Profinite number theory

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The factorial number system

Each $n \in \mathbf{Z}_{\geq 0}$ has a unique representation

$$n = \sum_{i=1}^{\infty} c_i i! \quad \text{with } c_i \in \mathbf{Z},$$

$$0 \leq c_i \leq i, \quad \#\{i : c_i \neq 0\} < \infty.$$

In factorial notation:

$$n = (\ldots c_3 c_2 c_1)_!.$$

Examples: $25 = (1001)_!$, $1001 = (121221)_!$.

Note: $c_1 \equiv n \bmod 2$.

Conversion

Given n, one finds all c_i by

$$c_1 = (\text{remainder of } n_1 = n \text{ upon division by } 2),$$

$$c_i = (\text{remainder of } n_i = \frac{n_{i-1} - c_{i-1}}{i} \text{ upon division by } i+1),$$

until $n_i = 0$.

Knowing $c_1, c_2, \ldots, c_{k-1}$ is equivalent to knowing n modulo $k!$.

Profinite numbers

If one starts with $n = -1$, one finds $c_i = i$ for all i:

$$-1 = (\dots 54321)_!.$$

In general, for a negative integer n one finds $c_i = i$ for almost all i.

A *profinite integer* is an infinite string $(\dots c_3 c_2 c_1)_!$ with each $c_i \in \mathbf{Z}$, $0 \le c_i \le i$.

Notation: $\hat{\mathbf{Z}} = \{\text{profinite integers}\}$.

A citizen of the world

Features of $\hat{\mathbf{Z}}$:

- it has an *algebraic structure*,
- it comes with a *topology*,
- it occurs in *Galois theory*,
- it shows up in *arithmetic geometry*,
- it connects to *ultrafilters*,
- it carries *"analytic" functions*,
- and it knows *Fibonacci numbers*!

Addition and multiplication

For any k, the last k digits of $n + m$ depend only on the last k digits of n and of m.

Likewise for $n \cdot m$.

Hence one can also define the sum and the product of *any* two profinite integers, and $\hat{\mathbf{Z}}$ is a *commutative ring*.

Ring homomorphisms

Call a profinite integer $(\ldots c_3 c_2 c_1)_!$ *even* if $c_1 = 0$ and *odd* if $c_1 = 1$.

The map $\hat{\mathbf{Z}} \to \mathbf{Z}/2\mathbf{Z}$, $(\ldots c_3 c_2 c_1)_! \mapsto (c_1 \bmod 2)$, is a ring homomorphism. Its kernel is $2\hat{\mathbf{Z}}$.

More generally, for any $k \in \mathbf{Z}_{>0}$, one has a ring homomorphism $\hat{\mathbf{Z}} \to \mathbf{Z}/k!\mathbf{Z}$ sending $(\ldots c_3 c_2 c_1)_!$ to $(\sum_{i<k} c_i i! \bmod k!)$, and it has kernel $k!\hat{\mathbf{Z}}$.

Visualising profinite numbers

Define $\mathrm{v}\colon \hat{\mathbf{Z}} \to [0,1]$ by

$$\mathrm{v}((\ldots c_3 c_2 c_1)_!) = \sum_{i \geq 1} \frac{c_i}{(i+1)!}.$$

Then $\mathrm{v}(2\hat{\mathbf{Z}}) = [0, \frac{1}{2}]$, $\mathrm{v}(1+2\hat{\mathbf{Z}}) = [\frac{1}{2}, 1]$, $\mathrm{v}(1+6\hat{\mathbf{Z}}) = [\frac{1}{2}, \frac{2}{3}]$.

One has

$$\#\mathrm{v}^{-1} r = 2 \text{ for } r \in \mathbf{Q} \cap (0,1),$$
$$\#\mathrm{v}^{-1} r = 1 \text{ for all other } r \in [0,1].$$

Examples:

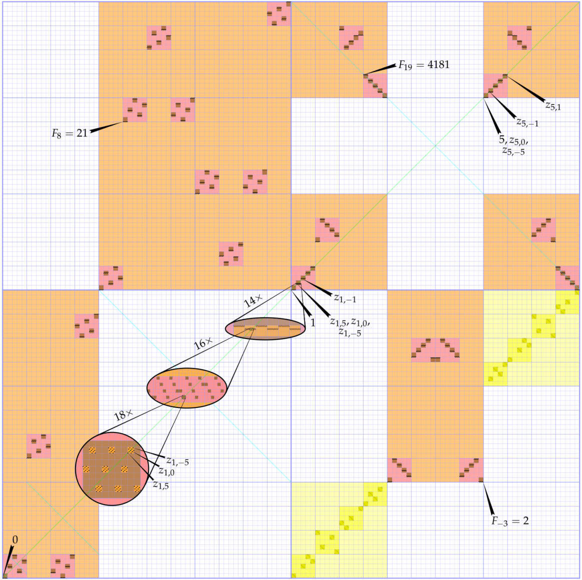
$$\mathrm{v}^{-1}\tfrac{1}{2} = \{-2, 1\}, \quad \mathrm{v}^{-1}\tfrac{2}{3} = \{-5, 3\}, \quad \mathrm{v}^{-1}1 = \{-1\}.$$

Graphs

For graphical purposes, we represent $a \in \hat{\mathbf{Z}}$ by $\mathrm{v}(a) \in [0,1]$.

We visualise a function $f\colon \hat{\mathbf{Z}} \to \hat{\mathbf{Z}}$ by representing its graph $\{(a, f(a)) : a \in \hat{\mathbf{Z}}\}$ in $[0,1] \times [0,1]$.

Illustration by Willem Jan Palenstijn



Four functions

In green: the graph of $a \mapsto a$.

In blue: the graph of $a \mapsto -a$.

In yellow: the graph of $a \mapsto a^{-1} - 1$ ($a \in \hat{\mathbf{Z}}^*$).

In orange/red/brown: the graph of $a \mapsto F(a)$, the "a-th Fibonacci number".

A formal definition

A more satisfactory definition is

$$\hat{\mathbf{Z}} = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} (\mathbf{Z}/n\mathbf{Z}) : n|m \Rightarrow a_m \equiv a_n \bmod n\}.$$

This is a subring of $\prod_{n=1}^{\infty}(\mathbf{Z}/n\mathbf{Z})$.

Its unit group $\hat{\mathbf{Z}}^*$ is a subgroup of $\prod_{n=1}^{\infty}(\mathbf{Z}/n\mathbf{Z})^*$.

Alternative definition: $\hat{\mathbf{Z}} = \mathrm{End}(\mathbf{Q}/\mathbf{Z})$, the *endomorphism ring* of the abelian group $\mathbf{Q}/\mathbf{Z}$. Then $\hat{\mathbf{Z}}^* = \mathrm{Aut}(\mathbf{Q}/\mathbf{Z})$.

Basic facts

The ring $\hat{\mathbf{Z}}$ is *uncountable*, it is *commutative*, and it has $\mathbf{Z}$ as a subring. It has lots of zero-divisors.

For each $m \in \mathbf{Z}_{>0}$, there is a ring homomorphism

$$\hat{\mathbf{Z}} \to \mathbf{Z}/m\mathbf{Z}, \quad a = (a_n)_{n=1}^{\infty} \mapsto a_m,$$

which together with the group homomorphism $\hat{\mathbf{Z}} \to \hat{\mathbf{Z}}$, $a \mapsto ma$, fits into a short exact sequence

$$0 \to \hat{\mathbf{Z}} \xrightarrow{m} \hat{\mathbf{Z}} \to \mathbf{Z}/m\mathbf{Z} \to 0.$$

Profinite rationals

Write

$$\hat{\mathbf{Q}} = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} (\mathbf{Q}/n\mathbf{Z}) : n|m \Rightarrow a_m \equiv a_n \bmod n\mathbf{Z}\}.$$

The additive group $\hat{\mathbf{Q}}$ has exactly one ring multiplication extending the ring multiplication on $\hat{\mathbf{Z}}$.

It is a commutative ring, with $\mathbf{Q}$ and $\hat{\mathbf{Z}}$ as subrings, and

$$\hat{\mathbf{Q}} = \mathbf{Q} + \hat{\mathbf{Z}} = \mathbf{Q} \cdot \hat{\mathbf{Z}} \cong \mathbf{Q} \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$$

(as rings).

Topology

If each $\mathbf{Z}/n\mathbf{Z}$ has the discrete topology and $\prod_{n=1}^{\infty}(\mathbf{Z}/n\mathbf{Z})$ the product topology, then $\hat{\mathbf{Z}}$ is *closed* in $\prod_{n=1}^{\infty}(\mathbf{Z}/n\mathbf{Z})$.

One can define the topology on $\hat{\mathbf{Z}}$ by the metric

$$\begin{aligned} \mathrm{d}(x,y) &= \frac{1}{\min\{k \in \mathbf{Z}_{>0} : x \not\equiv y \bmod (k+1)!\}} \\ &= \frac{1}{\min\{k \in \mathbf{Z}_{>0} : c_k \neq d_k\}} \end{aligned}$$

if $x = (\ldots c_3 c_2 c_1)_!$, $y = (\ldots d_3 d_2 d_1)_!$, $x \neq y$.

More topology

Fact: $\hat{\mathbf{Z}}$ is a compact Hausdorff totally disconnected topological ring.

One can make the map $\mathrm{v}\colon \hat{\mathbf{Z}} \to [0,1]$ into a homeomorphism by "cutting" $[0,1]$ at every $r \in \mathbf{Q} \cap (0,1)$.

A neighborhood base of 0 in $\hat{\mathbf{Z}}$ is $\{m\hat{\mathbf{Z}} : m \in \mathbf{Z}_{>0}\}$.

With the same neighborhood base, $\hat{\mathbf{Q}}$ is also a topological ring. It is *locally* compact, Hausdorff, and totally disconnected.

Amusements for algebraists

We have $\hat{\mathbf{Z}} \subset \mathrm{A} = \prod_{n=1}^{\infty} (\mathbf{Z}/n\mathbf{Z})$.

Theorem. *One has* $\mathrm{A}/\hat{\mathbf{Z}} \cong \mathrm{A}$ *as additive topological groups.*

Proof (Carlo Pagano): write down a surjective continuous group homomorphism $\epsilon\colon \mathrm{A} \to \mathrm{A}$ with $\ker \epsilon = \hat{\mathbf{Z}}$.

Theorem. *One has* $\mathrm{A} \cong \mathrm{A} \times \hat{\mathbf{Z}}$ *as groups but not as topological groups.*

Here the axiom of choice comes in.

Profinite groups

In infinite Galois theory, the Galois groups that one encounters are *profinite groups*.

A profinite group is a topological group that is isomorphic to a closed subgroup of a product of finite discrete groups.

Equivalent definition: it is a compact Hausdorff totally disconnected topological group.

Examples: the additive group of $\hat{\mathbf{Z}}$ and its unit group $\hat{\mathbf{Z}}^*$ are profinite groups.

$\hat{\mathbf{Z}}$ as the analogue of $\mathbf{Z}$

Familiar fact. For each group G and each $\gamma \in G$ there is a unique group homomorphism $\mathbf{Z} \to G$ with $1 \mapsto \gamma$, namely $n \mapsto \gamma^n$.

Analogue for $\hat{\mathbf{Z}}$. For each profinite group G and each $\gamma \in G$ there is a unique group homomorphism $\hat{\mathbf{Z}} \to G$ with $1 \mapsto \gamma$, and it is continuous. Notation: $a \mapsto \gamma^a$.

Examples of infinite Galois groups

For a field k, denote by $\bar{k}$ an algebraic closure.

Example 1: with p prime and $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ one has

$$\hat{\mathbf{Z}} \cong \mathrm{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p), \quad a \mapsto \mathrm{Frob}^a,$$

where $\mathrm{Frob}(\alpha) = \alpha^p$ for all $\alpha \in \bar{\mathbf{F}}_p$.

Example 2: with

$$\mu = \{\text{roots of unity in } \bar{\mathbf{Q}}^*\} \cong \mathbf{Q}/\mathbf{Z}$$

one has

$$\mathrm{Gal}(\mathbf{Q}(\mu)/\mathbf{Q}) \cong \mathrm{Aut}\,\mu \cong \hat{\mathbf{Z}}^*$$

as topological groups.

Radical Galois groups

Example 3. For $r \in \mathbf{Q}$, $r \notin \{-1, 0, 1\}$, put

$$\sqrt[\infty]{r} = \{\alpha \in \bar{\mathbf{Q}} : \exists n \in \mathbf{Z}_{>0} : \alpha^n = r\}.$$

Theorem (Abtien Javanpeykar). *Let G be a profinite group. Then there exists $r \in \mathbf{Q} \backslash \{-1, 0, 1\}$ with $G \cong \mathrm{Gal}(\mathbf{Q}(\sqrt[\infty]{r})/\mathbf{Q})$ (as topological groups) if and only if there is a non-split exact sequence*

$$0 \to \hat{\mathbf{Z}} \xrightarrow{\iota} G \xrightarrow{\pi} \hat{\mathbf{Z}}^* \to 1$$

of profinite groups such that

$$\forall a \in \hat{\mathbf{Z}}, \gamma \in G : \gamma \cdot \iota(a) \cdot \gamma^{-1} = \iota(\pi(\gamma) \cdot a).$$

Arithmetic geometry

Given $f_1, \ldots, f_k \in \mathbf{Z}[X_1, \ldots, X_n]$, one wants to solve the system $f_1(x) = \ldots = f_k(x) = 0$ in $x = (x_1, \ldots, x_n) \in \mathbf{Z}^n$.

Theorem. (a) *There is a solution* $x \in \mathbf{Z}^n \Rightarrow$ *for each* $m \in \mathbf{Z}_{>0}$ *there is a solution modulo* $m \Leftrightarrow$ *there is a solution* $x \in \hat{\mathbf{Z}}^n$.

(b) *It is decidable whether a given system has a solution* $x \in \hat{\mathbf{Z}}^n$.

p-adic numbers

Let p be prime. The *ring of p-adic integers* is

$$\mathbf{Z}_p = \{(b_i)_{i=0}^{\infty} \in \prod_{i=0}^{\infty}(\mathbf{Z}/p^i\mathbf{Z}) : i \leq j \Rightarrow b_j \equiv b_i \bmod p^i\}.$$

Just as $\hat{\mathbf{Z}}$, it is a compact Hausdorff totally disconnected topological ring.

It is also a *principal ideal domain*, with $p\mathbf{Z}_p$ as its only non-zero prime ideal. Its field of fractions is written $\mathbf{Q}_p$.

All ideals of $\mathbf{Z}_p$ are *closed*, and of the form $p^h\mathbf{Z}_p$ with $h \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$, where $p^\infty\mathbf{Z}_p = \{0\}$.

The Chinese remainder theorem

For $n = \prod_{p \text{ prime}} p^{i(p)}$ one has

$$\mathbf{Z}/n\mathbf{Z} \cong \prod_{p \text{ prime}} (\mathbf{Z}/p^{i(p)}\mathbf{Z}) \qquad \text{(as rings).}$$

In the limit:

$$\hat{\mathbf{Z}} \cong \prod_{p \text{ prime}} \mathbf{Z}_p \qquad \text{(as topological rings).}$$

For each p, the projection map $\hat{\mathbf{Z}} \to \mathbf{Z}_p$ induces a ring homomorphism $\pi_p \colon \hat{\mathbf{Q}} \to \mathbf{Q}_p$.

Profinite number theory

The isomorphism $\hat{\mathbf{Z}} \cong \prod_p \mathbf{Z}_p$ reduces most questions that one may ask about $\hat{\mathbf{Z}}$ to similar questions about the much better behaved rings $\mathbf{Z}_p$.

Profinite number theory studies the exceptions. Many of these are caused by the set $\mathcal{P}$ of primes being *infinite*.

Ideals of $\hat{\mathbf{Z}}$

For an ideal $\mathfrak{a} \subset \hat{\mathbf{Z}} = \prod_p \mathbf{Z}_p$, one has:

$$\mathfrak{a} \text{ is closed} \Leftrightarrow \mathfrak{a} \text{ is finitely generated} \Leftrightarrow \mathfrak{a} \text{ is principal}$$

$$\Leftrightarrow \ \mathfrak{a} = \prod_p \mathfrak{a}_p \text{ where each } \mathfrak{a}_p \subset \mathbf{Z}_p \text{ an ideal.}$$

The set of closed ideals of $\hat{\mathbf{Z}}$ is in bijection with the set $\{\prod_p p^{h(p)} : h(p) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}\}$ of *Steinitz numbers*.

Most ideals of $\hat{\mathbf{Z}}$ are not closed.

The spectrum and ultrafilters

The *spectrum* $\operatorname{Spec} R$ of a commutative ring R is its set of prime ideals. *Example*: $\operatorname{Spec} \mathbf{Z}_p = \{\{0\}, p\mathbf{Z}_p\}$.

With each $\mathfrak{p} \in \operatorname{Spec} \hat{\mathbf{Z}}$ one associates the *ultrafilter*

$$\Upsilon(\mathfrak{p}) = \{S \subset \mathcal{P} : e_S \in \mathfrak{p}\}$$

on the set $\mathcal{P}$ of primes, where $e_S \in \prod_{p \in \mathcal{P}} \mathbf{Z}_p = \hat{\mathbf{Z}}$ has coordinate 0 at $p \in S$ and 1 at $p \notin S$.

Then $\mathfrak{p}$ is closed if and only if $\Upsilon(\mathfrak{p})$ is principal, and

$$\Upsilon(\mathfrak{p}) = \Upsilon(\mathfrak{q}) \Leftrightarrow \mathfrak{p} \subset \mathfrak{q} \text{ or } \mathfrak{q} \subset \mathfrak{p}.$$

The logarithm

$u \in \mathbf{R}_{>0} \Rightarrow \log u = (\frac{\mathrm{d}}{\mathrm{d}x} u^x)_{x=0} = \lim_{\epsilon \to 0} \frac{u^\epsilon - 1}{\epsilon}$.

Analogously, define $\log \colon \hat{\mathbf{Z}}^* \to \hat{\mathbf{Z}}$ by

$$\log u = \lim_{n \to \infty} \frac{u^{n!} - 1}{n!}.$$

This is a well-defined continuous group homomorphism.

Its kernel is $\hat{\mathbf{Z}}^*_{\mathrm{tor}}$, which is the closure of the set of elements of finite order in $\hat{\mathbf{Z}}^*$.

Its image is $2\mathrm{J} = \{2x : x \in \mathrm{J}\}$, where $\mathrm{J} = \bigcap_p p\hat{\mathbf{Z}}$ is the *Jacobson radical* of $\hat{\mathbf{Z}}$.

Structure of $\hat{\mathbf{Z}}^*$

The logarithm fits in a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \hat{\mathbf{Z}}^*_{\mathrm{tor}} \arrow[r] \arrow[d, "\wr"] & \hat{\mathbf{Z}}^* \arrow[r, "\log"] \arrow[d, equal] & 2\mathrm{J} \arrow[r] & 0 \\
1 & (\hat{\mathbf{Z}}/2\mathrm{J})^* \arrow[l] & \hat{\mathbf{Z}}^* \arrow[l] & 1 + 2\mathrm{J} \arrow[l] \arrow[u, "\wr"'] & 1 \arrow[l]
\end{tikzcd}$$

of profinite groups, where the other horizontal maps are the natural ones, the rows are exact, and the vertical maps are *isomorphisms*.

Corollary: $\hat{\mathbf{Z}}^* \cong (\hat{\mathbf{Z}}/2\mathrm{J})^* \times 2\mathrm{J}$ *(as topological groups).*

More on $\hat{\mathbf{Z}}^*$

Less canonically, with $\mathrm{A} = \prod_{n \geq 1} (\mathbf{Z}/n\mathbf{Z})$:

$$2\mathrm{J} \cong \hat{\mathbf{Z}},$$

$$(\hat{\mathbf{Z}}/2\mathrm{J})^* \cong (\mathbf{Z}/2\mathbf{Z}) \times \prod_p (\mathbf{Z}/(p-1)\mathbf{Z}) \cong \mathrm{A},$$

$$\hat{\mathbf{Z}}^* \cong \mathrm{A} \times \hat{\mathbf{Z}},$$

as topological groups, and

$$\hat{\mathbf{Z}}^* \cong \mathrm{A}$$

as groups.

Power series expansions

The inverse isomorphisms

$$\log\colon 1 + 2\mathrm{J} \xrightarrow{\sim} 2\mathrm{J}$$
$$\exp\colon 2\mathrm{J} \xrightarrow{\sim} 1 + 2\mathrm{J}$$

are given by power series expansions

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad \exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

that converge for all $x \in 2\mathrm{J}$.

The logarithm is analytic on all of $\hat{\mathbf{Z}}^*$ in a weaker sense.

Analyticity

Let $x_0 \in D \subset \hat{\mathbf{Q}}$. We call $f \colon D \to \hat{\mathbf{Q}}$ *analytic in* x_0 if there is a sequence $(a_n)_{n=0}^{\infty} \in \hat{\mathbf{Q}}^{\infty}$ such that one has

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n$$

in the sense that for each prime p there is a neighborhood U of x_0 in D such that for all $x \in U$ the equality

$$\pi_p(f(x)) = \sum_{n=0}^{\infty} \pi_p(a_n) \cdot (\pi_p(x) - \pi_p(x_0))^n$$

is valid in the topological field $\mathbf{Q}_p$.

Examples of analytic functions

The map $\log\colon \hat{\mathbf{Z}}^* \to \hat{\mathbf{Z}} \subset \hat{\mathbf{Q}}$ is analytic in each $x_0 \in \hat{\mathbf{Z}}^*$, with expansion

$$\log x = \log x_0 - \sum_{n=1}^{\infty} \frac{(x_0 - x)^n}{n \cdot x_0^n}.$$

For each $u \in \hat{\mathbf{Z}}^*$, the map

$$\hat{\mathbf{Z}} \to \hat{\mathbf{Z}}^* \subset \hat{\mathbf{Q}}, \qquad x \mapsto u^x$$

is analytic in each $x_0 \in \hat{\mathbf{Z}}$, with expansion

$$u^x = \sum_{n=0}^{\infty} \frac{(\log u)^n \cdot u^{x_0} \cdot (x - x_0)^n}{n!}.$$

A Fibonacci example

Define $F\colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ by

$$F(0) = 0, \quad F(1) = 1, \quad F(n+2) = F(n+1) + F(n).$$

Theorem. *The function F has a unique continuous extension $\hat{\mathbf{Z}} \to \hat{\mathbf{Z}}$, and it is analytic in each $x_0 \in \hat{\mathbf{Z}}$.*

Notation: F.

For $n \in \mathbf{Z}$, one has

$$F(n) = n \Leftrightarrow n \in \{0, 1, 5\}.$$

Up to eleven

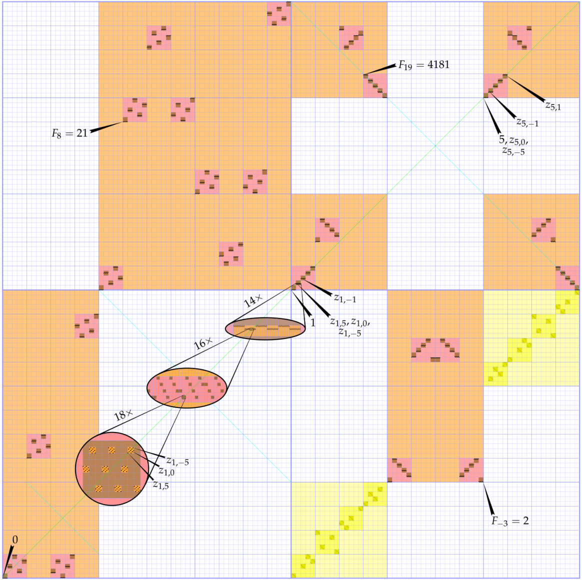
One has $\#\{x \in \hat{\mathbf{Z}} : F(x) = x\} = 11$.

The only *even* fixed point of F is 0, and for each $a \in \{1, 5\}$, $b \in \{-5, -1, 0, 1, 5\}$ there is a unique fixed point $z_{a,b}$ with

$$z_{a,b} \equiv a \bmod \bigcap_{n=0}^{\infty} 6^n \hat{\mathbf{Z}}, \quad z_{a,b} \equiv b \bmod \bigcap_{n=0}^{\infty} 5^n \hat{\mathbf{Z}}.$$

Examples: $z_{1,1} = 1$, $z_{5,5} = 5$.

Illustration by Willem Jan Palenstijn



Graphing the fixed points

The graph of $a \mapsto F(a)$ is shown in orange/red/brown.

Intersecting the graph with the diagonal one obtains the fixed points 0 and $z_{a,b}$, for $a = 1, 5$, $b = -5, -1, 0, 1, 5$.

Surprise: one has $z_{5,-5}^2 - 25 = \sum_{i=1}^{\infty} c_i i!$ with $c_i = 0$ for $i \leq 200$ and $c_{201} \neq 0$.

Larger cycles

I believe:

$$\#\{x \in \hat{\mathbf{Z}} : F(F(x)) = x\} = 21,$$
$$\#\{x \in \hat{\mathbf{Z}} : F^n(x) = x\} < \infty \quad \text{for each } n \in \mathbf{Z}_{>0}.$$

Question: does F have cycles of length greater than 2?

Other linear recurrences

If $E\colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}$, $t \in \mathbf{Z}_{>0}$, $d_0, \ldots, d_{t-1} \in \mathbf{Z}$ satisfy

$$\forall n \in \mathbf{Z}_{\geq 0} : E(n+t) = \sum_{i=0}^{t-1} d_i \cdot E(n+i),$$
$$d_0 \in \{1, -1\},$$

then E has a unique continuous extension $\hat{\mathbf{Z}} \to \hat{\mathbf{Z}}$. It is analytic in each $x_0 \in \hat{\mathbf{Z}}$.

Finite cycles

Suppose also $X^t - \sum_{i=0}^{t-1} d_i X^i = \prod_{i=1}^{t} (X - \alpha_i)$, where

$$\alpha_1, \ldots, \alpha_t \in \mathbf{Q}(\sqrt{\mathbf{Q}}),$$
$$\alpha_j^{24} \neq \alpha_k^{24} \qquad (1 \leq j < k \leq t).$$

Tentative theorem. *If $n \in \mathbf{Z}_{>0}$ is such that the set*

$$S_n = \{x \in \hat{\mathbf{Z}} : E^n(x) = x\}$$

is infinite, then $S_n \cap \mathbf{Z}_{\geq 0}$ contains an infinite arithmetic progression.

This would imply that $\{x \in \hat{\mathbf{Z}} : F^n(x) = x\}$ is finite for each $n \in \mathbf{Z}_{>0}$.

Who's who

Fibonacci, Italian mathematician, $\sim$1170–$\sim$1250.

Évariste Galois, French mathematician, 1811–1832.

Ferdinand Georg Frobenius, German mathematician, 1849–1917.

Felix Hausdorff, German mathematician, 1868–1942.

Ernst Steinitz, German mathematician, 1871–1928.

Nathan Jacobson, American mathematician, 1910–1999.

Willem Jan Palenstijn, Dutch mathematician, 1980.

Abtien Javanpeykar, Dutch mathematics student, 1989.

Carlo Pagano, Italian mathematics student, 1990.