b$, $k = c$, and $n = 2$. Therefore in the time that the average movement of the sun advances an infinitely small angle dv, the pole of the earth will be moved from A to a, in a meridian perpendicular to the meridian of AOF, where the sun is found, in such a way that this change will be

$$Aa \quad \text{or} \quad Aca = \frac{3}{732\frac{50}{81}} \cdot N\,dv\,\sin\varphi\,\cos\varphi$$

supposing that the rotational movement of the earth is made in the direction EHF, and this will be the momentary effect of the force of the sun.

Now if the moon is located at O, of which the distance of the earth $= h$, and the force $= \frac{kk}{hh} - \delta$, as I have supposed above (24), and the value of $\frac{(n+1)k^n}{h^{n+1}}$ will change to $\frac{3kk}{h^3} - \frac{\delta}{h}$. Also let the arc $AO = \varphi$, and while the mean motion of the sun $= dv$, the force of the moon will move the pole of the earth from A to a also in a meridian perpendicular to AOF, in such a way that Aa, or

$$ACa = \frac{1}{732\frac{50}{81}} \left(\frac{3kk}{h^3} - \frac{\delta}{h}\right) \frac{b^3}{cc} N\,dv\,\sin\varphi\,\cos\varphi.$$

Now we must note here that $\frac{3kk}{h^3} - \frac{\delta}{h}$ is to $\frac{3cc}{b^3}$ as the force of the moon to cause the tides is to that of the sun, a ratio which Newton has established as $4\frac{1}{2}$ to 1. But as this determination is not too certain, let us suppose that in the production of the flux and reflux of the ocean, the force of the moon is to that of the sun as m to 1, and we will have

$$\frac{3kk}{h^3} - \frac{\delta}{h} = \frac{3mcc}{b^3},$$

and consequently the change of the pole caused by this force of the moon will be:

$$Aa \quad \text{or} \quad Aca = \frac{3m}{732\frac{50}{81}} \cdot N\,dv\,\sin\varphi\,\cos\varphi$$

Q.E.D.

Corollary 1

33. If we let

$$\frac{3}{732\frac{50}{81}}N = \lambda$$

for the sake of simplicity, the sun being supposed at O in such a way that $AO = \varphi$, the change of the pole will be $Aa = \lambda dv \sin\varphi\cos\varphi$. Now if we suppose the moon at O, then the change of the pole will be $Aa = \lambda m dv \sin\varphi\cos\varphi$.

Corollary 2

34. Thus if we should find by observing the changes of the pole of the earth the value of the number λ, we will derive from this $N = 732\frac{50}{81}\cdot\frac{1}{3}\lambda$; and from this value we would be able to determine the core of the earth, since

$$N = \frac{a^3(ee - aa) + \nu\alpha^3(\epsilon\epsilon - \alpha\alpha)}{a^r + \nu\alpha^5}.$$

Remark

35. If the earth had no core at all, but were formed from a homogenous matter, the value of N would be known. For since the diameter of the equator surpasses the axis by a 200^{th} part, as was concluded by the observations done in the North, in France, and in Peru, we have $a : e = 200 : 201$, and in this case we will have

$$N = \left(\frac{201}{200}\right)^2 - 1 = \frac{1}{100} \quad \text{and so} \quad \lambda = \frac{3}{100\cdot 732\frac{50}{81}} = \frac{1}{24,421}.$$

Problem 5

36. *Supposing that the sun moves according to its mean motion, to find the changes that its force will cause in the poles of the earth.*

Solution

Let (Fig. 4) $\Upsilon\Xi\Lambda$[1] be the ecliptic and Π be the pole of the ecliptic; however Υ does represent the point of the equinox, but rather some fixed point in the

[1]In the original, Euler uses the zodiacal symbols for Aries, Cancer and Libra here, as in Figure 4.

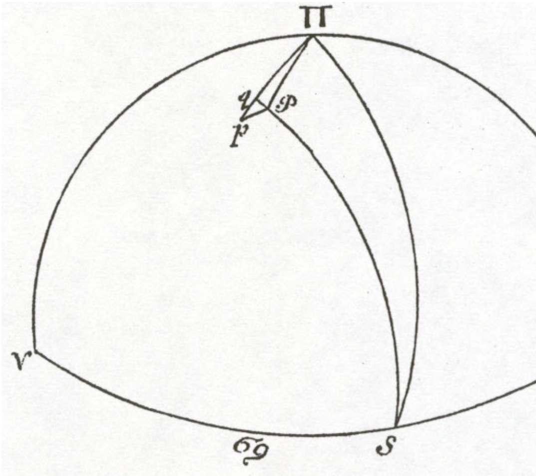

Fig. 4

heavens, such as $1 * \Upsilon^2$. Let the longitude of the sun at S from this fixed point be $\Upsilon S = p$, and let at this time the pole of the earth be found at P, and let $\Upsilon\Pi P = x$, $\Pi P = y$. Let us draw the arcs of the great circles ΠS, PS, and in the spherical triangle $P\Pi S$ we will have $P\Pi S = p - x$; $P\Pi = y$ and $\Pi S = 90°$, from which we derive

$$\cos PS = \cos(p-x)\sin y \quad \text{and} \quad \cot \Pi PS = -\frac{\cos(p-x)\cos y}{\sin(p-x)}.$$

Now setting $PS = \varphi$, the pole will be moved from P to p, so that Pp will be perpendicular to PS and $Pp = \lambda dv \sin\varphi\cos\varphi$, while the sun advances through the angle dv; we will therefore have $dp = dv$, and having drawn Πp, we have $\Xi\Pi p = x - dx^3$, and $\Pi p = y + dy$. Thus dropping the perpendicular Pq to ΠP, we will have $pq = dy$ and $Pq = -dx\sin y$. Now the angle pPq being the compliment of the angle ΠPS in two right angles, we will have

$$\cot pPq = \frac{\cos(p-x)\cos y}{\sin(p-x)} \quad \text{and} \quad \tan Ppq = \frac{\cos(p-x)\cos y}{\sin(p-x)}.$$

Consequently we will have:

$$Pq = -dx\sin y = Pp\cdot\sin Pqp \quad \text{and} \quad pq = dy = Pp\cos Pqp.$$

[2]Again, Euler uses the symbol for Aries here, in referring to a particular star that constellation.

[3]In the original edition, this was $x + dx$; this corection was made by Leo Courvoisier, editor of vol. II.30 of Euler's *Opera Omnia*.

Now since $\cos\varphi = \cos(p-x)\sin y$; we will have

$$\sin Pqp = \frac{\cos(p-x)\cos y}{\sin y} \quad \text{and} \quad \cos Pqp = \sin(p-x)\sin\varphi.$$

Therefore since $Pp = \lambda dp \sin\varphi\cos\varphi$, we will obtain

$$-dx \sin y = \lambda dp \cos\varphi \cos(p-x)\cos y = \lambda dp \cos^2(p-x)\cos y \sin y$$

and

$$dy = \lambda dp \cos\varphi \sin(p-x) = \lambda dp \sin(p-x)\cos(p-x)\sin y.$$

But since $\sin(p-x)\cos(p-x) = \frac{1}{2}\sin 2(p-x)$ and $\cos^2(p-x) = \frac{1}{2}+\frac{1}{2}\cos 2(p-x)$, these equations will take the following form:

$$\begin{aligned} -dx &= \frac{1}{2}\lambda dp(1+\cos 2(p-x))\cos y \\ \text{and} \quad dy &= \frac{1}{2}\lambda dp \sin 2(p-x)\sin y. \end{aligned}$$

Now since it is easy to see that the variability of x and y is infinitely small with respect to that of p, in the integration we will be able to consider x and y as though they were constant quantities, and so the integrals will be:

$$\begin{aligned} x &= C - \frac{1}{2}\lambda\left(p+\frac{1}{2}\sin 2(p-x)\right)\cos y \\ y &= C - \frac{1}{2}\lambda\cos 2(p-x)\sin y \end{aligned}$$

from which we will know at any given time the location of the pole of the earth in the heavens.
Q.E.D.

Corollary 1

37. Since $p-x$ denotes $P\Pi S$, and the circle ΠP passes through the summer solstice, this angle $p-x$ will express the longitude of the sun measured from the summer solstice. Thus if we set the longitude for the sun from the vernal equinox $= p$, we will have $p - x = p - 90^\circ$, and $2(p-x) = 2p - 180^\circ$. From this we will have

$$x = C - \frac{1}{2}\lambda(p - \frac{1}{2}\sin 2p)\cos y \quad \text{and} \quad y = C + \frac{1}{4}\lambda\cos 2p \sin y.$$

Corollary 2

38. In these formulas y denotes the mean distance of the pole of the earth from that of the ecliptic. Therefore setting this mean distance of the poles $= \vartheta$, we will have

$$\begin{aligned} x = C \quad &- \quad \frac{1}{2}\lambda v \cos\vartheta + \frac{1}{4}\lambda \sin 2p \cos\vartheta \\ \text{and} \quad y \quad &= \quad \vartheta + \frac{1}{4}\lambda \cos 2p \sin\vartheta. \end{aligned}$$

Now the value of ϑ is approximately $23°28'30''$.

Corollary 3

39. Setting $\vartheta = 23°28'30''$, we will have, reducing the sine and cosine of ϑ to angles expressed in seconds[4],

$$\begin{aligned} x &= C - 0.458617''\lambda v + 47,298''\lambda \sin 2p \\ y &= \vartheta + 20,541''\lambda \cos 2p. \end{aligned}$$

Corollary 4

40. From this it is clear that the force of the sun makes the pole of the earth recede; for after one year, when v becomes $v + 360°$, the longitude of the pole for $1 * \Upsilon$ will be

$$x = C - 0.458617\lambda(v + 360°) + 47,298''\lambda \sin 2p.$$

Thus during one year the pole of the earth recedes in longitude by the quantity $= 0.458617\lambda(v + 360°)$.

Corollary 5

41. Thus if the earth were formed from a uniform matter, in such a way that $\lambda = \frac{1}{24421}$, the annual precession of the pole and also of the equinoxes would

[4]The second coefficient on the next line was 46,222 in the original edition. We use the value 47,298 as given by Courvoisier in the *Opera Omnia*

be $= 24\frac{1}{3}$ seconds; under the assumption that it be caused only by the action of the sun. And under the same hypothesis, after A years, we will have

$$\begin{aligned} x &= C - 24\frac{1}{3}A'' + 1\frac{8}{9}'' \sin 2p \\ y &= \vartheta + \frac{5}{6}'' \cos 2p. \end{aligned}$$

Therefore in addition to the mean motion, the pole will be drawn away from its mean location by almost $2''$ in longitude, and by almost a second in latitude.

Problem 6

42. *Supposing the motion of the moon to be uniform about the earth, to find the changes that the force of the moon will cause in the poles of the earth.*

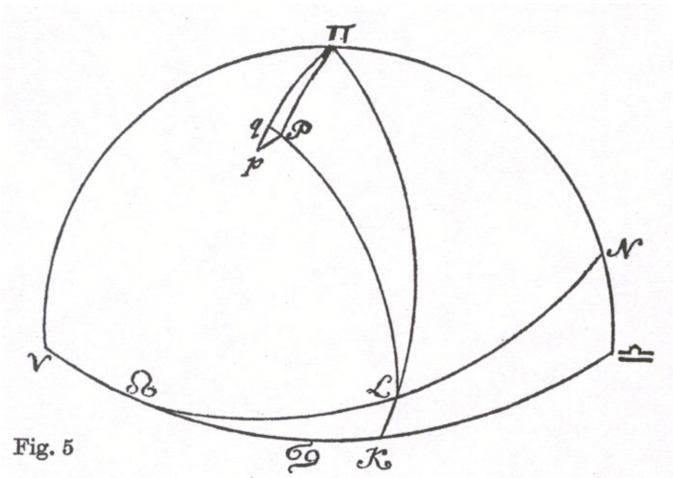

Fig. 5

Solution

Letting $\Upsilon\Xi\Lambda$ (Fig. 5) be the ecliptic as before, and Π its pole, and let the pole of the earth be found at this hour at P, and let us denote as before the angle $\Upsilon\Pi P = x$, $\Pi P = y$. At the same time let the moon be at L, and letting its distance to the pole $LP = \varphi$, we have seen from the above that as long as the mean motion of the sun is $= dv$, the pole P will be moved to p, along the element Pp perpendicular to PL, in such a way that

$$Pp = \lambda m dv \sin\varphi \cos\varphi,$$

from which we conclude as in the preceding problem:

$$\begin{aligned} -dx \sin y &= \lambda m dv \sin\varphi \cos\varphi \sin Ppq \\ \text{and} \quad dy &= \lambda m dv \sin\varphi \cos\varphi \cos Ppq. \end{aligned}$$

It is therefore a matter of determining the angles φ and Ppq. To this end, let ΩLN[5] be the orbit of the moon and from Ω the ascending node with longitude $\Upsilon\Omega = r$, and the inclination of the lunar orbit to the ecliptic $L\Omega K = \varrho$. Since we are supposing the motion of the moon to be uniform, it will not matter if we attribute this uniformity to the movement in the orbit itself, or to a movement in longitude, because the difference is extremely small. Therefore let q be the longitude of the moon $= \Upsilon K$ in such a way that the angle $\Upsilon\Pi L = q$, and we will have $\Omega K = q - r$. From this we will find the latitude LK, which I will denote by $s = LK$, and we will have $\tan s = \sin(q-r)\tan\varrho$. Now in the triangle $P\Pi L$ having $P\Pi = y$, $P\Pi L = q - x$ and $\Pi L = 90° - s$ we will have:

$$\cos PL = \cos\varphi = \cos(q-x)\sin y \cos s + \cos y \sin s$$

and

$$\cot \Pi PL = \frac{\sin y \sin s - \cos(q-x)\cos y \cos s}{\sin(q-x)\cos s} = -tanPqp$$

so that

$$\tan Ppq = \frac{\cos(q-x)\cos y \cos s - \sin y \sin s}{\sin(q-x)\cos s},$$

and from there we will conclude:

$$\sin Ppq = \frac{\cos(q-x)\cos y \cos s - \sin y \sin s}{\sin\varphi}$$

$$\text{and} \quad \cos Pqp = \frac{\sin(q-x)\cos s}{\sin\varphi}.$$

[5]In the original, Euler used the zodiacal symbol for Leo here, as in Figure 5.

Substituting these values will give:

$$\begin{aligned} -dx\sin y &= \lambda m dv \cos\varphi(\cos(q-x)\cos y\cos s - \sin y \sin s) \\ dy &= \lambda m dv\cos\varphi\sin(q-x)\cos s \end{aligned}$$

or also substituting for $cos\varphi$ its value:

$$\begin{aligned} -dx\sin y &= -\lambda m dv(\cos(q-x)\sin y\cos s + \cos y\sin s) \\ &\qquad \times(\cos(q-x)\cos y\cos s - \sin y\sin s) \\ dy &= \lambda m dv(\cos(q-x)\sin y\cos s + \cos y \sin s)\sin(q-x)\cos s. \end{aligned}$$

Now

$$\begin{aligned} \sin s &= \frac{\tan\varrho\sin(q-r)}{\sqrt{1+\tan^2\varrho\sin^2(q-r)}} \\ \text{and} \quad \cos s &= \frac{1}{\sqrt{1+\tan^2\varrho\sin^2(q-r)}} \end{aligned}$$

which gives:

$$\begin{aligned} dx\sin y &= \frac{-\lambda m dv}{\sqrt{1+\tan^2\varrho\sin^2(q-r)}} \\ &\quad \cdot \Big[\cos^2(q-x)\sin y\cos y + \cos(q-x)\cos^2 y\tan\varrho\sin(q-r) \\ &\quad - \sin y\cos y\tan^2\varrho\sin^2(q-r) - \cos(q-x)\sin^2 y\tan\varrho\sin(q-r)\Big] \end{aligned}$$

and

$$dy = \frac{\lambda m dv[\sin(q-x)\cos(q-x)\sin y + \sin(q-x)\cos y\tan\varrho\sin(q-r)]}{\sqrt{1+\tan^2\varrho\sin^2(q-r)}}$$

To integrate these formulas we must note that we may consider x, y, and ϱ to be constant quantities. Therefore let $\tan\varrho = \gamma$, which is a very small quantity, and let ϑ be the mean distance from the pole of the earth to the pole of the ecliptic. Thus we will have:

$$dx\sin\vartheta = -\frac{1}{2}\lambda m dv \cdot$$

$$\frac{\sin 2\vartheta\cos^2(q-x) + 2\gamma\cos 2\vartheta\cos(q-x)\sin(q-r) - \gamma^2\sin 2\vartheta\sin^2(q-r)}{1+\gamma\gamma\sin^2(q-r)}$$

$$dy = \lambda mdv \cdot \frac{\sin\vartheta \sin(q-x)\cos(q-x) + \gamma\cos\vartheta \sin(q-x)\sin(q-r)}{1+\gamma\gamma\sin^2(q-r)}$$

Now since γ is a very small number, knowing that it is the tangent of an angle of about 5°, we may replace these denominators by performing a long division, and it will be permissible to neglect those terms in which the power of γ is greater than two. We will thus have:

$$\begin{aligned}
dx\sin\vartheta &= -\frac{1}{2}\lambda mdv \Big[\sin 2\vartheta\cos^2(q-x) + 2\gamma\cos 2\vartheta\cos(q-x)\sin(q-r) \\
&\quad -\gamma\gamma\sin 2\vartheta\sin^2(q-r)(1+\cos(q-x)^2)\Big] \\
dy &= \lambda mdv \left[\sin\vartheta\sin(q-x)\cos(q-x) + \gamma\cos\vartheta\sin(q-x)\sin(q-r)\right. \\
&\quad \left.\gamma\gamma\sin\vartheta\sin(q-x)\cos(q-x)\sin^2(q-r)\right]
\end{aligned}$$

Now we need to reduce these products of the sines and cosines of the variable angles to the sines and cosines of simple angles, and since $\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$; $\sin A\cos A = \frac{1}{2}\sin 2A$ and $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$, we will have:

$$\begin{aligned}
dx\sin\vartheta &= -\frac{1}{2}\lambda mdv \left[\frac{1}{2}\sin 2\vartheta + \frac{1}{2}\sin 2\vartheta\cos 2(q-x)\right. \\
&\quad +2\gamma\cos 2\vartheta\cos(q-x)\sin(q-r) \\
&\quad \left.-\gamma\gamma\sin 2\vartheta\sin 2(q-x)\left(\frac{1}{2}-\frac{1}{2}\cos 2(q-r)\right)\left(\frac{3}{2}+\frac{1}{2}\cos 2(q-x)\right)\right] \\
dy &= \lambda mdv \left[\frac{1}{2}\sin\vartheta\sin 2(q-x)\right. \\
&\quad +\gamma\cos\vartheta\sin(q-x)\sin(q-r) \\
&\quad \left.-\frac{1}{2}\gamma\gamma\sin\vartheta\sin 2(q-x)\left(\frac{1}{2}-\frac{1}{2}\cos 2(q-r)\right)\right].
\end{aligned}$$

Moreover since

$$\begin{aligned}
\cos A\sin B &= \frac{1}{2}\sin(A+B) - \frac{1}{2}\sin(A-B) \\
\sin A\sin B &= \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B) \\
\cos A\cos B &= \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)
\end{aligned}$$

after we have made all these reductions we will have:

$$dx\sin\vartheta = -\frac{1}{2}\lambda mdv \left[\frac{1}{2}\sin\vartheta + \frac{1}{2}\sin\vartheta\cos 2(q-x)\right.$$

$$
\begin{aligned}
& +\gamma\gamma\cos 2\vartheta\sin(2q-r-x) - \gamma\cos 2\vartheta\sin(r-x) \\
& -\frac{3}{4}\gamma\gamma\sin 2\vartheta + \frac{3}{4}\gamma\gamma\sin 2\vartheta\cos 2(q-r) - \frac{1}{4}\gamma\gamma\sin 2\vartheta\cos 2(q-x) \\
& +\frac{1}{8}\gamma\gamma\sin 2\vartheta\cos 2(r-x) + \frac{1}{8}\gamma\gamma\sin 2\vartheta\cos 2(2q-r-x)\Big] \\
dy \;=\; & \lambda m dv\left[\frac{1}{2}\sin\vartheta\sin 2(q-x) + \frac{1}{2}\gamma\cos\vartheta\cos(r-x)\right. \\
& -\frac{1}{2}\gamma\cos\vartheta\cos(2q-r-x) - \frac{1}{4}\gamma\gamma\sin\vartheta\sin 2(q-x) \\
& \left.+\frac{1}{8}\gamma\gamma\sin\vartheta\sin 2(2q-r-x) + \frac{1}{8}\gamma\gamma\sin\vartheta\sin 2(r-x)\right]
\end{aligned}
$$

To integrate these equations we must take note that the differentials of q and r are given with regard to the differential dv, which are those of the mean motion of the moon and of the retrograde motion of the node to the mean motion of the sun. Therefore let:

$$dq = \mu dv \quad \text{and} \quad dr = -\kappa dv$$

and the desired integrals will be:

$$
x\sin\vartheta = C - \frac{1}{2}\lambda m \left\{
\begin{array}{l}
\frac{1}{2}v\sin 2\vartheta + \frac{1}{4\mu}\sin 2\vartheta\sin 2(q-x) \\
-\frac{\gamma}{2\mu+\kappa}\cos 2\vartheta\cos(2q-r-x) \\
-\frac{\gamma}{\kappa}\cos 2\vartheta\cos(r-s) - \frac{3}{4}\gamma\gamma v\sin 2\vartheta \\
+\frac{3\gamma\gamma}{8(\mu+\kappa)}\sin 2\vartheta\sin 2(q-r) \\
-\frac{\gamma\gamma}{8\mu}\sin 2\vartheta\sin 2(q-x) \\
-\frac{\gamma\gamma}{16\kappa}\sin 2\vartheta\sin 2(r-x) \\
+\frac{\gamma\gamma}{16(2\mu+\kappa)}\sin 2\vartheta\sin 2(2q-r-x)
\end{array}\right.
$$

$$
y = \vartheta + \lambda m \left\{
\begin{array}{l}
-\frac{1}{4\mu}\sin 2\vartheta\cos 2(q-x) - \frac{\gamma}{2\kappa}\cos\vartheta\sin(r-x) \\
-\frac{\gamma}{2(2\mu+\kappa)}\cos\vartheta\sin(2q-r-x) \\
+\frac{\gamma\gamma}{8\mu}\sin\vartheta\cos 2(q-x) \\
-\frac{\gamma\gamma}{16(2\mu+\kappa)}\sin\vartheta\cos 2(2q-r-x) \\
+\frac{\gamma\gamma}{16\kappa}\sin\vartheta\cos 2(r-x)
\end{array}\right.
$$

Q.E.D.

Corollary 1

43. Since ϑ is more or less $23°28'30''$ and γ is the tangent of the inclination of the orbit of the moon to the ecliptic, the mean value of which is $= 5°9'$,

these expressions reduced to seconds will be[6]:

$$
\begin{aligned}
x &= C - 0.45302\lambda mv - \frac{46685}{\mu}\lambda m \sin 2(q-x) \\
&\quad + \frac{16952}{2\mu+\kappa}\lambda m \cos(2q-r-x) + \frac{16952}{\kappa}\lambda m \cos(r-x) \\
&\quad - \frac{576}{\mu+\kappa}\lambda m \sin(q-r) + \frac{96}{\kappa}\lambda m \sin(r-x) \\
&\quad - \frac{96}{2\mu+\kappa}\lambda m \sin 2(2q-r-x) \\
y &= \vartheta - \frac{20008}{\mu}\lambda m \cos 2(q-x) - \frac{8525}{\kappa}\lambda m \sin(r-x) \\
&\quad - \frac{8525}{2\mu+\kappa}\lambda m \sin(2q-r-x) - \frac{40}{2\mu+\kappa}\lambda m \cos 2(2q-r-x) \\
&\quad + \frac{40}{\kappa}\lambda m \cos 2(r-x)
\end{aligned}
$$

Corollary 2

44. Now since $\mu : 1$ as the mean motion of the moon is to that of the sun, we will have $\mu = 13,368$, and since $\kappa : 1$ as the mean retrograde motion of the node is to the motion of the sun, we will have $\kappa = \frac{1}{18,616}$. These values being substituted, our integral equations will become:

$$
\begin{aligned}
x = C &- 0.45302\lambda mv - 3,492\lambda m \sin 2(q-x) + 594\lambda m \cos(2q-r-x) \\
&+ 296,535\lambda m \cos(r-x) - 43\lambda m \sin 2(q-r) \\
&+ 1,788\lambda m \sin 2(r-x) - 4\lambda m \sin 2(2q-r-x) \\
y = \vartheta &- 1,497\lambda m \cos 2(q-x) - 158,718\lambda m \sin(r-x) \\
&- 363\lambda m \sin(2q-r-x) - 2\lambda m \cos 2(2q-r-x) \\
&+ \lambda m \cos 2(r-x)
\end{aligned}
$$

Corollary 3

45. From this we clearly see that if the terms $\cos(r-x)$ and $\sin(r-x)$ are not too large, that is if they do not exceed $100''$, which we recognize immediately,

[6] The factors λm were omitted from the final term in the expression for y in the version printed in Euler's *Opera Omnia*

then all the other terms can be ignored without error due to their extremely small magnitude, and consequently our formulas will become much simpler.

$$\begin{aligned} x &= C - 0.45302\lambda mv + 286,535\lambda m\cos(r-x) \\ &= \vartheta + 158,718\lambda m\sin(r-x) \end{aligned}$$

Corollary 4

46. Here v denotes the mean motion of the sun counting from a given epoch, where the location of the pole is assumed to be known, and $r-x$ denotes the longitude of the ascending node from the present location of the pole, that is to say, from the summer solstice. Thus, if we let the longitude of the ascending node from the vernal equinox $= u$, we will have $r-x = u-90°$ and our formulas will be:

$$\begin{aligned} x &= C - 0.45302\lambda mv + 296,535\lambda m\sin u \\ y &= \vartheta + 158,718\lambda m\cos u \end{aligned}$$

Remark

47. Since all the terms that depend on the mean motion of the moon were taken out of our formulas because of their smallness, we will easily understand that if I had introduced into the calculation the true motion of the moon, having taken into account its apogee and the position of the sun, all the terms that would have been introduced into our formulas, would have vanished for even stronger reasons. For then in place of q we would have had $q+A+B+C+\ldots$, where A, B, C, denote angles dependent on the anomaly of the moon and its elongation from the sun; and it is clear that the terms containing $q+A+B+C+\ldots$ would be taken out of the calculation in the same way as those in our formulas that contained q. Now, it is not the same for the terms which contain the position of the ascending node, because the latitude of the moon depends on them, which particularly affects the calculation. From there we understand that if the movement of the node were faster than it actually is, the inequalities in the movement of the pole would be smaller for the same reason; and these inequalities would be especially large, if the movement of the nodes were slower. Now if the nodes remained fixed, the mean motion of the pole would be slowed down by it, and because r is constant, dx would no longer be almost infinitely small in relation to dr

but rather infinitely large, and therefore we would be obliged to look for the integral of our formulas by an entirely different method.

Problem 7

48. *To determine the variation that we need to find in the position of the poles of the earth, given that they are caused conjointly by the forces of the sun and of the moon.*

Solution

Let ϑ be the mean distance from the north pole of the earth to the pole of the ecliptic, and for a given epoch let the mean longitude of the pole of the earth, from a fixed star, such as $1 * \Upsilon$, be $= \zeta$. At present it is a matter of determining the position of the north pole of the earth, letting its true longitude $= x$ and its true latitude $= 90^\circ - y$[7]; and that from the given epoch up till now, let the mean motion of the sun $= v$. Further for the present time, let the longitude of the sun from the vernal equinox $= p$ and the longitude of the ascending node $= u$; where we need to suppose that the position of the pole is already approximately known. This established, we have seen that by virtue of the force of the sun we will have[8]:

$$\begin{aligned} x &= C - 0.458617\lambda v + 46,222\lambda \sin 2p \\ y &= \vartheta + 20,541\lambda \cos 2p. \end{aligned}$$

Now the force of the moon gives

$$\begin{aligned} x &= C - 0.45302\lambda m v + 296,535\lambda m \sin u \\ y &= \vartheta + 158,718\lambda m \cos u \end{aligned}$$

Therefore these two forces working together will produce the effect contained in the following formulas[9]

$$x = \zeta \quad - \quad 0.45862\lambda v + 46,222\lambda \sin 2p$$

[7]In the original edition, this was y. The correction was made by Courvoisier.

[8]The last coefficient in the first expression for x was corrected from 46,222 to 47,298 by Courvoisier in §39, but the value of 46,222 appears twice in this paragraph, so we have reproduced that value from the *Opera Omnia* here. Also, the first coefficient of y was actually given as 20,641 in the *Opera Omnia* edition; we have changed this to the correct value of 20,541.

[9]The coefficient 46,222 was mistakenly given as 56,222 in the *Opera Omnia*.

$$\begin{aligned} & - \quad 0.45302\lambda mv + 296,535\lambda m \sin u \\ y & = \vartheta + 20,541\lambda \cos 2p + 158,718\lambda m \cos u \end{aligned}$$

Q.E.D.

Corollary 1

49. Here we can distinguish two sorts of movement. The first contains the mean movement of the pole, and is given by these two formulas

$$x = \zeta - 0.45872\lambda v - 0.45302\lambda mv \quad \text{and} \quad y = \vartheta$$

by which we will find the mean position of the pole, for any given time. Then the other terms make up the equations, which are needed to correct the mean longitude as well as the mean latitude of the pole.

Corollary 2

50. Regarding the mean position of the pole, we see that its distance to the pole of the ecliptic is always the same, but that the longitude decreases more and more; that is to say the movement of the pole in longitude will be retrograde in relation to the fixed starts, and this retrogression in a year will be

$$\text{of} \quad 1,296,000\lambda(0.45862 + 0.45302m) \quad \text{seconds}$$

$$\text{or of} \quad 594,371\lambda + 587,114\lambda m \quad \text{seconds.}$$

Corollary 3

51. Neglecting the inequalities that depend on the location of the sun, since they are very small, the true longitude of the pole will be greater than the mean, when the ascending node is found in the boreal signs; but if this node is in the austral signs, then the true longitude will be smaller than the mean. Now the difference between the true longitude and the mean longitude of the pole will be the greatest when the node is in the solisticial points; and this difference will consist of $296,537\lambda m$ seconds.

Corollary 4

52. The distance of the pole of the earth to that of the ecliptic will be greater when the ascending node is found in the vernal equinox: but if this node is in the autumnal equinox, then the pole of the earth will be found closer to that of the ecliptic. In these cases the greatest difference between the true latitude and the mean latitude of the pole will be of $158,718\lambda m$ seconds.

Remark 1

53. These corollaries are generally in agreement with the observations, by which we know that the movement of the earth's pole in the longitude is retrograde, and that every year this precession is equivalent to $50''$ or $51''$. Moreover Mr. Bradley has just discovered that the distance of the poles from the equator and from the ecliptic is greatest when the ascending node is at the beginning of Aries and smallest when this node enters into Libra and that the greatest difference between the true distance and the mean is $9''$. Also, what he observed in relation to the inequalities of the longitude is in complete agreement with what the theory just derived. Now as we do not know the true value of the quantities λ and m, these observations will enable us to determine these values, since we must have:

$$594,371\lambda + 587,113\lambda m = 50\frac{1}{2}''$$

and

$$158,718\lambda m = 9''$$

from there we conclude

$$\frac{594,371 + 587,113m}{158,718} = \frac{101}{18}$$

and hence m is more or less equal to 2. Here we should note that if the greater aberration in latitude were $11''$ instead of $9''$, we would find $m = 4$. Now $m : 1$ denotes the relation of the force of the moon to that of the sun in the production of the tides, and Newton had established this relation as 4 or $4\frac{1}{2}$ to 1. Mr. Daniel Bernoulli made this relationship much smaller by determining m as only $2\frac{1}{2}$; and this value is much closer to the one I have just found than to that of Newton, which is new proof not only for the belief of Mr. Bernoulli, which seems moreover very well founded, but also for the theory by which I have just determined the movement of the pole of the earth. For if we had found for m a value, whether too large of too small, it would have been a clear sign that the movement of the pole does not depend solely on the forces of the sun and the moon, but on centrifugal forces, not canceling themselves perfectly, also contributing something to it.

Remark 2

54. However it does not seem that we can diminish the value of m beyond $2\frac{1}{2}$, and it is rather probable that the greatest difference between the true latitude and mean latitude of the pole of the earth is a little larger than $9''$. Also Mr. Bradley himself acknowledged that this determination is not so precise that he could find it within a half-second. Thus, let us suppose according to Mr. Bernoulli that $m = 2\frac{1}{2}$, and the greatest difference between the mean latitude and true latitude of the pole of the earth will become $= 9.75$ or $9\frac{3}{4}$ seconds. Now if we had supposed the mean annual movement of the pole only $50''$ instead of $50\frac{1}{2}''$, we would have found 9.62 instead of 9.75, from which it seems that we may confidently agree with Mr. Bradley that the annual movement of the pole of the earth is $50.3''$, and with Mr. Bernoulli that $m = 2\frac{1}{2}$. From there we will find that value of λ to be

$$\lambda = \frac{50.3}{594,371 + 587,113 \cdot 2\frac{1}{2}} = \frac{1}{40,997}$$

and this value is considerably smaller than that which results from the homogenous earth hypothesis. From that it follows that the value of

$$\frac{a^3(ee - aa) + \nu\alpha^3(\epsilon\epsilon - \alpha\alpha)}{a^5 + \nu\alpha^5} \quad \text{is smaller than} \quad \frac{ee - aa}{aa}$$

in the ratio of $40,997$ to $24,421$ or of 5 to 3. And from there we must conclude that the earth is not homogenous, but that it has a much denser core at its center. Let us suppose that the his core spherical, and since $ee - aa = \frac{1}{100}aa$, we will have

$$\frac{\frac{1}{100}a^5}{a^5 + \nu\alpha^5} : \frac{1}{100} = 3 : 5^{10}, \quad \text{or} \quad 2a^5 = 3\nu\alpha^5.$$

Therefore if the density of this core were known, we could thus determine the radius of the core. Let us suppose, for example, that the core were 10 times

[10] 5:3 in the original edition. Correction due to Courvoisier.

denser than the crust, and we will have $\nu = 9$, and $\alpha^5 = \frac{2}{27}a^5$ or $\alpha = \frac{3}{5}a$. This does not seem contrary at all to observations which look at the interior of the earth; it seems instead that such a core is very consistent with the principles of physics.

Problem 8

55. *For an arbitrary given time to determine the longitude and latitude of the north pole of the earth, measuring the longitude from a given fixed star.*

Solution

Having drawn up the table for the mean retrograde movement in longitude of the pole at a rate of $50.3''$ per year, we will find for each given time the mean longitude of the pole, measured from a given fixed star such as $1 * \Upsilon$, provided that this longitude has been determined by observation at a given epoch. Therefore let η be this mean longitude of the pole from $1 * \Upsilon$ for the given time, and ϑ will be the mean distance of this pole from the pole of the elliptic, which is estimated as $23°28'30''$. Then let x be the true longitude of the pole from $1 * \Upsilon$, and y the true distance to the pole of the ecliptic. Given this, since we have $\lambda = \frac{1}{40{,}997}$ and $\lambda m = \frac{1}{16399}$, the formulas that will give us the true location of the pole for the given time will be:

$$\begin{aligned} x &= \zeta + 18.08'' \sin u + 1.13'' \sin 2p \\ y &= \vartheta + 9.68'' \cos u + 0.50'' \cos 2p \end{aligned}$$

where u denotes the longitude of the ascending node of the moon counted from the vernal equinox, and p is the longitude of the sun counted from the same origin. For as we already have the mean longitude of the pole, we will also have that of the equinoctial point, which precedes that by 3 signs. Now we see clearly that it is sufficient to know the longitudes u and p approximately, and that the error would be insignificant even if we should be mistaken by several degrees.

Corollary 1

56. Since x denotes the longitude of the pole from the first star of Aries, and since x corresponds to the point of the summer solstice, $x - 90°$ will express the longitude of the point of the vernal equinox from $1 * \Upsilon$. And from this

knowing the mean longitude of $1 * \Upsilon$ from the equinox for a given time, which is presumed to be $50.3''$ per year, which I will set $= E$, for the same time the true longitude of the first star of Aries, the beginning of the sign of the ram, will be

$$E - 18.08'' \sin u - 1/13'' \sin 2p.$$

Corollary 2

57. We may thus regard the equinoctial point as fixed in the Heavens and attribute its movement to the fixed stars in the opposite direction, and hence the longitude of the fixed stars will not only increase regularly by $50.3''$ per year, but it will be moreover subjected to a double inequality, of which one depends on the longitude of the ascending node of the moon, and the other on the longitude of the sun.

Corollary 3

58. The primary effect, which causes the stars to advance $50.3''$ in longitude per year, is the one which is called in astronomy the precession of the equinoxes, but the rules that the astronomers have derived from it only give the mean longitude of the stars. Now having found by this method the mean longitude of an arbitrary star, which is $= L$, its true longitude will be

$$L - 18.08'' \sin u - 1.13'' \sin 2p.$$

Corollary 4

59. Since y denotes the distance between the poles of the earth and the ecliptic, this will also be the value of the obliquity of the ecliptic. Thus letting the mean obliquity of the ecliptic $= \vartheta$, which we estimate at $23°28'30''$, for each proposed time the true obliquity of the ecliptic will be

$$\vartheta + 9.68'' \cos u + 0.50 \cos 2p.$$

Now it is evident that if the mean obliquity of the ecliptic were a little larger or a little smaller, the corrections would yet remain the same. Thus, if because of other forces the mean obliquity of the ecliptic were variable these same corrections would nevertheless give the true value, provided that we always put for ϑ the mean obliquity.

Corollary 5

60. The latitude of the fixed stars will not undergo any change from this side, since their distance from the ecliptic always remains the same. Thus the mobility of the pole of the earth only produces two effects, of which the first consists in the variation of the longitude of the fixed stars, and the other in the variation of the obliquity of the ecliptic.

Remark 1

61. Having properly established the true quantity of the mean precession of the equinoxes by comparing ancient and modern observations, it will be easy to determine, for any given time, the mean longitude of all the fixed stars, given that we have properly determined the longitude once. I will suppose therefore that we know the mean longitude of a star at a given time and that we want to determine the true longitude of it for the same time. This will be done by the method of two equations that the following two tables will supply.

First correction to the mean longitude of the fixed stars
Given *the longitude of the ascending node of the moon*

	Sign 0 subtract	Sign 1 subtract	Sign 2 subtract	Sign 3 subtract	Sign 4 subtract	Sign 5 subtract	
0°	$0'', 0'''$	$9'', 2'''$	$15'', 40'''$	$18'', 5'''$	$15'', 40'''$	$9'', 2'''$	30
5°	$1'', 34'''$	$10'', 22'''$	$16'', 23'''$	$18'', 0'''$	$14'', 50'''$	$7'', 38'''$	25
10°	$3'', 8'''$	$11'', 37'''$	$17'', 0'''$	$17'', 48'''$	$13'', 51'''$	$6'', 11'''$	20
15°	$4'', 41'''$	$12'', 47'''$	$17'', 19'''$	$17'', 29'''$	$12'', 47'''$	$4'', 41'''$	15
20°	$6'', 1'''$	$13'', 51'''$	$17'', 48'''$	$17'', 0'''$	$11'', 37'''$	$3'', 8'''$	10
25°	$7'', 38'''$	$14'', 50'''$	$18'', 0'''$	$16'', 23'''$	$10'', 22'''$	$1'', 34'''$	5
30°	$9'', 2'''$	$15'', 40'''$	$18'', 5'''$	$15'', 40'''$	$9'', 2'''$	$0'', 0'''$	0
	add Sign 11	add Sign 10	add Sign 9	add Sign 8	add Sign 7	add Sign 6	

Second correction to the mean longitude of the fixed stars
Given *the longitude of the sun*

	Sign 0	Sign 1	Sign 2	Sign 3	Sign 4	Sign 5	
	subtract	subtract	subtract	add	add	add	
0°	$0'', 0'''$	$0'', 59'''$	$0'', 59'''$	$0'', 0'''$	$0'', 59'''$	$0'', 59'''$	30
5°	$0'', 22'''$	$1'', 4'''$	$0'', 52'''$	$0'', 12'''$	$1'', 4'''$	$0'', 52'''$	25
10°	$0'', 23'''$	$1'', 7'''$	$0'', 44'''$	$0'', 23'''$	$1'', 7'''$	$0'', 44'''$	20
15°	$0'', 34'''$	$1'', 8'''$	$0'', 34'''$	$0'', 34'''$	$1'', 8'''$	$0'', 34'''$	15
20°	$0'', 44'''$	$1'', 7'''$	$0'', 23'''$	$0'', 44'''$	$1'', 7'''$	$0'', 23'''$	10
25°	$0'', 52'''$	$1'', 4'''$	$0'', 12'''$	$0'', 52'''$	$1'', 4'''$	$0'', 12'''$	5
30°	$0'', 59'''$	$0'', 59'''$	$0'', 0'''$	$0'', 59'''$	$0'', 59'''$	$0'', 0'''$	0
	add	add	add	subtract	subtract	subtract	
	Sign 11	Sign 10	Sign 9	Sign 8	Sign 7	Sign 6	

Corollary 6

62. The true longitude of the fixed stars will thus be the greatest with respect to the mean, when the ascending node of the moon is found at the beginning of Capricorn, and when the sun is either in Leo 15° or in Aquarius 15°. And then the difference between the true longitude and the mean longitude will be of $19''13'''$.

Corollary 7

63. Now the true longitude of the fixed stars will be the smallest with respect to the mean longitude when the ascending node is at the beginning of Cancer, and when the sun is found either in Taurus 15° or in Scorpio 15°. Then the difference between the true longitude and the mean longitude will be of $19''13'''$. Therefore the difference between the largest and smallest true longitude could rise to $38''26'''$.

Corollary 8

64. But when the ascending node is found in the one of equinoctial points or when the sun is either in the equinoxes or in the solstices, then the mean longitude of the stars will not differ at all from the true value. This will therefore be the best time to observe the mean longitude of the stars.

Remark 2

65. It will be the same for the obliquity of the ecliptic, which will also require a double correction, of which the one depends on the location of the ascending node of the moon, and the other on the location of the sun, from which I have formed the following two tables.

First correction to the mean obliquity of the ecliptic
Given *the longitude of the ascending node of the moon*

	Sign 0	Sign 1	Sign 2	Sign 3	Sign 4	Sign 5	
	add	add	add	subtract	subtract	subtract	
0°	$9'', 41'''$	$8'', 24'''$	$4'', 50'''$	$0'', 0'''$	$4'', 50'''$	$4'', 24'''$	30
5°	$9'', 39'''$	$7'', 57'''$	$4'', 5'''$	$0'', 50'''$	$5'', 33'''$	$8'', 45'''$	25
10°	$9'', 32'''$	$7'', 25'''$	$3'', 19'''$	$1'', 41'''$	$6'', 14'''$	$9'', 5'''$	20
15°	$9'', 21'''$	$6'', 51'''$	$2'', 30'''$	$2'', 30'''$	$6'', 51'''$	$9'', 21'''$	15
20°	$9'', 5'''$	$6'', 14'''$	$1'', 41'''$	$3'', 19'''$	$7'', 25'''$	$9'', 32'''$	10
25°	$8'', 45'''$	$5'', 33'''$	$0'', 50'''$	$4'', 5'''$	$7'', 57'''$	$9'', 39'''$	5
30°	$8'', 24'''$	$4'', 50'''$	$0'', 0'''$	$4'', 50'''$	$8'', 24'''$	$9'', 41'''$	0
	add	add	add	subtract	subtract	subtract	
	Sign 11	Sign 10	Sign 9	Sign 8	Sign 7	Sign 6	

Second correction to the mean obliquity of the ecliptic
Given *the longitude of the sun*

	Sign 0	Sign 1	Sign 2	Sign 3	Sign 4	Sign 5	
	add	add	subtract	subtract	subtract	add	
0°	$30'''$	$15'''$	$15'''$	$30'''$	$15'''$	$15'''$	30
5°	$30'''$	$10'''$	$19'''$	$30'''$	$10'''$	$19'''$	25
10°	$28'''$	$5'''$	$22'''$	$28'''$	$5'''$	$22'''$	20
15°	$25'''$	$0'''$	$25'''$	$25'''$	$0'''$	$25'''$	15
20°	$22'''$	$5'''$	$28'''$	$22'''$	$5'''$	$28'''$	10
25°	$19'''$	$10'''$	$30'''$	$19'''$	$10'''$	$30'''$	5
30°	$15'''$	$15'''$	$30'''$	$15'''$	$15'''$	$30'''$	0
	add	subtract	subtract	subtract	add	add	
	Sign 11	Sign 10	Sign 9	Sign 8	Sign 7	Sign 6	

Corollary 9

66. The obliquity of the ecliptic therefore will be the largest when the ascending node of the moon is found in the point of the vernal equinox and when the sun is also in the equinoxes. For then the true obliquity of the ecliptic will surpass the mean by $10''11'''$.

Corollary 10

67. Now the obliquity of the ecliptic will be the smallest when the ascending node of the moon is in the point of the autumnal equinox, and when the sun is found in one or the other solstice. Then the true obliquity of the ecliptic will be surpassed by the mean by $10''11'''$, and the variations of the obliquity of the ecliptic will rise to $20''22'''$.

Corollary 11

68. But if the ascending node is in one or the other solstice, and if the location of the sun is either Taurus $15°$ or Leo $15°$ or Scorpio $15°$ or Aquarius $15°$, then the true obliquity of the ecliptic will not differ at all from the mean obliquity.

Problem 9

69. *To determine the true quantity of the precession of the equinoxes during the space of a given year.*

Solution

Let us look for the longitude of the ascending node for the middle of the year during which we wish to know the precession of the equinoxes, and let s be this longitude of the node, which corresponds to the middle of the proposed year. Since the annual movement of the nodes is $19°20'$, at the beginning of our year the longitude of the ascending node will have been $s + 9°40'$, and at the end $s + 9°40'$, and since the longitude of the sun is the same at the beginning of the year as at the end, it will contribute nothing to the annual precession.

Therefore at the beginning of the year the longitude of the star having been

$$L - 18.08'' \sin(s + 9°40') - 1.13'' \sin 2p$$

at the end of this year it will be

$$L + 50.3'' - 18.08'' \sin(s - 9°40') - 1.13'' \sin 2p.$$

hence the precession of the equinoxes during this year

$$= 50.3'' - 18.08'' \sin(s - 9°40') + 18.08'' \sin(s + 9°40').$$

This precession will thus be

$$50.3'' + 36.16'' \cos s \sin 9°40'$$

or $50.3'' + 6.07'' \cos s$.

Therefore having determined the longitude of the ascending node for the middle of the proposed year, we will easily find how much the equinoxes will recede during each year.
Q.E.D.

Corollary 1

70. The precession of the equinoxes will thus be greatest in those years during the course of which the ascending node is found at the beginning of Aries, and in these years the precession of the equinoxes will be $56.37''$ or $56''22'''$.

Corollary 2

71. Now the precession of the equinoxes will be the least in those years, during the course of which the ascending node is found at the beginning of Libra, and then the precession will only be $44.23''$ or $44''14'''$. Therefore the difference between the greatest and least annual precession will be of $12''8'''$.

Corollary 3

72. If we know the longitude of the ascending node for the beginning of the proposed year, and if we let it $= u$, and since $u = s + 9°40'$, we will have $s = u - 9°40'$, and the precession of the equinoxes during this year will be

$$50.3'' + 6.07'' \cos(u - 9°40').$$

Remark

73. From this I have calculated the following table, by which we will find the precession of the equinoxes for each year, for the beginning of which we know the longitude of the ascending node of the moon.

Tabulation of the quantity of the precession of
the equinoxes for each given year
Given *the longitude of the ascending node of the moon at the beginning of the year*

	Sign 0 (Aries)	Sign 1 (Taurus)	Sign 2 (Gemini)	Sign 3 (Cancer)	Sign 4 (Leo)	Sign 5 (Virgo)
0°	$56'', 17'''$	$56'', 0'''$	$54'', 12'''$	$51'', 20'''$	$48'', 14'''$	$45'', 39'''$
5°	$56'', 21'''$	$55'', 48'''$	$53'', 48'''$	$50'', 49'''$	$47'', 44'''$	$45'', 20'''$
10°	$56'', 22'''$	$55'', 34'''$	$53'', 20'''$	$50'', 18'''$	$47'', 16'''$	$45'', 2'''$
15°	$56'', 21'''$	$55'', 18'''$	$52'', 52'''$	$49'', 48'''$	$46'', 48'''$	$44'', 48'''$
20°	$56'', 17'''$	$54'', 57'''$	$52'', 22'''$	$49'', 16'''$	$46'', 24'''$	$44'', 36'''$
25°	$56'', 10'''$	$54'', 30'''$	$51'', 52'''$	$48'', 44'''$	$45'', 59'''$	$44'', 26'''$
30°	$56'', 0'''$	$54'', 12'''$	$51'', 20'''$	$48'', 14'''$	$45'', 39'''$	$44'', 19'''$

Continuation of this table for the southern signs

	Sign 0 (Libra)	Sign 1 (Scorpio)	Sign 2 (Sagittarius)	Sign 3 (Capricorn)	Sign 4 (Aquarius)	Sign 5 (Pisces)
0°	$44'', 19'''$	$44'', 36'''$	$46'', 24'''$	$49'', 16'''$	$52'', 22'''$	$54'', 57'''$
5°	$44'', 15'''$	$44'', 48'''$	$46'', 48'''$	$49'', 48'''$	$52'', 52'''$	$55'', 18'''$
10°	$44'', 14'''$	$45'', 2'''$	$47'', 16'''$	$50'', 18'''$	$53'', 20'''$	$55'', 34'''$
15°	$44'', 15'''$	$45'', 20'''$	$47'', 44'''$	$50'', 49'''$	$53'', 48'''$	$55'', 48'''$
20°	$44'', 19'''$	$45'', 39'''$	$48'', 14'''$	$51'', 20'''$	$54'', 12'''$	$56'', 0'''$
25°	$44'', 26'''$	$45'', 59'''$	$48'', 44'''$	$51'', 52'''$	$54'', 36'''$	$56'', 10'''$
30°	$44'', 36'''$	$46'', 24'''$	$49'', 16'''$	$52'', 22'''$	$54'', 57'''$	$56'', 17'''$

Corollary 4

74. Let us take for example the year 1750, at the beginning of which the location of the ascending node is found at Capricorn $10°17'$. Therefore from the first of January 1750 to the first of January 1751 the precession of the equinoxes will be $50''18'''$, or it will be about equal to the average annual precession.

Corollary 5

75. Since at the beginning of the year 1746 the position of the ascending node was close to the vernal equinox, known to be at Pisces $27°40'$, the annual precession of the equinoxes for the year 1745 and the following years will be[11]:

Year	Annual Precession	Year	Annual Precession
1745	$56''20'''$	1765	$56''\ 0'''$
1746	$56''15'''$	1766	$55''\ 0'''$
1747	$55''25'''$	1767	$53''20'''$
1748	$54''10'''$	1768	$51''40'''$
1749	$52''20'''$	1769	$49''40'''$
1750	$50''20'''$	1770	$47''30'''$
1751	$48''20'''$	1771	$45''40'''$
1752	$46''30'''$	1772	$44''50'''$
1753	$45''10'''$	1773	$44''15'''$
1754	$44''20'''$	1774	$44''20'''$
1755	$44''15'''$	1775	$45''10'''$
1756	$44''50'''$	1776	$46''30'''$
1757	$45''50'''$	1777	$48''30'''$
1758	$47''40'''$	1778	$50''20'''$
1759	$49''40'''$	1779	$52''20'''$
1760	$51''50'''$	1780	$54''10'''$
1761	$53''30'''$	1781	$55''30'''$
1762	$55''10'''$	1782	$56''15'''$
1763	$56''\ 0'''$	1783	$56''20'''$
1764	$56''22'''$	1784	$55''40'''$

We can use this table to see how much the observations on the mobility of the axis of the earth will be in agreement with this Theory.

[11] In the *Opera Omnia*, the annual precession given for 1774 is $40''20'''$, but it has been corrected here.